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C 17
V. 22

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CONTENTS

VOL. XXII

PAGE

On certain nets of plane curves. By Mr F. P. **White**, St John's College . . . 1

The conics through fives of six points. By Mr F. P. **White**, St John's College . . . 11

On the stability of the periodic states of the triode oscillator. By Mr W. M. H. **Greaves**, St John's College . . . 16

Involutions on a normal quartic curve in space of four dimensions. By Mr C. G. F. **James**, Trinity College . . . 24

On errors of observation. By Dr W. **Burnside**, Pembroke College . . . 26

On the generalisation of a theorem of Steiner. By Professor H. F. **Baker** . . . 28

On an extension of Wallace's pedal property of the circumcircle. By Mr H. W. **Richmond**, King's College . . . 34

On the real twisted cubic curves which are geodesic upon a quadric surface. By Mr H. W. **Richmond**, King's College . . . 42

On the number of lines which meet four regions in hyper-space. By R. **Vaidyanathaswamy**. (Communicated by Mr H. W. **Turnbull**, Trinity College) . . . 49

The mechanism of the rusting of iron. By Mr U. R. **Evans**, King's College . . . 54

Angular momentum and electron impact. By P. M. S. **Blackett**, King's College. (Four Text-figures) . . . 56

Hankel transforms. By E. C. **Titchmarsh**. (Communicated by Mr G. H. **Hardy**) . . . 67

On a modified electrostatic oscillograph. By Dr J. A. **Crowther**, St John's College. (One Text-figure) . . . 68

An oscillographic study of the production of X-rays by a Coolidge tube. By Dr J. A. **Crowther**, St John's College. (Plates I, II) . . . 70

An early formulation by Stokes of the theories of the rotatory polarizations of light. By Sir **Joseph Larmor**, Lucasian Professor . . . 76

On Graeffe's method for complex roots of algebraic equations. By Mr S. **Brodetsky**, Trinity College, and G. **Smeal** . . . 83

On the problem of four particles. By Mr J. **Brill**, St John's College . . . 88

Canonical forms of the quaternary cubic associated with arbitrary quadrics. By Mr H. W. **Turnbull**, Trinity College . . . 92

On protecting and sensitising colloidal sols. By Mr **Eric K. Rideal** . . . 101

The equation of state of a gas. By J. E. **Jones**, 1851 Exhibition Senior Research Student, Trinity College, Cambridge. (Communicated by Mr R. H. **Fowler**) . . . 105

On the differential invariants of the Laguerre group. By **Tadahiko Kubota**. (Communicated by Professor H. F. **Baker**) . . . 113

PAGE

On the penetration of waves and their rays into a medium of gradually varying properties. By J. A. Wilcken, B.Sc. (Copenhagen), Ph.D. (La Plata); late of Christ's College. (Communicated by Professor Sir Joseph Larmor) 124

Dissociation under a temperature gradient. By P. A. M. Dirac, St John's College, Cambridge. (Communicated by Mr R. H. Fowler) . . 132

The electromagnetic equations as basis of Einstein's quadratic form. By Mr R. Hargreaves, St John's College. (One Text-figure) . . 138

On the polygons inscribed in one conic and circumscribed to another. By Dr W. Burnside, Pembroke College, Cambridge 163

The problems of random flight and conduction of heat. By Dr W. Burnside, Pembroke College, Cambridge 167

The integral expansions of arbitrary functions connected with integral equations. By J. Hyslop, St John's College. (Communicated by Professor E. W. Hobson) 169

On the angle of incidence in soaring flight. By Dr E. H. Hankin, St John's College, and J. D. North. (One Text-figure) 186

On five lines, in space of four dimensions, which lie upon a quadric threefold and the normal quartic curves of which they are chords. By F. Bath, King's College. (Communicated by Mr H. W. Richmond, King's College) 189

Complexes of conics and the Weddle surface. By Mr C. G. F. James, Trinity College 201

Trajectories of small horizontal velocity in a resisting medium. By Mr J. E. Littlewood, Trinity College, Cayley Lecturer 217

Two notes on the Riemann zeta-function. By Mr J. E. Littlewood, Trinity College, Cayley Lecturer 234

On the applicability and deformation of surfaces. By B. M. Sen, King's College. (Communicated by Professor H. F. Baker) 243

The quadratic form for radial acceleration, in the theory of relativity. By Mr R. Hargreaves, St John's College 248

The statistical theory of dissociation and ionization by collision, with applications to the capture and loss of electrons by α-particles. By Mr R. H. Fowler, Trinity College 253

Integrals of systems of ordinary differential equations. By Dr T. M. Cherry, Trinity College 273

A system of linear equations with an infinity of unknowns. By E. C. Titchmarsh. (Communicated by Mr G. H. Hardy, Trinity College) 282

On Poincaré's theorem of "the non-existence of uniform integrals of dynamical equations." By Dr T. M. Cherry, Trinity College . 287

On the zeros of the Riemann zeta-function. By Mr J. E. Littlewood, Trinity College, Cayley Lecturer 295

On loci which have two systems of generating spaces. By Mr H. W. Richmond and F. Bath, King's College 319

PAGE

On integrals developable about a singular point of a Hamiltonian system of differential equations. By Dr T. M. Cherry, Trinity College . 325

On a case of steady flow of a gas in two dimensions. By Mr S. Lees, St John's College. (One Text-figure) 350

The thallous thallic halides. By Mr A. J. Berry, Downing College . 363

The high energy groups in the magnetic spectrum of the radium C β-rays. By Dr C. D. Ellis, Trinity College. (One Text-figure) 369

The relative absorbing powers of the L-levels for radiation of varying wave-length. By H. W. B. Skinner, Trinity College. (Two Text-figures) 379

Some electrical properties of liquid sulphur. By D. H. Black, Emmanuel College. (Communicated by Dr F. W. Aston.) (Four Text-figures) 393

The number of β-particles from Radium E. By K. G. Emeléus, B.A., St John's College. (Communicated by Professor Sir E. Rutherford.) (Two Text-figures) 400

On the measurement of the critical potentials of gases. By Mr E. G. Dymond, St John's College. (Two Text-figures) 405

The spectra of some lithium-like and sodium-like atoms. By Mr D. R. Hartree, St John's College 409

Improvements in the apparatus for measuring the velocity of very rapid chemical reactions. By Dr H. Hartridge, King's College, and Mr F. J. W. Roughton, Trinity College. (Three Text-figures) . 426

Note on the Doppler principle and Bohr's frequency condition. By P. A. M. Dirac, St John's College. (Communicated by Mr R. H. Fowler) 432

Some experiments with an α-particle counter. By Mr E. V. Appleton, K. G. Emeléus, B.A., and M. A. F. Barnett. (Eleven Text-figures) 434

A helical method of focussing β-rays. By R. A. R. Tricker, B.A., Trinity College. (Communicated by Professor Sir E. Rutherford.) (Plate III and Three Text-figures) 454

Some methods of estimating the successive ionisation potentials of any element. By Mr D. R. Hartree, St John's College 464

An optical interference method of measuring Young's modulus for rods. By Dr G. F. C. Searle, Peterhouse. (Four Text-figures) . . 475

On the vector algebra of eight associated points of three quadric surfaces. By Mr H. W. Turnbull, Trinity College 481

Why wireless electric rays can bend round the earth (Abstract). By Sir Joseph Larmor, F.R.S., Lucasian Professor 488

Note on the chemical constant of chlorine. By R. R. S. Cox, B.A., Christ's College. (Communicated by Mr R. H. Fowler) 491

Dissociative equilibrium in an external field of force. By Mr E. A. Milne, Trinity College 493

PAGE

Integrals developable about a singular point of a Hamiltonian system of differential equations. Part II. By Dr T. M. CHERRY, Trinity College. (One Text-figure) 510

The acceleration of β-particles in strong electric fields such as those of thunderclouds. By Professor C. T. R. WILSON, Sidney Sussex College 534

The determination of the frequency of an alternating current supply by the vibrations of rods. By Dr G. F. C. SEARLE, Peterhouse. (Three Text-figures) 539

Photographic plates for the detection of mass rays. By Dr F. W. ASTON, Trinity College. (Plate IV) 548

The kinetic theory of metallic conduction. By E. L. DAVIES and Mr G. H. LIVENS 555

The decay constant of radium emanation. By L. BASTINGS. (Communicated by Professor Sir E. RUTHERFORD) 562

On the current induced in a wireless telegraph receiving antenna. By Mr E. B. MOULLIN, Downing College. (Communicated by Mr E. V. APPLETON.) (Two Text-figures) 567

On the positive flash in vacuum discharge tubes. (Preliminary note.) By Mr R. WHIDDINGTON, St John's College. (Plate V) . . . 574

The structure of radiation. By Dr EDMUND C. STONER, Emmanuel College 577

Note on the heating effect of the γ-rays from RaB and RaC. By Dr C. D. ELLIS, Trinity College, and W. A. WOOSTER, B.A., Peterhouse . 595

A preliminary note on a direct determination of the distribution of intensity in the natural β-ray spectrum of RaB and RaC. By Dr L. F. CURTISS. (Communicated by Professor Sir E. RUTHERFORD.) (Three Text-figures) 597

A generalization of Feuerbach's theorem. By Mr J. P. GABBATT, Peterhouse 601

Characteristics of complexes of conics in space of four dimensions. By Mr C. G. F. JAMES, Trinity College 621

Note on the Lorentz group. By Mr J. BRILL, St John's College . . . 630

The symmetric functions of which the general determinant is a particular case. By Major P. A. MACMAHON, Sc.D., F.R.S., St John's College 633

On simplexes doubly incident with a quadric. By R. VAIDYANATHASWAMY. (Communicated by Professor H. F. BAKER) 655

On the phrase "equally probable." By Mr W. BURNSIDE, Honorary Fellow of Pembroke College 669

A note on wireless signal strength measurements made during the solar eclipse of 24 January, 1925. By Mr E. V. APPLETON and M. A. F. BARNETT. (One Text-figure) 672

The action of the electrical counter. By K. G. EMELÉUS, B.A., St John's College. (Communicated by Dr J. CHADWICK.) (Four Text-figures) 676

An extension of Wallace's, Miquel's and Clifford's theorems on circles. By Mr F. P. WHITE, St John's College 684

PAGE

Triple binary forms; the complete system for a single (1, 1, 1) *form with its geometrical interpretation.* By Mr W. SADDLER, St John's College. (Communicated by Mr H. W. TURNBULL) 688

A geometrical treatment of the correspondence between lines in threefold space and points of a quadric in fivefold space. By Mr H. W. TURNBULL, Trinity College 694

Theory of statistical estimation. By Mr R. A. FISHER, Gonville and Caius College 700

On the idea of frequency. By Mr W. BURNSIDE, Honorary Fellow of Pembroke College 726

The bursting of soap-bubbles in a uniform electric field. By Professor C. T. R. WILSON, and Mr G. I. TAYLOR, Trinity College. (Plates VI, VII) 728

The passage of electrons through small apertures. By J. F. LEHMANN and T. H. OSGOOD. (Communicated by Professor Sir E. RUTHERFORD, F.R.S.) (Four Text-figures) 731

An apparatus for the investigation of the effect of poisonous substances, and mixed vapours on catalytic activity. By F. H. CONSTABLE, B.A., Strathcona Research Student, St John's College. (Three Text-figures) 738

On pedal quadrics in non-euclidean hyperspace. By Mr J. P. GABBATT, Peterhouse 751

The temperature distribution in a transformer or other laminated core of rectangular cross section in which heat is generated at a uniform rate. By J. D. COCKCROFT, B.A., St John's College. (Five Text-figures) . 759

The theory of the influence of magnetic fields on the stopping power of gases for α-particles. By R. DE L. KRONIG. (Communicated by Mr R. H. FOWLER) 773

On Hausdorff's proof of the extended Riesz-Fischer theorem. By Mr S. POLLARD, Trinity College 777

On the representation of the modular group of order $\frac{1}{2}p(p^2-1)$ *as a group of linear substitutions on* $\frac{1}{2}(p-1)$ *symbols, when* p *is a prime of the form* $4n+3$. By Mr W. BURNSIDE, Honorary Fellow of Pembroke College 779

On groups of linear substitutions which contain irreducible metacyclical subgroups. By Mr W. BURNSIDE, Honorary Fellow of Pembroke College 788

A theoretical study of the stopping power of hydrogen atoms for α-particles. By Mr R. H. FOWLER, Trinity College. (One Text-figure) . . 793

Proceedings at the Meetings held during the Session 1923–1924 . . 804

Statement of Accounts of the Cambridge Philosophical Society, from January 1, 1924, to December 31, 1924 814

Notes on optical constants. By Mr C. G. DARWIN, F.R.S., Tait Professor of Natural Philosophy in the University of Edinburgh. I. *The optical behaviour of certain atomic models* 817

PAGE

Notes on optical constants. By Mr C. G. Darwin, F.R.S., Tait Professor of Natural Philosophy in the University of Edinburgh. II. *The lateral scattering from a gas* 824

The β-ray spectrum of the natural L-radiation from radium B. By D. H. Black, M.Sc., Emmanuel College. (Communicated by Professor Sir E. Rutherford) 832

The natural X-ray spectrum of radium B. By Professor Sir Ernest Rutherford, F.R.S., and W. A. Wooster, B.A., Peterhouse. (One Text-figure) 834

The analysis of the β-ray spectrum due to the natural L radiation of radium B. By D. H. Black, M.Sc., Emmanuel College. (Communicated by Professor Sir E. Rutherford) 838

The atomic number of a radioactive element at the moment of emission of the γ-rays. By Dr C. D. Ellis, Trinity College, and W. A. Wooster, B.A., Peterhouse. (One Text-figure) 844

The β-ray type of disintegration. By Dr C. D. Ellis, Trinity College, and W. A. Wooster, B.A., Peterhouse 849

Assemblies of imperfect gases by the method of partition functions. By Mr R. H. Fowler, Trinity College 861

An extended form of Kronecker's theorem with an application which shows that Burgers' theorem on adiabatic invariants is statistically true for an assembly. By L. H. Thomas, B.A., Trinity College. (Communicated by Mr R. H. Fowler) 886

Doublet and triplet separations in optical spectra as evidence whether orbits penetrate into the core. By Mr D. R. Hartree, St John's College. (Three Text-figures) 904

On the theorem of Pappus. By Mr M. H A. Newman, St John's College 919

On differentiation with respect to a function. By Mr E. C. Francis, Peterhouse 924

The Lebesgue-Stieltjes integral. By Mr E. C. Francis, Peterhouse . . 935

The enumeration of the partitions of multipartite numbers. By Major P. A. MacMahon, Sc.D., St John's College 951

On the substitution of Wallis's postulate of similarity for Euclid's postulate of parallels. By Professor M. J. M. Hill, Peterhouse. (Eight Text-figures) 964

Proceedings at the Meetings held during the Session 1924–1925 . . 970

Index to Vol. XXII 979

PLATES

Plates I, II. To illustrate Dr Crowther's paper 70

Plate III. To illustrate Mr Tricker's paper 454

Plate IV. To illustrate Dr Aston's paper 548

Plate V. To illustrate Mr Whiddington's paper 574

Plates VI, VII. To illustrate Messrs Wilson and Taylor's paper . . 728

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PROCEEDINGS

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On certain nets of plane curves. By Mr F. P. WHITE, St John's College.

[*Read* 29 October 1923.]

1. The plane quartic curves which pass through twelve fixed points g, of which no three lie on a straight line, no six on a conic and no ten on a cubic, form a net of quartics represented by the equation

$$\lambda_0 S_0 + \lambda_1 S_1 + \lambda_2 S_2 = 0.$$

Any two curves of the net meet in four points besides in the points g, the sixteen points forming the base-points of a pencil of quartics contained in the net. If for any such pencil three of the four points lie on a straight line, then a similar property holds for every pencil of the net, the fourth point is fixed, and we have a net of a special character with *thirteen* base-points instead of twelve. For this to happen eleven of the twelve points g may be taken arbitrarily; there is then a single infinity of such nets, and the other two base-points are paired and lie on a definite hyperelliptic curve of order 7 which has the eleven points as double-points[*].

Consideration of such nets of quartics arises naturally in the investigation of the sextic surface in four dimensions which is generated by the points of intersection of corresponding hyperplanes of four projective sheaves[†]. Such a surface may be birationally represented upon a plane, hyperplane sections corresponding to quartic curves through ten base-points. The surface has a triple infinity of trisecant lines, and thus there are ∞^2 triads of points in the plane such that any triad forms with the ten base-points a set of thirteen points which are the base-points of a *net* and not merely a *pencil* of quartics. Moreover, through any point

---

[*] See C. Küpper, "Über geometrische Netze" [*Abh. d. k. böhm. Gesell. d. Wiss.*, Prag, (7) I (1886)]. The existence of such nets affords an example of the exceptions pointed out by Bacharach [*Math. Ann.* XXVI (1886), 275] to a theorem of Cayley's [*Papers*, I, 25] that a curve of order $p = m + n - \gamma$ passing through

$$mn - \tfrac{1}{2}(\gamma - 1)(\gamma - 2)$$

of the points of intersection of two curves of orders m and n passes also through the remaining $\frac{1}{2}(\gamma - 1)(\gamma - 2)$. This no longer holds if these $\frac{1}{2}(\gamma - 1)(\gamma - 2)$ points lie on a curve of order $\gamma - 3$. In our case $m = n = \gamma = 4$. This example has been given for many years by Mr Richmond in his lectures on Plane Algebraic Geometry at Cambridge.

[†] See F. P. White, *Proc. Camb. Phil. Soc.* XXI (1922), 216.

P of the surface passes a single infinity of such trisecants, the points of intersection with the surface forming a curve of order 8 with a double-point at P. Thus, adding any point of the plane (representing the point P) to the ten base-points, we get a set of eleven points to which a pair of points may be added in a single infinity of ways to make up a set of thirteen with the prescribed property; the hyperelliptic curve on which the pair lie representing the curve of order 8 on the surface[*].

By considering surfaces in space of higher dimensions we are easily led to nets of curves of any order with similar special properties.

2. In a space S_n of n dimensions ($n > 3$) take n sheaves of S_{n-1},

$$\lambda p^{(k)} + \mu q^{(k)} + \nu r^{(k)} = 0, \quad k = 1, 2 \dots n.$$

Corresponding S_{n-1} will, in general, meet in a point and the locus of this point will be a surface F_2, whose equations are given by the vanishing of all determinants of order 3 contained in the matrix

$$\| \, p^{(k)} \quad q^{(k)} \quad r^{(k)} \, \|.$$

The surface will thus be the intersection of $(n-2)$ cubic varieties $F^3{}_{n-1}$ having in common a cubic variety $F^3{}_{n-2}$.

To find its order we have to find in how many points it is met by an arbitrary S_{n-2}, say $x_{n+1} = x_n = 0$. We have n equations

$$\sum_{1}^{n-1} \{\lambda p_l{}^{(k)} + \mu q_l{}^{(k)} + \nu r_l{}^{(k)}\} \, x_l = 0$$

to be satisfied for non-zero values of $x_1, x_2, \dots x_{n-1}$. The condition is the vanishing of the determinants of order $(n-1)$ of a matrix with n rows and $(n-1)$ columns, i.e. two conditions and, interpreting the λ, μ, ν as the coordinates of points in a plane, the number of solutions is the number of intersections of two curves of order $n-1$, less the number of intersections of two curves of order $n-2$, less the number of intersections of two curves of order $n-3$ ....

The order of the surface is thus

$$(n-1)^2 - (n-2)^2 + (n-3)^2 \dots \pm 1 = \tfrac{1}{2} n (n-1).$$

To any point of the surface corresponds in general one set of ratios $\lambda : \mu : \nu$ and conversely. Hence the surface is represented bi-rationally upon the plane (λ, μ, ν).

To an arbitrary straight line in the plane, $A\lambda + B\mu + C\nu = 0$, corresponds the locus of intersections of corresponding S_{n-1} of

---

[*] Projecting from P on to a hyperplane, we get a quintic surface in ordinary space with a double twisted cubic curve; this surface was investigated years ago by Clebsch [*Math. Ann.* I (1869), 284]; the hyperelliptic curve is the representation of the double curve.

n pencils of S_{n-1}, i.e. a rational normal curve C_n of order n, whose equations are

$$\begin{Vmatrix} p^{(1)} & q^{(1)} & r^{(1)} \\ p^{(2)} & q^{(2)} & r^{(2)} \\ & \vdots & \\ p^{(n)} & q^{(n)} & r^{(n)} \\ A & B & C \end{Vmatrix} = 0.$$

Any two such curves on the surface thus meet in *one* point, and two points of the surface determine *one* such curve.

Sections of the surface by general S_{n-1} are thus represented by curves in the plane, which are met by any line in points corresponding to the intersections of the S_{n-1} with the corresponding C_n, i.e. in n points; so that the curves in the plane are of order n.

The three ∞^{n-1} systems

$$\left.\begin{array}{l} a_1 p^{(1)} + a_2 p^{(2)} + \ldots + a_n p^{(n)} = 0 \\ a_1 q^{(1)} + a_2 q^{(2)} + \ldots + a_n q^{(n)} = 0 \\ a_1 r^{(1)} + a_2 r^{(2)} + \ldots + a_n r^{(n)} = 0 \end{array}\right\}$$

give, by the intersection of corresponding S_{n-1}, an ∞^{n-1} system of S_{n-3}.

Take, for instance, the S_{n-3} given by $p^{(1)} = q^{(1)} = r^{(1)} = 0$. It will meet the surface where it meets the F_3 generated by $n-1$ sheaves of S_{n-1}, which is of order $\frac{1}{2}(n-1)(n-2)$. Thus it meets the surface in $\frac{1}{2}(n-1)(n-2)$ points.

Hence we have a system ∞^{n-1} of S_{n-3} each of which meets the surface in $\frac{1}{2}(n-1)(n-2)$ points.

The surface will contain straight lines, represented by points of the plane (λ, μ, ν), if the n corresponding S_{n-1} meet in a *line* instead of a point.

The condition for this is that the matrix of elements

$$\lambda p_l^{(k)} + \mu q_l^{(k)} + \nu r_l^{(k)} \quad (k = 1, 2 \ldots n;\ l = 1, 2 \ldots n+1)$$

should be of characteristic $(n-2)$, i.e. all determinants of order $(n-1)$ contained therein must vanish. The number of sets of values $\lambda : \mu : \nu$ is thus

$$n^2 - (n-1)^2 + \ldots \pm 1 = \tfrac{1}{2}n(n+1).$$

There are thus $\frac{1}{2}n(n+1)$ lines upon the surface, and hyperplane sections are represented upon the plane by curves of order n having $\frac{1}{2}n(n+1)$ points in common. Such curves form a system of dimension $\frac{1}{2}n(n+3) - \frac{1}{2}n(n+1) = n$, representing the ∞^n system of S_{n-1} sections; and any two meet in

$$n^2 - \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n(n-1)$$

further points, corresponding to the points in which the surface is met by an S_{n-2}.

Consider now two points on the surface represented by $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$. The curve C_n which joins them is generated by the pencils

$$\lambda_1 p^{(k)} + \mu_1 q^{(k)} + \nu_1 r^{(k)} + \theta \{\lambda_2 p^{(k)} + \mu_2 q^{(k)} + \nu_2 r^{(k)}\} = 0, \quad (k = 1, 2 \ldots n).$$

The S_{n-2} whose equations are

$$a_1 \{\lambda_1 p^{(1)} + \mu_1 q^{(1)} + \nu_1 r^{(1)}\} + a_2 \{\lambda_1 p^{(2)} + \mu_1 q^{(2)} + \nu_1 r^{(2)}\} + \ldots + a_n \{\lambda_1 p^{(n)} + \mu_1 q^{(n)} + \nu_1 r^{(n)}\} = 0,$$

$$a_1 \{\lambda_2 p^{(1)} + \mu_2 q^{(1)} + \nu_2 r^{(1)}\} + a_2 \{\lambda_2 p^{(2)} + \mu_2 q^{(2)} + \nu_2 r^{(2)}\} + \ldots + a_n \{\lambda_2 p^{(n)} + \mu_2 q^{(n)} + \nu_2 r^{(n)}\} = 0,$$

where the a's are arbitrary, meets this curve in $n - 1$ points.

Next take an arbitrary secant S_{n-3} of the surface given by

$$\left.\begin{aligned} a_1 p^{(1)} + a_2 p^{(2)} + \ldots + a_n p^{(n)} &= 0 \\ a_1 q^{(1)} + a_2 q^{(2)} + \ldots + a_n q^{(n)} &= 0 \\ a_1 r^{(1)} + a_2 r^{(2)} + \ldots + a_n r^{(n)} &= 0 \end{aligned}\right\}$$

This will contain the $(n - 1)$-secant S_{n-2}

$$\left.\begin{aligned} \sum_1^n a_k (\lambda_1 p^{(k)} + \mu_1 q^{(k)} + \nu_1 r^{(k)}) &= 0 \\ \sum_1^n a_k (\lambda_2 p^{(k)} + \mu_2 q^{(k)} + \nu_2 r^{(k)}) &= 0 \end{aligned}\right\}$$

Moreover, if we keep $(\lambda_1, \mu_1, \nu_1)$ fixed and vary $(\lambda_2, \mu_2, \nu_2)$, this S_{n-2} will always lie in the S_{n-1}

$$\sum_1^n a_n (\lambda_1 p^{(k)} + \mu_1 q^{(k)} + \nu_1 r^{(k)}) = 0,$$

which contains the point corresponding to $(\lambda_1, \mu_1, \nu_1)$ for which

$$\lambda_1 p^{(k)} + \mu_1 q^{(k)} + \nu_1 r^{(k)} = 0, \quad (k = 1, 2 \ldots n).$$

Thus, if through an arbitrary secant S_{n-3} (c) of the surface we draw the $(n - 1)$-secant S_{n-2} to all curves C_1^n through a point P of the surface, they will all lie in a fixed S_{n-1} containing P and c.

Moreover, the S_{n-1} may be taken arbitrarily through P, the constants a_k being at our disposal, and then the secant S_{n-3} (c) will be determined in it by P.

Thus, for any S_{n-1} section of the surface, which is a curve $C_1^{\frac{1}{2}n(n-1)}$ of S_{n-1}, we have a $(1, 1)$ correspondence between its points and its $\frac{1}{2}(n - 1)(n - 2)$-secant S_{n-3}.

Also, which is our immediate concern, since $\infty^2 S_{n-1}$ pass through an S_{n-3}, we have the following theorem in plane curves:

(a) On a curve of order n take $\frac{1}{2}n(n+1)$ general fixed points g. Take also a fixed point P and a variable straight line through P meeting the curve again in $(n-1)$ points $Q_1 \ldots Q_{n-1}$. These points Q, together with g, $\frac{1}{2}n(n+3)-1$ in all, determine a pencil of C_n of which the given curve is one. They meet further in $\frac{1}{2}(n-1)(n-2)$ points R of C_n. Then these points R are the same whatever line through P be taken, and together with g form a set of n^2-n+1 points through which pass a *net* of curves of order n.

[In general $\frac{1}{2}n(n+3)-2=\frac{1}{2}(n-1)(n+4)$ points determine a net of C_n, so that there are $\frac{1}{2}(n-2)(n-3)$ extra points.]

The theorem may also be expressed as follows:

(b) If two curves of a net of curves of order n, with

$$\tfrac{1}{2}(n-1)(n+4)$$

base-points, meet again in points of which $(n-1)$ lie on a straight line, then the same is true for every two curves of the net, and the remaining $\frac{1}{2}(n-2)(n-3)$ points of intersection of any two curves are fixed and lie on every curve of the net, which thus has (n^2-n+1) base-points.

3. The above theorem is easily verified by the methods of plane algebraic geometry.

(a) Take a second line through P meeting C_n in $(n-1)$ further points Q'. Then on the curve C_n, the sets of points Q and Q' are both residual to P. Moreover, the set Q is residual to the set composed of the points g and the points R lying with them on a curve C_n'. Hence by the theorem of residuation (*der Restsatz*) Q' is also residual to $g+R$, forming with them the complete intersection of another curve of order n with C_n.

Suppose now we have a net of curves C_n, with $\frac{1}{2}(n-1)(n+4)$ base-points g, of which two particular curves $C_1 . C_2$, and therefore the pencil $\lambda_1 C_1+\lambda_2 C_2$ meet again in points of which $(n-1)$, $(Q_1 \ldots Q_{n-1})$, lie on a straight line L. Call the remaining points of intersection of C_1 and C_2, $\frac{1}{2}(n-2)(n-3)$ in number, the points R. Take any point Q_1' in the plane, and let C_1 be that curve of the pencil $\lambda_1 C_1+\lambda_2 C_2$ which passes through Q_1'. L meets it in the points Q and a further point P. PQ_1' meets C_1 again in points $Q_2' \ldots Q'_{n-1}$, which by the theorem (a) lie on another curve of order n through the points g and R. This curve is a curve of the net not belonging to the pencil $\lambda_1 C_1+\lambda_2 C_2$; hence all curves of the net pass through the points R as well as the points g, and any two curves of it meet again in $n-1$ points of a straight line.

4. If we have such a net of curves C_n with (n^2-n+1) base-points, it is clear that any point Q_1 determines one set of $(n-1)$

collinear points $Q_1, Q_2 \dots Q_{n-1}$ which are the further intersections of a pencil of the net.

Also on any straight line L there is one such set of $(n-1)$ points; for let Q' be the set determined by any point Q_1', let L' be the line on which it lies, and let L, L' meet in P. Then that curve of the pencil through Q' which passes through P will clearly meet L in the set Q which lie upon it. Two sets Q cannot lie upon the same line L, for any two sets Q lie on one curve of the net, which would thus have to contain the line L, which is not in general possible.

In the S_n these theorems are:

(1) There is one S_{n-2} which joins any secant S_{n-3} to any point of the surface; and the $(n-1)$ points in which it meets the surface lie on a curve C_1^n of the surface;

(2) Any curve C_1^n of the surface has one $(n-1)$-secant S_{n-2} through a given secant S_{n-3},

both of which are clear enough from paragraph 2.

5. Consider now the system ∞^{n-1} of $\frac{1}{2}(n-1)(n-2)$-secant S_{n-3};

$$\left.\begin{aligned} \alpha_1 p^{(1)} + \alpha_2 p^{(2)} + \dots + \alpha_n p^{(n)} &= 0 \\ \alpha_1 q^{(1)} + \alpha_2 q^{(2)} + \dots + \alpha_n q^{(n)} &= 0 \\ \alpha_1 r^{(1)} + \alpha_2 r^{(2)} + \dots + \alpha_n r^{(n)} &= 0 \end{aligned}\right\} \qquad \dots\dots(5\cdot1).$$

For a secant S_{n-3} to pass through an arbitrary point y of S_n we have three conditions for the α's, obtained by substituting $y_1, y_2 \dots y_{n+1}$ in the above equations for the current coordinates $x_1, x_2 \dots x_{n+1}$.

If, however, the point y lie upon the surface $F_2^{\frac{1}{2}n(n-1)}$, the equations so obtained will be connected by a linear relation, so that there are effectively only *two* conditions for a secant S_{n-3} to pass through a point of the surface. Similarly, any further general point of the surface will impose two more conditions.

We have thus to distinguish two cases, n odd and n even.

(1) *n odd.*

One secant S_{n-3} can be made to pass through $\frac{1}{2}(n-1)$ general points of the surface.

(2) *n even.*

A single infinity of secant S_{n-3} pass through $\frac{1}{2}(n-2)$ general points of the surface.

Case 1. $n = 2m+1$.

In this case

$$\tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n-1) = \tfrac{1}{2}(n^2+2n-1) = \tfrac{1}{2}n(n+3) - \tfrac{1}{2}(n+1)$$
$$= \tfrac{1}{2}n(n+3) - 2 - \tfrac{1}{2}(n-3)$$

of the base-points may be taken at random, and the other $\frac{1}{2}(n-1)(n-3)$ are thereby determined.

This may be verified in the plane as follows:

The arbitrary base-points determine a linear system of curves C_n of dimensions $\frac{1}{2}(n+1)$.

Take any line L and on it two points f_1, f_2.

The curves of the system which pass through f_1 form a linear system of dimensions $\frac{1}{2}(n-1)$ which cut out on L a linear series of sets of $(n-1)$ points of freedom $\frac{1}{2}(n-1)$, given by an equation

$$\lambda_0 T_0 + \lambda_1 T_1 + \dots + \lambda_{\frac{1}{2}(n-1)} T_{\frac{1}{2}(n-1)} = 0,$$

where the T's are polynomials of order $(n-1)$ in t, the parameter determining the position of a point on the line.

Similarly the curves through f_2 determine another series of sets of points on the line given by an equation

$$\mu_0 T_0' + \mu_1 T_1' + \dots + \mu_{\frac{1}{2}(n-1)} T'_{\frac{1}{2}(n-1)} = 0.$$

There is one and only one set of values of the ratios of the λ's, and of the μ's, such that the two sets of $(n-1)$ points coincide. For this requires that

$$\lambda_0 T_0 + \lambda_1 T_1 + \dots + \lambda_{\frac{1}{2}(n-1)} T_{\frac{1}{2}(n-1)} = \rho\{\mu_0 T_0' + \mu_1 T_1' + \dots + \mu_{\frac{1}{2}(n-1)} T_{\frac{1}{2}(n-1)}\}$$

as an identity in t, which gives n linear equations, from which, in general, the $n+1$ ratios of $\lambda_0, \lambda_1 \dots \lambda_{\frac{1}{2}(n-1)}, \rho\mu_0, \rho\mu_1 \dots \rho\mu_{\frac{1}{2}(n-1)}$ are uniquely determined.

One curve can thus be drawn through the arbitrary base-points and through f_1, and one through the base-points and f_2, such that $(n-1)$ of their further intersections lie on the straight line L.

We have thus determined the unique pencil of the system which has $(n-1)$ of its base-points on L. The remaining base-points, $\frac{1}{2}(n-1)(n-3)$ in number, will thus form with the $\frac{1}{2}(n^2+2n-1)$, with which we started, the base-points of a net of curves.

Case 2. $n = 2m$.

$\frac{1}{2}n(n+1) + \frac{1}{2}(n-2) = \frac{1}{2}(n^2+2n-2)$ of the base-points may be taken at random. The remaining $\frac{1}{2}(n-2)^2$ points then lie on a curve of order $(2n-1)$ with double-points at the $\frac{1}{2}(n^2+2n-2)$ base-points.

Take any line L and on it two fixed points f_1, f_2 and a further point Q_1. The curves of order n through the arbitrary base-points and through f_1, Q_1 form a linear system of dimensions $\frac{1}{2}(n-2)$, which cuts out on L a linear series of sets of $(n-2)$ points of this freedom.

We get a similar series of sets of points from curves through the base-points and through f_2, Q_1. By the same argument as in Case 1 these two series have one common set. Hence there is one curve through f_1 and one through f_2 such that $(n-1)$ of their intersections $Q_1, Q_2 \dots Q_{n-1}$ lie on L, one of them being at the arbitrary point Q_1 of L. The remaining intersections, i.e. the $\frac{1}{2}(n^2+2n-2)$ arbitrary base-points, and $\frac{1}{2}(n-2)^2$ others R, thus form the base-points of a net.

Taking a second point Q_1' on L we likewise get a set of points R' forming with the arbitrary base-points the base-points of a net; there is one curve through f_1 and one through f_2 meeting in $(n-1)$ points Q' of L, of which Q_1' is one.

The two curves which we have obtained through f_1 give a pencil, and the two curves through f_2 form another pencil, which can be projectively related so that corresponding curves intersect in $(n-1)$ points of L.

The locus of the intersections of corresponding curves of two projectively related pencils of curves of order n is a curve of order $2n$ passing through the base-points of each pencil. In our case part of the locus is clearly the line L, the remainder, the locus of the points R, is thus a curve of order $2n-1$ with double-points at the $\frac{1}{2}(n^2+2n-2)$ arbitrary base-points. Its genus is thus $\frac{1}{2}(3n^2-12n+8)$. It is not the most general curve of its order with this number of double-points; it contains a linear series of sets of $\frac{1}{2}(n-2)^2$ points of freedom one.

6. Interpreting the steps of the procedure in the preceding paragraph in space of n dimensions, we get the following theorems concerning rational normal curves.

(1) *n odd.*

Through any $\frac{1}{2}(n-1)$ general points not on the curve we can draw *one* $(n-1)$-secant S_{n-2} of the curve.

(2) *n even.*

Through any $\frac{1}{2}(n-2)$ general points Q not on the curve we can draw a single infinity of $(n-1)$-secant S_{n-2}. Joining these to any point of the curve P we get a single infinity of S_{n-1} *which form a pencil.*

In other words a point P on the curve and $\frac{1}{2}(n-2)$ points Q not on the curve determine a pencil of S_{n-1} each of which meets the curve again in $(n-1)$ points which lie in a S_{n-2} with the $\frac{1}{2}(n-2)$ points Q.

The S_{n-2} which is the base of this pencil is uniquely determined by the points P and Q taken; e.g. in S_4, a point P on a normal quartic curve and a point Q not on it determine a *plane*, any hyperplane through which meets the curve again in three points lying in a plane with Q.

These results are easily verified.

The rational normal curve C_1^n of S_n is obtained as the locus of intersections of corresponding S_{n-1} of n projective pencils of S_{n-1}

$$\lambda p^{(k)} + \mu q^{(k)} = 0, \quad (k = 1, 2 \dots n).$$

The equations of the curve are

$$\frac{p^{(1)}}{q^{(1)}} = \frac{p^{(2)}}{q^{(2)}} = \dots = \frac{p^{(n)}}{q^{(n)}}.$$

The two systems of S_{n-1}

$$\left.\begin{array}{l} a_1 p^{(1)} + a_2 p^{(2)} + \dots + a_n p^{(n)} = 0 \\ a_1 q^{(1)} + a_2 q^{(2)} + \dots + a_n q^{(n)} = 0 \end{array}\right\}$$

give the ∞^{n-1} system of $(n-1)$-secant S_{n-2}.

For such a secant S_{n-2} to pass through an arbitrary point of S_n, two equations must be satisfied, but if the point lie on the curve one condition only is required.

Thus if n is odd, for the secant S_{n-2} through $\frac{1}{2}(n-1)$ arbitrary points we have $(n-1)$ linear equations for the homogeneous quantities $a_1, a_2 \dots a_n$, whose ratios are thus uniquely determined.

If n is even, the secant S_{n-2} through $\frac{1}{2}(n-2)$ arbitrary points Q are a single infinity and form a quadric variety whose equation is given by the vanishing of a determinant of order n, of which $\frac{1}{2}n$ rows are made up of $p^{(1)}, p^{(2)} \dots p^{(n)}$ and the constants obtained by substituting in them for the current coordinates the coordinates of the $\frac{1}{2}(n-2)$ points; and the other $\frac{1}{2}n$ rows are similarly obtained from the q's.

This quadric variety is clearly a cone, the "vertex" being an S_{n-4}. Projecting from this S_{n-4} upon an S_3, we get a rational curve of order n lying upon a quadric, in ordinary space, meeting every generator of one system in $(n-1)$ points. A point P on the curve in S_n projects into a point P' on this quadric, and the generator of the other system through P' lies in a plane with each generator of the first system. Joining these to the S_{n-4} we get an S_{n-2}, every S_{n-1} through which contains a secant S_{n-2} through the S_{n-4}, and thus through the points Q.

7. *Particular Examples.*

The case $n = 4$ has been sufficiently dealt with.

$n = 5$.

We get nets of quintics with twenty-one base-points, i.e. three more than is normal. Of these, seventeen may be taken arbitrarily and the four others are then determined without ambiguity.

In S_5 we get a surface of order 10 with a quadruple infinity of sex-secant planes, one of which passes through two arbitrary points of the surface. The surface has fifteen lines, and, in the representation on a plane, sections by S_4 are represented by quintics through fifteen base-points.

Projecting from two points of the surface on to an S_3, we get a surface of order 8 in ordinary space, with a quadriplanar point.

In the plane representation to plane sections correspond quintics through seventeen base-points. Since a plane section, of order 8, is of genus 6, it must have fifteen double-points. The surface therefore has a double curve of order 15. It has seventeen lines.

The S_4 section of the surface in S_5 is a curve of order 10 and genus 6, with a single infinity of sex-secant planes, of which six pass through any point of the curve. Projecting from a point of the curve we get, in S_3, a curve of order 9 and genus 6 with six quinqui-secant lines.

$n = 6$.

We have nets of sextics with thirty-one base-points, i.e. six more than normal. Of these, twenty-three may be taken arbitrarily, the remaining eight then lie on a curve of order 11 with double-points at the twenty-three base-points. These sets of eight points form a linear series of freedom 1 cut out by adjoint sextics through twelve fixed points of the curve.

In S_6 we have a surface of order 15 with ∞^5 10-secant S_3, of which a single infinity pass through two points of the surface, forming a cubic variety of dimension 4, a cone with a line for "vertex."

Projecting on to an S_4, we get a surface of order 13 with a single infinity of lines meeting it in eight points, forming a ruled cubic surface.

The conics through fives of six points. By Mr F. P. WHITE, St John's College.

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The theorem, due to Miquel, that the foci of the five parabolas which touch fours of five straight lines lie on a circle, when generalised projectively and dualised becomes the theorem: If six arbitrary points 1, 2, 3, 4, 5, 6 be taken and the five conics passing respectively through the five points obtained by omitting in turn 1, 2, 3, 4, 5, then there exists a conic touching two arbitrary lines through the point 6 and triangularly inscribed to these five conics. It appears, however, that the relation is symmetrical* and that the conic obtained is also triangularly inscribed to the conic passing through the points 1, 2, 3, 4, 5. If the condition of touching the two arbitrary straight lines through the point 6 be omitted, we have a doubly infinite system of conics triangularly inscribed to the six conics passing through fives of six points. It does not immediately appear how this family of conics depends upon the two parameters involved, and the following direct analytical investigation of the general symmetrical figure was undertaken with a view to deciding this point.

1. *The equations of the six conics.*

Let three of the points 1, 2, 3 be taken as the vertices of the triangle of reference, and let 4, 5, 6 be respectively (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3).

Denote the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by Δ, and its first minors by A_1, B_1, C_1, etc.

The three conics passing through 1, 2, 3 and respectively through 56, 64, 45 are

$$S_4 \equiv A_1 a_2 a_3 yz + B_1 b_2 b_3 zx + C_1 c_2 c_3 xy = 0,$$
$$S_5 \equiv A_2 a_3 a_1 yz + B_2 b_3 b_1 zx + C_2 c_3 c_1 xy = 0,$$
$$S_6 \equiv A_3 a_1 a_2 yz + B_3 b_1 b_2 zx + C_3 c_1 c_2 xy = 0.$$

The straight lines 56, 64, 45 have equations

$$L \equiv A_1 x + B_1 y + C_1 z = 0,$$
$$M \equiv A_2 x + B_2 y + C_2 z = 0,$$
$$N \equiv A_3 x + B_3 y + C_3 z = 0,$$

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* This is recognised by Mr Wakeford, *Proc. Lond. Math. Soc.*, XV (1916), p. 341.

and the three conics passing through 4, 5, 6 and two of 1, 2, 3 are

$$S_1 \equiv a_1 B_1 C_1 MN + a_2 B_2 C_2 NL + a_3 B_3 C_3 LM = 0,$$
$$S_2 \equiv b_1 C_1 A_1 MN + b_2 C_2 A_2 NL + b_3 C_3 A_3 LM = 0,$$
$$S_3 \equiv c_1 A_1 B_1 MN + c_2 A_2 B_2 NL + c_3 A_3 B_3 LM = 0.$$

2. The condition that the conic S should be triangularly inscribed to the conic S', namely $\Theta^2 = 4\Delta\Theta'$, becomes, in the case in which $S' \equiv \lambda yz + \mu zx + \nu xy = 0$ and

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$BC\lambda^2 + CA\mu^2 + AB\nu^2 + 2AF\mu\nu + 2BG\nu\lambda + 2CH\lambda\mu = 0.$$

Thus conics triangularly inscribed to the three conics S_4, S_5, S_6 of the previous paragraph are such that BC, CA, AB, $2AF$, $2BG$, $2CH$ are the coefficients in the equation of a conic passing through the three points $(A_1 a_2 a_3, B_1 b_2 b_3, C_1 c_2 c_3)$, $(A_2 a_3 a_1, B_2 b_3 b_1, C_2 c_3 c_1)$, $(A_3 a_1 a_2, B_3 b_1 b_2, C_3 c_1 c_2)$. Hence BC, etc., may be expressed as sums of multiples of the coefficients of any three linearly independent conics through these three points.

It is convenient to take conics through these points and through two of the three vertices of the triangle of reference.

The conic through y, z is found to be

$$\Sigma_1 \equiv \Delta b_1 b_2 b_3 c_1 c_2 c_3 x^2 + a_1 a_2 a_3 (\alpha . \alpha)\, yz + b_1 b_2 b_3 (\alpha . \beta)\, zx + c_1 c_2 c_3 (\alpha . \gamma)\, xy = 0,$$

where

$$(\alpha . \alpha) = A_1 b_1 c_1 (b_2 c_3 + b_3 c_2) + A_2 b_2 c_2 (b_3 c_1 + b_1 c_3) + A_3 b_3 c_3 (b_1 c_2 + b_2 c_1),$$
$$(\alpha . \beta) = A_1 c_1 a_1 (c_2 a_3 + c_3 a_2) + A_2 c_2 a_2 (c_3 a_1 + c_1 a_3) + A_3 c_3 a_3 (c_1 a_2 + c_2 a_1),$$
$$(\alpha . \gamma) = A_1 a_1 b_1 (a_2 b_3 + a_3 b_2) + A_2 a_2 b_2 (a_3 b_1 + a_1 b_3) + A_3 a_3 b_3 (a_1 b_2 + a_2 b_1).$$

Similarly the other conics are

$$\Sigma_2 \equiv \Delta c_1 c_2 c_3 a_1 a_2 a_3 y^2 + a_1 a_2 a_3 (\beta . \alpha)\, yz + b_1 b_2 b_3 (\beta . \beta)\, zx + c_1 c_2 c_3 (\beta . \gamma)\, xy = 0,$$

and $\Sigma_3 \equiv \Delta a_1 a_2 a_3 b_1 b_2 b_3 z^2 + a_1 a_2 a_3 (\gamma . \alpha)\, yz + b_1 b_2 b_3 (\gamma . \beta)\, zx + c_1 c_2 c_3 (\gamma . \gamma)\, xy = 0,$

where the meaning of the symbols $(\beta . \alpha)$, etc., is clear.

Now it is easily seen that

$$(\alpha . \alpha) = A_1 A_2 A_3, \quad (\beta . \beta) = B_1 B_2 B_3, \quad (\gamma . \gamma) = C_1 C_2 C_3,$$

while a symbol such as $(\alpha . \beta)$, being symmetrical with regard to the suffixes 1, 2, 3, can be written in one of the three forms:

$$(\alpha . \beta) = A_1 B_2 B_3 + \Delta a_1 c_2 c_3 = A_2 B_3 B_1 + \Delta a_2 c_3 c_1 = A_3 B_1 B_2 + \Delta a_3 c_1 c_2,$$

and similarly with the others.

Conics through the three points in question being of the form

$$l\Sigma_1 + m\Sigma_2 + n\Sigma_3 = 0,$$

we have, from the coefficients,

$$\rho BC = \Delta b_1 b_2 b_3 c_1 c_2 c_3 l, \quad \rho CA = \Delta c_1 c_2 c_3 a_1 a_2 a_3 m, \quad \rho AB = \Delta a_1 a_2 a_3 b_1 b_2 b_3 n,$$
$$2\rho AF = a_1 a_2 a_3 \{(\alpha . \alpha)\, l + (\beta . \alpha)\, m + (\gamma . \alpha)\, n\},$$
$$2\rho BG = b_1 b_2 b_3 \{(\alpha . \beta)\, l + (\beta . \beta)\, m + (\gamma . \beta)\, n\},$$
$$2\rho CH = c_1 c_2 c_3 \{(\alpha . \gamma)\, l + (\beta . \gamma)\, m + (\gamma . \gamma)\, n\},$$

whence, taking $\rho\Delta lmn = 1$,

$$A = \Delta a_1 a_2 a_3 mn, \quad B = \Delta b_1 b_2 b_3 nl, \quad C = \Delta c_1 c_2 c_3 lm,$$

$$2F = l\,\{(\alpha.\alpha)\,l + (\beta.\alpha)\,m + (\gamma.\alpha)\,n\},$$

$$2G = m\,\{(\alpha.\beta)\,l + (\beta.\beta)\,m + (\gamma.\beta)\,n\},$$

$$2H = n\,\{(\alpha.\gamma)\,l + (\beta.\gamma)\,m + (\gamma.\gamma)\,n\},$$

and the tangential equation of a conic S triangularly inscribed to the three conics S_4, S_5, S_6 is:

$$\begin{aligned}\Sigma = \Delta a_1 a_2 a_3 mn\lambda^2 &+ \Delta b_1 b_2 b_3 nl\mu^2 + \Delta c_1 c_2 c_3 lm\nu^2 \\ &+ l\,\{(\alpha.\alpha)\,l + (\beta.\alpha)\,m + (\gamma.\alpha)\,n\}\,\mu\nu + m\,\{(\alpha.\beta)\,l + (\beta.\beta)\,m + (\gamma.\beta)\,n\}\,\nu\lambda \\ &+ n\,\{(\alpha.\gamma)\,l + (\beta.\gamma)\,m + (\gamma.\gamma)\,n\}\,\lambda\mu = 0,\end{aligned}$$

where l, m, n are arbitrary parameters.

3. Making use of the various forms for $(\alpha.\beta)$, etc., it is at once clear that Σ can be written in either of the forms*:

$$\begin{aligned}&\Delta\,(a_1\lambda + b_1\mu + c_1\nu)\,(a_2 a_3 mn\lambda + b_2 b_3 nl\mu + c_2 c_3 lm\nu) \\ &\qquad + (A_1 l + B_1 m + C_1 n)\,(A_2 A_3 \mu\nu l + B_2 B_3 \nu\lambda m + C_2 C_3 \lambda\mu n) = 0,\end{aligned}$$

$$\begin{aligned}&\Delta\,(a_2\lambda + b_2\mu + c_2\nu)\,(a_3 a_1 mn\lambda + b_3 b_1 nl\mu + c_3 c_1 lm\nu) \\ &\qquad + (A_2 l + B_2 m + C_2 n)\,(A_3 A_1 \mu\nu l + B_3 B_1 \nu\lambda m + C_3 C_1 \lambda\mu n) = 0,\end{aligned}$$

$$\begin{aligned}&\Delta\,(a_3\lambda + b_3\mu + c_3\nu)\,(a_1 a_2 mn\lambda + b_1 b_2 nl\mu + c_1 c_2 lm\nu) \\ &\qquad + (A_3 l + B_3 m + C_3 n)\,(A_1 A_2 \mu\nu l + B_1 B_2 \nu\lambda m + C_1 C_2 \lambda\mu n) = 0.\end{aligned}$$

4. We have next to find the tangential equation of conics triangularly inscribed to the three conics S_1, S_2, S_3.

These are of the form

$$\lambda MN + \mu NL + \nu LM = 0,$$

and the problem is exactly similar to that of the previous paragraphs, except that the triangle of reference is now 456.

The values of λ, μ, ν for the conics S_1, S_2, S_3 may be obtained from those for the conics S_4, S_5, S_6 by the substitution of

$$A_1, A_2, A_3, \quad B_1, B_2, B_3, \quad C_1, C_2, C_3$$

respectively for $\quad a_1, b_1, c_1, \quad a_2, b_2, c_2, \quad a_3, b_3, c_3,$

and conversely, a Δ being put in where necessary to preserve homogeneity.

Thus, using the results of paragraph 3, the tangential equation of a conic triangularly inscribed to S_1, S_2, S_3 can be written in the form

$$\begin{aligned}&(A_1\lambda' + A_2\mu' + A_3\nu')\,(B_1 C_1 m'n'\lambda' + B_2 C_2 n'l'\mu' + B_3 C_3 l'm'\nu') \\ &\qquad + \Delta\,(a_1 l' + a_2 m' + a_3 n')\,(b_1 c_1 \mu'\nu'l' + b_2 c_2 \nu'\lambda'm' + b_3 c_3 \lambda'\mu'n') = 0,\end{aligned}$$

or in two similar forms obtained from this by a cyclical permutation of the letters a, b, c.

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* This was pointed out to me by Professor Baker, who obtained these forms from a geometrical proof of the general theorem, which utilizes a construction in three dimensions.

The three forms represent the same conic as a result of identities of the type

$$A_2B_3C_3+\Delta a_2b_1c_1=B_2C_3A_3+\Delta b_2c_1a_1=C_2A_3B_3+\Delta c_2a_1b_1.$$

Also $\lambda'=0$ is the tangential equation of the point 4, so that

$$\lambda'=a_1\lambda+b_1\mu+c_1\nu,$$

and similarly. $\mu'=a_2\lambda+b_2\mu+c_2\nu,\quad \nu'=a_3\lambda+b_3\mu+c_3\nu.$

The arbitrary parameters are here l', m', n'.

5. We can now verify that the equation of the conic in the last paragraph is another form of the equations of paragraph 3, where l', m', n' are linear functions of l, m, n and λ', μ', ν' have the values given above, so that we may say that the general conic triangularly inscribed to three of the original six conics is triangularly inscribed to the other three.

To carry out this verification take the equation in the form given at the end of paragraph 2 and substitute for l, m, n values obtained from

$$l'=A_1l+B_1m+C_1n,\quad m'=A_2l+B_2m+C_2n,\quad n'=A_3l+B_3m+C_3n,$$

i.e. $\Delta l=a_1l'+a_2m'+a_3n',\quad \Delta m=b_1l'+b_2m'+b_3n',\quad \Delta n=c_1l'+c_2m'+c_3n'.$

Using the symmetrical expression for $(\alpha.\alpha)$, $(\alpha.\beta)$, etc., and taking account of the relations

$$a_1A_1+b_1B_1+c_1C_1=\Delta,\quad a_2A_1+b_2B_1+c_2C_1=0,\ \text{etc.},$$

we find

$$(\alpha.\alpha)\,l+(\beta.\alpha)\,m+(\gamma.\alpha)\,n=b_1c_1(b_2c_3+b_3c_2)\,l'+b_2c_2(b_3c_1+b_1c_3)\,m' \\ +b_3c_3(b_1c_2+b_2c_1)\,n',$$

and two similar expressions.

Thus

$$\Delta\Sigma=a_1a_2a_3(b_1l'+b_2m'+b_3n')(c_1l'+c_2m'+c_3n')\lambda^2+\dots+\dots \\ +(a_1l'+a_2m'+a_3n')\{b_1c_1(b_2c_3+b_3c_2)\,l'+b_2c_2(b_3c_1+b_1c_3)\,m'+b_3c_3(b_1c_2+b_2c_1)\,n'\} \\ +\dots+\dots.$$

The coefficient of l'^2 is $a_1b_1c_1\mu'\nu'$.

The coefficient of $m'n'$ is

$$\lambda'\{a_2a_3(b_2c_3+b_3c_2)\lambda+b_2b_3(c_2a_3+c_3a_2)\mu+c_2c_3(a_2b_3+a_3b_2)\nu\},$$

wherein the second factor, multiplied by Δ, can be written

$$(1.1)\,\lambda'+(2.1)\,\mu'+(3.1)\,\nu',$$

where (1.1), (2.1), (3.1) are obtained respectively from $(\alpha.\alpha)$, $(\beta.\alpha)$, $(\gamma.\alpha)$ by the interchange of rows and columns of the determinant Δ

Thus

$$(1.1)=A_1B_1C_1,\quad (2.1)=A_2B_1C_1+\Delta a_2b_3c_3,\quad (3.1)=A_3B_1C_1+\Delta a_3b_2c_2.$$

Hence we have

$$\begin{aligned}\Delta^2\Sigma &= \Delta a_1 b_1 c_1 \mu'\nu' l'^2 + \Delta a_2 b_2 c_2 \nu'\lambda' m'^2 + \Delta a_3 b_3 c_3 \lambda'\mu' n'^2 \\ &\quad + \lambda' \left\{(A_1\lambda' + A_2\mu' + A_3\nu')\, B_1 C_1 + \Delta\,(a_2 b_3 c_3 \mu' + a_3 b_2 c_2 \nu')\right\} m'n' + \ldots + \ldots, \\ &= (A_1\lambda' + A_2\mu' + A_3\nu')\,(B_1 C_1 m'n'\lambda' + B_2 C_2 n'l'\mu' + B_3 C_3 l'm'\nu') \\ &\quad + \Delta\,(a_1 l' + a_2 m' + a_3 n')\,(b_1 c_1 \mu'\nu' l' + b_2 c_2 \nu'\lambda' m' + b_3 c_3 \lambda'\mu' n'),\end{aligned}$$

which is the form found in paragraph 4.

6. We have thus obtained the tangential equation of the conics which are triangularly inscribed to all six conics. It depends upon the ratios of three parameters, entering homogeneously to the second degree, and can thus be made to satisfy two further conditions.

In particular. four such conics can be found to touch two straight lines, in general. If, however, these two lines pass through one of the six given points, three of these conics will degenerate and there will be a unique solution.

On the Stability of the Periodic States of the Triode Oscillator. By Mr W. M. H. Greaves, St John's College.

[*Received* 10 October 1923.]

1. Appleton and van der Pol have shown* that in a simple Triode or Dynatron generating circuit the anode potential v is related to the time t by a differential equation of the type

$$\frac{d^2v}{dt^2} + f(v)\frac{dv}{dt} + \omega^2 v = 0 \qquad \text{......(1)},$$

where $f(v)$ is a power series in v, and may be written

$$f(v) = -\alpha + \beta v + \gamma v^2 + \delta v^3 + \epsilon v^4 + \dots \qquad \text{......(2)}.$$

In a practical case $f(v)$ is usually small compared with ω.

The equation (1) is also representative of the Duddell and Poulsen arcs†.

It is a well-known experimental fact that all these generators settle down into a state in which their behaviour is purely periodic, the radiation emitted consisting of a fundamental wave and higher harmonics. Since in practice we are concerned chiefly with the behaviour of the generator in its final periodic state, we may in the first place confine our analytical investigations to the possible *periodic* solutions of equation (1), the general solution being of secondary importance. These periodic solutions have been developed formally in a recent paper by Dr E. V. Appleton and the writer‡, and the convergence of the series obtained has since been established for sufficiently small values of the quantities α, β, γ, etc., or for sufficiently large values of ω§.

In sections 2 to 5 of the present paper we shall be concerned with an investigation of the stability of these periodic solutions. A criterion of stability has already been given by Appleton and van der Pol and we shall arrive at the same criterion in the present paper, but the method used below in addition to being more satisfactory from the mathematical point of view, possesses the advantage of showing that in the neighbourhood of a stable periodic solution of (1), the general solution tends exponentially to a periodic one, there being no possibility of small undamped vibrations about a periodic state‖. In the final section we shall show by means of a certain method of approximation that, if a system represented by (1) be started in any manner, it will eventually arrive in the immediate neighbourhood of one of the stable periodic states, and the previous result then shows that it will settle down exponentially into such a state. We shall thus arrive mathematically at the well-known experimental result which has served as the basis of previous investigations, namely that generators represented by an equation of the type (1) settle down into a periodic state.

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* *Phil. Mag.*, Ser. 6, Vol. 43, No. 253, p. 179, January 1922.

† *Phil. Mag.*, Ser. 6, Vol. 45, No. 267, p. 402, March 1923.

‡ *Phil. Mag.*, Vol. 45, pp. 401–414, March 1923.

§ *Proc. Roy. Soc.*, A., Vol. 103, pp. 516–524, 1923.

‖ This possibility is not definitely excluded by the first investigation of Appleton and van der Pol, which was intended purely as a first approximation and in which a whole series of periodic terms is neglected.

2. Let us write

$$\alpha/\omega = a_0\mu, \quad \beta/\omega = a_1\mu, \quad \gamma/\omega = a_2\mu, \text{ etc.} \qquad \text{......(3)},$$

where μ is small, and change the independent variable from t to τ where τ is given by

$$\omega t = \tau \qquad \text{......(4)}.$$

The equation (1) then becomes

$$\frac{d^2v}{d\tau^2} + \mu\phi(v)\frac{dv}{d\tau} + v = 0 \qquad \text{......(5)},$$

where $$\phi(v) = -a_0 + a_1 v + a_2 v^2 + \ldots \qquad \text{......(6)}.$$

Now write $$\left.\begin{aligned} v &= \sqrt{x}\cos y \\ \frac{dv}{d\tau} &= \sqrt{x}\sin y \end{aligned}\right\} \qquad \text{......(7)*}$$

and (5) reduces to the simultaneous first order equations

$$\left.\begin{aligned} \frac{dx}{d\tau} &= -\mu\xi \\ \frac{dy}{d\tau} &= -1 - \mu\eta \end{aligned}\right\} \qquad \text{......(8)},$$

where $$\left.\begin{aligned} \xi &= 2x\sin^2 y\,.\,\phi(\sqrt{x}\cos y) \\ \eta &= \tfrac{1}{2}\sin 2y\,.\,\phi(\sqrt{x}\cos y) \end{aligned}\right\} \qquad \text{......(9)}.$$
and

ξ and η are developable in the form of Fourier series in sines and cosines of multiples of y, the coefficients being functions of x. Let $\xi_0(x)$ denote the non-periodic part of the Fourier expansion of ξ so that

$$\xi_0(x) = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 y\,.\,\phi(\sqrt{x}\cos y)\,dy \qquad \text{......(10)}.$$

It has been shown in the papers referred to above, that corresponding to every real root $x = a$ of the equation $\xi_0(x) = 0$, there exists, provided that μ is sufficiently small, and that the derivative

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[*] For details of this transformation see *Proc. Roy. Soc*, A., Vol. 103, p. 517, 1923.

$\xi_0'(a)$ is not zero, a group of periodic solutions of equations (8) and that this group of solutions may be expressed in the form

$$\left.\begin{aligned} x &= a + \mu x_1 + \mu^2 x_2 + \dots \\ y &= -\tau' + \mu y_1 + \mu^2 y_2 + \dots \end{aligned}\right\} \qquad \text{......(11)},$$

where $$\tau' = (\tau + \epsilon)(1 + \mu\delta_1 + \mu^2\delta_2 + \dots) \qquad \text{......(12)}.$$

In these equations δ_1, δ_2, etc., are constant coefficients and x_1, x_2, etc., y_1, y_2, etc., are periodic functions of τ' with period 2π, the y_1, y_2, etc., vanishing for $\tau' = 0$. ϵ is an arbitrary additive constant of integration. The solution (11) is then periodic with period 2π in τ'*. The series (11) and (12) are convergent for sufficiently small values of μ.

From (8) we obtain $$\frac{dx}{dy} = \frac{\mu\xi}{1 + \mu\eta},$$

and on expanding in powers of μ this relation becomes

$$\frac{dx}{dy} = \mu P_1 + \mu^2 P_2 + \mu^3 P_3 + \dots \qquad \text{......(13)},$$

where P_1, P_2, P_3 are expressible as Fourier series in sines and cosines of multiples of y, the coefficients being functions of x so that we may write

$$P_r = \sum_{s=-\infty}^{s=+\infty} P_{rs} e^{siy},$$

$P_{rs}(x)$ being a function of x.

It is obvious that $$P_{10}(x) = \xi_0(x) \qquad \text{.....(14)}.$$

The second of the equations (11) gives on reversion†,

$$\tau' = -y + \mu\tau_1 + \mu^2\tau_2 + \dots,$$

where τ_1, τ_2, etc., are periodic functions of y with period 2π.

Substituting this value of τ' in the first of the equations (11) we obtain an equation of the form

$$x = a + \mu X_1 + \mu^2 X_2 + \dots \qquad \text{......(15)},$$

where X_1, X_2, etc., are periodic functions of y with period 2π.

It is clear that (15) is a solution of the differential equation (13), and is further a periodic solution. So that to every group of periodic solutions of (8) (there being one such group for every real root of the equation $\xi_0(x) = 0$), there corresponds a periodic solution of (13).

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* It was explained in one of the previous papers (*Proc. Roy. Soc.*, A, Vol. 103, p. 517, 1923) that a solution of an equation such as (8) is to be regarded as periodic if at the end of each period x resumes its original value and y changes by a multiple of 2π. Such a solution will be strictly periodic when expressed in terms of the original dependent variable v.

† The actual details of the reversion are not needed here. It may be carried out if desired with the aid of Lagrange's formula. (See, for instance, Goursat, *Cours d'Analyse*, t. I, ch. IX, p. 481.)

3. We have seen (equation (15)), that in any of the periodic solutions corresponding to the root $x = a$ of the equation $\xi_0(x) = 0$, x and y are connected by the relation

$$x = a + \mu X_1 + \mu^2 X_2 + \dots = \theta(y), \text{ say} \quad \text{......(15').}$$

Now suppose that in the neighbourhood of such a periodic solution we have

$$x = \theta(y) + \rho \qquad \text{......(16),}$$

where ρ is small.

Substituting the value (16) of x in the equation (13) and remembering that $x = \theta(y)$ is a solution of this equation we get, squares and higher powers of ρ being neglected in the usual way*,

$$\frac{d\rho}{dy} = \rho\left[\mu\frac{\partial P_1}{\partial x} + \mu^2\frac{\partial P_2}{\partial x} + \dots\right] \qquad \text{......(17),}$$

where in the expressions $\frac{\partial P_1}{\partial x}$, $\frac{\partial P_2}{\partial x}$, etc., x is to be put equal to $\theta(y)$ after the differentiations have been performed.

On doing this and rearranging the right-hand side of the equation in powers of μ, (17) becomes

$$\frac{1}{\rho}\frac{d\rho}{dy} = \mu Q_1 + \mu^2 Q_2 + \mu^3 Q_3 + \dots \qquad \text{......(18),}$$

where Q_1, Q_2, etc., are periodic functions of y with period 2π so that we may write

$$Q_r = \sum_{s=-\infty}^{s=+\infty} Q_{rs} e^{siy}.$$

We clearly have $$Q_1 = \left(\frac{\partial P_1}{\partial x}\right)_{x=a}.$$

Integrating (18) we get

$$\rho = Ae^{cy}\psi(y) \qquad \text{......(19),}$$

where A is an arbitrary constant of integration and c and $\psi(y)$ are given by the formulae

$$c = \mu Q_{10} + \mu^2 Q_{20} + \mu^3 Q_{30} + \dots \qquad \text{......(20),}$$

and $$\psi(y) = \text{Exp.}\left[\sum_{r=1}^{r=\infty}\sum_{s=-\infty}^{s=+\infty}{}' \left\{\frac{\mu^r}{is} Q_{rs} e^{siy}\right\}\right],$$

the dash attached to the second sign of summation indicating that $s = 0$ is to be excluded.

Now the second of equations (8) shows that for sufficiently small values of μ, $y \to -\infty$ as $\tau \to +\infty$, that is as $t \to +\infty$. So that if $c < 0$ (19) indicates that ρ will increase with t and the group of periodic solutions corresponding to the root $x = a$ of $\xi_0(x) = 0$

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* See note at end of paper.

will be unstable. On the other hand, if $c > 0$, then as t increases indefinitely the solution (16) of equation (13) will tend asymptotically to the periodic solution (15), and the group of periodic solutions in question will be stable*.

For sufficiently small values of μ, c will be positive or negative according as Q_{10} is positive or negative, that is, according as $\left(\frac{d\xi_0(x)}{dx}\right)_{x=a}$ is positive or negative.

So that for sufficiently small values of μ, if $\xi_0'(a) < 0$ the group of periodic solutions in question will be unstable, but if $\xi_0'(a) > 0$, they will be stable.

4. Suppose that $c > 0$ so that the corresponding periodic solution of (13) is stable. It remains to be shown that in the neighbourhood of $x = a$ any solution of equations (8) tends exponentially to a periodic one as t increases indefinitely.

To prove this we substitute the value (16) of x in the second of equations (8) and on neglecting squares of ρ we get an equation of the form

$$\frac{d\tau}{dy} = -1 - \mu g(y) - \mu\rho h(y)$$

and we then get from (19)

$$\frac{d\tau}{dy} = -1 - \mu g(y) - \mu A e^{cy} h_1(y).$$

In these equations $g(y)$, $h(y)$ and $h_1(y)$ are periodic functions of y with period 2π developable in powers of μ.

We then get on integration

$$\tau + \epsilon = -(1 + \mu k)y - \mu g_1(y) - \mu A e^{cy} h_2(y),$$

where k is the non-periodic part of $g(y)$ and $g_1(y)$ and $h_2(y)$ are periodic functions of y with period 2π developable in powers of μ. ϵ is an arbitrary constant of integration.

Write $$\frac{\tau + \epsilon}{1 + \mu k} = \tau'.$$

The above equation then becomes

$$\tau' = -y - \mu g_2(y) - \mu A e^{cy} h_3(y).$$

where $$g_2(y) = \frac{g_1(y)}{1 + \mu k} \text{ and } h_3(y) = \frac{h_2(y)}{1 + \mu k}.$$

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* Strictly speaking we have only proved so far that (when $c > 0$) the periodic solution of (13) corresponding to $x = a$ is stable and we have not proved the stability of the corresponding group of solutions of (8). This, however, will be proved in the following section.

We then get on reversion (which may be carried out by Lagrange's formula), neglecting squares and higher powers of A,

$$y = -\tau' + \mu g_3(\tau') + \mu A e^{-c\tau'} h_4(\tau') \qquad \text{......(21)},$$

where $g_3(\tau')$ and $h_4(\tau')$ are periodic functions of τ' with period 2π developable in powers of μ.

Substituting this value of y in (16) we obtain x in the form

$$x = a + \mu g_4(\tau') + \mu A e^{-c\tau'} h_5(\tau') \qquad \text{......(22)},$$

$g_4(\tau')$ and $h_5(\tau')$ being periodic functions of τ' with period 2π developable in powers of μ.

(21) and (22) represent the general solution of equations (8) in the region under consideration. Now if we put $\rho = 0$ (so that $A = 0$) throughout the above work we obtain

$$\left.\begin{aligned} y &= -\tau' + \mu g_3(\tau') \\ x &= a + \mu g_4(\tau') \end{aligned}\right\} \qquad \text{......(23)},$$

where
$$\tau' = \frac{\tau + \epsilon}{1 + \mu k}.$$

The equations (23) are clearly the family of periodic solutions, involving one arbitrary constant, corresponding to the root $x = a$ of the equation $\xi_0(x) = 0$. It is clear that if $c > 0$, then as t increases indefinitely the general solution (21) and (22) approaches the periodic solution (23) exponentially.

5. We have seen that the criterion of stability is

$$\xi_0'(a) > 0 \qquad \text{......(24)}.$$

From (10) we get

$$\xi_0(a) = \frac{1}{\pi}\int_0^{2\pi} a \sin^2 y \, (-a_0 + a_1 a^{\frac{1}{2}} \cos y + a_2 a \cos^2 y + a_3 a^{\frac{3}{2}} \cos^3 y + \dots)\, dy,$$

which gives on reduction

$$\xi_0(a) = a\left[-a_0 + \sum_{r=1}^{r=\infty} \frac{(2r)!\, a_{2r} a^r}{2^{2r}\, r!\, (r+1)!}\right]$$

Hence (24) becomes

$$-a_0 + \sum_{r=1}^{r=\infty} \frac{(2r)!\, a_{2r}\, a^r}{2^{2r}\, (r!)^2} > 0 \qquad \text{......(25)}.$$

(25) is (in the notation of the present paper) the Appleton-van der Pol criterion for stability*.

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* *Loc. cit.*, equation (10), p. 182.

6. In the previous sections we have been considering the behaviour of the system in the neighbourhood of the periodic solutions. In order to investigate its behaviour in the general case we return to the equation (13)

$$\frac{dx}{dy} = \mu P_1 + \mu^2 P_2 + \mu^3 P_3 + \ldots .$$

We may obtain a convenient approximation to the general nature of the solution by neglecting μ^2 and higher powers of μ and at the same time replacing $\frac{dx}{dy}$ by its average value; that is, we retain only the non-periodic part of P_1*.

We then get the equation

$$\frac{dx}{dy} = \mu \xi_0 (x) \qquad \ldots\ldots(26).$$

Suppose that initially $p > x > q$ where p and q are two consecutive simple zeros of $\xi_0 (x)$.

Let us write

$$\xi_0 (x) = K (p - x) (x - q),$$

where K is a function of x which has no zeros and is thus one-signed for values of x between p and q.

The equation (26) may be replaced by the equations†

$$\frac{dx}{dy'} = \mu (p - x) (x - q) \qquad \ldots\ldots(27),$$

$$y = \int_0^{y'} \frac{dy'}{K} \qquad \ldots\ldots(28).$$

(27) gives on integration

$$\frac{x - q}{p - x} = Be^{\mu (p - q) y'} \qquad \ldots\ldots(29),$$

where B is an arbitrary constant of integration.

(29) shows that as $y' \rightarrow + \infty$, $x \rightarrow p$ and as $y' \rightarrow - \infty$, $x \rightarrow q$. We now have two cases to distinguish.

(*a*) Suppose $\xi' (p) > 0$ and $\xi' (q) < 0$. Then K will be negative for $p > x > q$ and (28) then shows that as $t \rightarrow + \infty$ so that $y \rightarrow - \infty$, $y' \rightarrow + \infty$ and so $x \rightarrow p$.

(*b*) Suppose $\xi' (p) < 0$ and $\xi' (q) > 0$. Then K will be positive for $p > x > q$ and we see that as $t \rightarrow + \infty$ so that $y \rightarrow - \infty$, $y' \rightarrow - \infty$ and $x \rightarrow q$.

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* This method of approximation is that which is used in celestial mechanics for the purpose of calculating the secular motions of the major planets. Cf. Tisserand, *Mécanique Céleste*, t. I, ch. XXVI.

† Cf. Charlier, *Mechanik des Himmels*, vol. I, p. 89, or the original source, Weierstrass, *Ges. Werke*, Bd. II, p. 1.

In each of these cases as t increases indefinitely the solution approaches the neighbourhood of a stable periodic solution and the previous sections then show that it will tend exponentially to such a solution.

Note. The solution (19) for ρ is approximate as it has been obtained by neglecting squares and higher powers of ρ. It may, however, be used as a basis from which the complete solutions in the neighbourhood of the periodic ones may be developed in the usual way[*]. These developments proceed in ascending positive integral powers of $e^{\alpha t}$ and the conclusion obtained above that in the neighbourhood of the stable periodic solutions the general solution tends exponentially to a periodic one is rigorously true to all orders of approximation.

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[*] Cf. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, t. I, ch. VII, pp. 335 *et seq.*

Involutions on a Normal Quartic Curve in Space of Four Dimensions. By Mr C. G. F. James, Trinity College.

[*Received* 2 December 1923.]

The object of this note is to correct an error in my paper[*] "Extensions of a theorem of Segre's...," the notation used being the same. The curve C_4 dealt with is regarded as given by its canonical representation

$$x_1 = 1, \quad x_2 = \theta, \quad x_3 = \theta^2, \quad x_4 = \theta^3, \quad x_5 = \theta^4;$$

and at one point in the paper we sought the locus of the lines analogous to the line A_2A_4 of the figure of reference for each of the ∞^2 representations of this type (p. 671, small print). In the space representation of the locus there is an additional principal curve

$$\psi = \theta^2 + \tau(\theta + \phi)$$

and the order of the locus must be reduced by that of the form corresponding to the points of this conic. The locus sought is in fact none other than the cubic form, locus of chords of C_4, the present system of lines being the directrix system[†]. This follows at once from the following results, which can be shown immediately using the above representation:

(1) The space joining such a line g to any tangent cuts the curve again in coincident points, and thus contains a second tangent;

(2) The line joining the points of contact of these tangents meets g, and the points give the involution

$$\theta + \phi = 0.$$

The system of such involutions is ∞^2 and hence *any involution on C_4 can be obtained in this way.*

The system of ∞^2 trisecant planes of C_4 meeting g pass by ∞^1's through the joining lines of the involution. This corrects a statement on p. 670. The joining lines generate the normal cubic scroll

$$\left\| \begin{matrix} x_1 & x_2 & x_3 \\ x_3 & x_4 & x_5 \end{matrix} \right\| = 0.$$

In fact, by a correspondence argument it can be shown that the joining lines for any involution form such a scroll, and from the existence of a directrix line on the scroll, the previous results follow almost immediately.

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[*] *Proc. Camb. Phil. Soc.* XXI, 664 (1923).

[†] Segre, "Sulle varietà cubiche...," *Mem. Torino*, ser. (2), XXXIX (1889).

Postscript. It has been brought to my notice that the initial theorem of my paper on trisecant planes of a normal quartic curve in space of four dimensions was actually first formulated by Segre in "Sui fochi di 2° ordine dei sistemi infiniti di piani...," *Rend. Accad. Lincei*, vol. xxx, ser. 5 (1921). The theorems of Pieri there quoted are also contained implicitly in my results, namely they arise as sections from the seven-dimensional figure.

On errors of observation. By Dr W. Burnside.

[*Received* 30 November 1923.]

The object of this note is to direct attention to an entirely false inference which was drawn from the analysis given in my previous paper with the same title.

To make the nature of this false inference clear it is necessary to restate shortly the conclusion that was arrived at in the paper referred to. It was as follows:

If $x_1, x_2, \ldots, x_n$ are n particular determinations of a physical constant, whose true unknown value is denoted by x_0, and if

$$\frac{1}{n}\Sigma x_i = \bar{x},\ \frac{1}{n}\Sigma x_i^2 - \frac{1}{n^2}(\Sigma x_i)^2 = s^2,$$

then the probability that $\dfrac{\bar{x} - x_0}{s}$

lies between α and β is

$$\int_\alpha^\beta \frac{dt}{(1+t^2)^{\frac{n}{2}+1}} \Big/ \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{\frac{n}{2}+1}}.$$

The assumptions made were (i) that the errors of observation in the determinations follow Gauss's Law, and (ii) that the precision constant, apart from any observations, was equally likely to have all positive values.

No assumption was made with respect to the probability of various values of x_0, apart from or before any observations.

In the analysis that leads up to the above (false) conclusion, x_0 is treated as a constant (though unknown) quantity. Further, the analysis involves integration over an infinite range with respect to $y_1, y_2, \ldots, y_n$ which denote the errors in a set of n observations.

The correct conclusion from the analysis is therefore that, when all possible sets of n observations are made, and for each set the values of $\bar{x}$ and s are calculated, the probability that the fraction

$$\frac{\bar{x} - x_0}{s},$$

where $\bar{x}$, s vary from one set of observations to another, while x_0 is a constant number, lies between α and β is

$$\int_\alpha^\beta \frac{dt}{(1+t^2)^{\frac{n}{2}+1}} \Big/ \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{\frac{n}{2}+1}}.$$

It is clear that this probability statement gives no direct justification for a probability statement which is concerned with one and only one set of *n* observations.

Until the above false inference is shown to be a logical consequence of the correct probability statement just made, it is entirely unsupported by the analysis, or by any analysis on similar lines. Many opinions seem to be held at the present time with regard to Bayes's formula for the probability of causes. I have, however, seen no support for the supposition that it is possible, after an observed event, to determine the respective probabilities of its various causes, in the case in which nothing is known about the corresponding probabilities before the event.

What I have called above the false inference from the analysis does in fact profess to determine, after a set of *n* observations, the probabilities of various values of x_0, without any assumption regarding their probabilities before the observations.

On the generalisation of a theorem of Steiner. By Prof. H. F. Baker.

[*Read* 3 December 1923.]

The theorem that, if four arbitrary lines be taken in a plane, the four circles about the triangles formed by threes of these lines, meet in a point, can be generalised to space of any even number of dimensions, as was recognised by Mr J. H. Grace in 1897*. In the plane case the centres of the four circles lie on another circle passing through the point of concurrence of these four; it has been sought to prove that the centres of the $n+2$ spheres, similarly arising in space of an even number, n, of dimensions, also lie on a sphere†.

It would seem that Herr Kühne's proof is invalid, the determinant Δ used† by him (p. 194) being zero because the elements of any two rows of the determinant $B_{\sigma f}$ are in the same ratios. In what follows a proof is given that the theorem in question is not true, except in the case of the plane. For clearness n is supposed to be 4, which is sufficiently representative.

We consider then, in space of five dimensions, six *flats* (linear spaces of four dimensions), drawn through a point, O, of a quadric, Ω; taking five of these, any four of these five give a line which meets Ω again in a point other than O, and the five points so obtained define a further flat. The six flats so obtained meet in a point Q of Ω, as will be proved. By projection from O upon an arbitrary flat, Π, Mr Grace's theorem above referred to is obtained, a sphere in Π being a quadric containing the quadric of two dimensions in which Π, Ω and the tangent flat of Ω at O, intersect. In fact, if the original flats drawn through O be written $x_1 = 0, \dots, x_6 = 0$, subject to $a_1x_1 + \dots + a_6x_6 = 0$, and $v = 0$ be a flat not passing through O, it is easy to see that the six flats which are to be proved to pass through a point Q of the quadric Ω are $\xi_1 = 0, \dots, \xi_6 = 0$, wherein $\xi_r = a_{r1}x_1 + a_{r2}x_2 + \dots + a_{r6}x_6 + a_r v$, and $a_{ij} = -a_{ji}$, and that the equation of Ω is of the form

$$\lambda_1 x_1 \xi_1 + \dots + \lambda_6 x_6 \xi_6 = 0.$$

The concurrence of $(\xi_1), \dots, (\xi_6)$ is an immediate consequence of the vanishing of a skew determinant of seven rows and columns. This form of the equations makes it immediately obvious that the relation of O, and the flats through it, with Q and the flats through this, is entirely reciprocal.

The theorem, now disputed, in regard to the centres of the spheres, would, it is easily seen, follow from, and require, the theorem, that the poles in regard to Ω, of the flats $(\xi_1), \ldots, (\xi_6)$, project from O into six points of Π lying on a sphere. From the reciprocity referred to, it follows that it is sufficient to examine the corresponding result for the poles of $(x_1), \ldots, (x_6)$. These all lie on the tangent flat of Ω at O. We define a sphere in this tangent flat as a quadric containing the quadric (∞^2) intersection of Ω with the tangent flats of Ω at O and Q, and the question is whether such a sphere contains the six poles of $(x_1), \ldots, (x_6)$.

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* *Camb. Phil. Trans.* XVI, 163.

† Kühne, *Crelle*, CXIX (1898), 186. Quoted by Segre, *Enzkl. Math. Wiss.* III, 2, p. 807, footnote 122.

For greater precision we may take $x_1 = 0, \ldots, x_5 = 0, v = 0$ as the fundamental flats for the fivefold space, choosing $v = 0$ so as to contain the five points of Ω in which it is met by the intersection lines of the fours of $x_1 = 0, \ldots, x_5 = 0$; thus $v = 0$ is the same as that above described as $\xi_6 = 0$. Then, using x, y, z, t, u in place of $x_1, \ldots, x_5$, the equation of Ω may be supposed to be of the form

$$fyz + gzx + hxy + pxt + qyt + rzt + (ax + by + cz + dt)\,u \\ + (x + y + z + t + u)\,v = 0,$$

which we may occasionally also write as

$$\sum_1^5 f_{rs} x_r x_s + (x_1 + \ldots + x_5)\,v = 0.$$

This is in fact the form taken by Herr Kühne, though with a quite different description. Then we may suppose $x_6 = 0$ to be

$$c_1 x_1 + \ldots + c_5 x_5 = 0.$$

A sphere of the tangent flat at O will then have an equation obtained from

$$T_Q\,(A_1 x_1 + \ldots + A_5 x_5 + Av) - \Omega = 0,$$

(where $T_Q = 0$ is the equation of the tangent flat of Ω at Q), by writing in this equation the condition, $x_1 + x_2 + \ldots + x_5 = 0$, for the coordinates of points in the tangent flat at O. We are to see whether the result can be satisfied by the coordinates of the poles of $(x_1), \ldots, (x_6)$. Now, if $(x_1^{(1)}, \ldots, x_5^{(1)}, v^{(1)})$ be the pole of $x_1 = 0$, and so on, the point $(x_1^{(5)}, \ldots, v^{(5)})$ being the pole of $x_5 = 0$, then the pole of $c_1 x_1 + \ldots + c_5 x_5 = 0$ is

$$(c_1 x_1^{(1)} + \ldots + c_5 x_1^{(5)}, \ldots, c_1 v^{(1)} + \ldots + c_5 v^{(5)}).$$

Wherefore, a *necessary* condition for the six poles to lie on a sphere, in the sense of the theorem, is the existence of the identity

$$c_1\,(\Omega/T_Q)_1 + \ldots + c_5\,(\Omega/T_Q)_5 = (\Omega/T_Q)_6,$$

where $(\Omega/T_Q)_r$ is the result of the substitution in Ω/T_Q of the coordinates of the pole of $x_r = 0$. Now, the result of substituting in T_Q, which is the polar of Q, the coordinates of the pole of $x_r = 0$, is the same as that of substituting in x_r the coordinates of Q; the six results so obtained are then $x_1', \ldots, x_5', c_1 x_1' + \ldots + c_5 x_5'$, where

$(x_1', \dots, x_5', x_6', v')$ are the coordinates of Q, of which $v' = 0$. Again, forming the determinantal discriminant of Ω, a determinant of six rows and columns, say Δ, and denoting the minors therein by F_{rs} $(r, s = 1, \dots, 6)$, the pole of $x_1 = 0$ is $(F_{11}, F_{12}, \dots, F_{15}, F_{16})$, and the result of substituting this in Ω is ΔF_{11}. The result of substituting in Ω, the six coordinates of the pole of $x_6 = 0$, namely

$$(c_1F_{11} + \dots + c_5F_{51},\ c_1F_{12} + \dots + c_5F_{52},\ \dots,\\ c_1F_{15} + \dots + c_5F_{55},\ c_1F_{16} + \dots + c_5F_{56})$$

is at once seen to be $\Delta\Sigma c_i c_j F_{ij}$ $(i, j = 1, \dots, 5)$. Thus the necessary condition in question is that the coordinates of Q satisfy the equation

$$\frac{c_1F_{11}}{x_1} + \dots + \frac{c_5F_{55}}{x_5} = \frac{\Sigma c_i c_j F_{ij}}{c_1x_1 + \dots + c_5x_5}.$$

It can however be proved that the coordinates of Q are in the ratios of the minors of the elements of any row of a certain determinant of five rows and columns of which the (i, j)th element is $c_i f_{ij}/(c_i - c_j)$, the equation of the flat above denoted by ξ_i being

$$\frac{c_1 f_{i1}}{c_i - c_1} x_1 + \frac{c_2 f_{i2}}{c_i - c_2} x_2 + \dots + \frac{c_5 f_{i5}}{c_i - c_5} x_5 - v = 0,$$

except that of ξ_6, which is $v = 0$. If then we write $X_1, \dots, X_5$ for $c_1x_1, \dots, c_1x_5$, we may conveniently write the above identity in the form

$$\frac{c_1^2F_{11}}{X_1} + \dots + \frac{c_5^2F_{55}}{X_5} = \frac{\Sigma c_i c_j F_{ij}}{X_1 + \dots + X_5}.$$

Herein $X_1, \dots, X_5$ are functions only of the coefficients f_{rs} in Ω and of the *differences* of $c_1, \dots, c_5$. The expression $\Sigma c_i c_j F_{ij}$ is also a function of these differences, and of f_{rs}, being in fact the negative of the determinant

$$\begin{vmatrix} 0 & f_{12} & f_{13} & f_{14} & f_{15} & 1 & c_1 \\ f_{21} & 0 & f_{23} & f_{24} & f_{25} & 1 & c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{51} & f_{52} & \cdots & \cdots & \cdots & 1 & c_5 \\ 1 & 1 & \cdots & \cdots & 1 & 0 & 0 \\ c_1 & c_2 & \cdots & \cdots & c_5 & 0 & 0 \end{vmatrix}$$

Wherefore, putting $c_i + \sigma$ for c_i in the supposed identity, we should require also the two identities

$$\frac{c_1F_{11}}{X_1} + \dots + \frac{c_5F_{55}}{X_5} = 0, \qquad \frac{F_{11}}{X_1} + \dots + \frac{F_{55}}{X_5} = 0.$$

Of these two, consider the second in the particular case when every $f_{ij} = 1$, so that $F_{11}, \dots, F_{55}$ are all equal. We should then require that, if $X_1, \ldots, X_5$ be in the ratios of the minors of the elements of any row of the skew determinant of five rows whose general element is $(c_i - c_j)^{-1}$, then $X_1^{-1} + \ldots + X_5^{-1} = 0$. This however is not true, as may be verified by considering particular numerical values for $c_1, \ldots, c_5$. (Taking these to be 0, 1, 2, 3, 4, it may be proved that $X_1, \ldots, X_5$ are in the ratios of 6.13, − 6.7, 73, − 6.7, 6.13, for example.)

Thus the truth of the theorem is disproved.

Note. For the case of the plane theorem, for which Ω is a quadric in three dimensions, we find $F_{11} = 2f$, $F_{22} = 2g$, $F_{33} = 2h$, $F_{23} = f - g - h$, $F_{31} = g - h - f$, $F_{12} = h - f - g$, and X_1, X_2, X_3 respectively equal to $f(c_2 - c_3)^{-1}$, $g(c_3 - c_1)^{-1}$, $h(c_1 - c_2)^{-1}$. The identity in question then holds. In this case the line OQ is a generator of the cubic cone, of vertex O,

$$(c_1 F_{11} yz + c_2 F_{22} zx + c_3 F_{33} xy)(c_1 x + c_2 y + c_3 z) - (\Sigma c_i c_j F_{ij})\, xyz = 0,$$

which is easily seen to contain the six lines of intersection of the pairs of planes $x = 0$, $y = 0$, $z = 0$, $c_1 x + c_2 y + c_3 z = 0$, and also the generators of the quadric Ω at O. The (? Steiner) theorem that the centres of the four circles lie on a circle, when applied to Q as centre of projection, instead of O, is thus seen to be the same theorem as that the four circles obtained by projection from O meet in a point (ascribed to Wallace), regarded as a particular case of a theorem for cubic curves through eight points, of which six are the intersections of four lines.

It may also be remarked that, for this plane problem, the point Q is obtained, on the quadric Ω, by this construction: Take, in the tangent plane at O, the poles, P_1, P_2, P_3, P_4, of the four planes drawn through O; let the conic $OP_1P_2P_3P_4$ meet the generators of Ω at O in I and J; then Q is the point of contact of the other tangent plane to Ω drawn from the line IJ.

Another remark, related to the problem considered, may also be made. In a plane, if A, B, C be three points of a conic, if a line, l, meet this conic in points I, J, and if P be any point of this conic, then the conic touching BC, CA, AB and the two lines PI, PJ, also touches the line l. In three dimensions, let A, B, C, D be four points of a quadric, let a plane, ϖ, meet this quadric in a conic, σ, and let P be any point of the quadric. A quadric can be drawn to touch the planes BCD, CAD, ABD, ABC, and to touch the planes through P which contain the tangent lines of the conic σ. But this quadric does not touch the plane ϖ unless P lie on a certain four-nodal cubic surface (being then on the curve of intersection of this with the original quadric).

We are accustomed to speak of the point equation of a cone, being the relation satisfied by the coordinates of all points on generators of this. We may similarly obtain an equation satisfied by the coordinates of all planes through the tangent lines of a conic. In the present case, if the original quadric be

$$(0, 0, 0, 0, f, g, h, p, q, r \between x, y, z, t)^2 = 0,$$

referred to A, B, C, D, the plane ϖ being

$$l_0 x + m_0 y + n_0 z + k_0 t = 0,$$

and (x_1, y_1, z_1, t_1) be any point of the conic σ, the general plane through the tangent line of this conic at (x_1, y_1, z_1, t_1) is of equation

$$x(hy_1 + gz_1 + pt_1 + \rho l_0) + \ldots + t(px_1 + qy_1 + rz_1 + \rho k_0) = 0,$$

where ρ is arbitrary. If this be $lx + my + nz + kt = 0$, we have the four equations

$$hy_1 + gz_1 + pt_1 + \rho l_0 + \sigma l = 0, \ldots, px_1 + qy_1 + rz_1 + \rho k_0 + \sigma k = 0,$$

for a suitable value of σ. We also have

$$l_0 x_1 + m_0 y_1 + n_0 z_1 + k_0 t_1 = 0, \quad lx_1 + \ldots + kt_1 = 0.$$

Whence

$$\begin{vmatrix} \cdot & h & g & p & l_0 & l \\ h & \cdot & f & q & m_0 & m \\ g & f & \cdot & r & n_0 & n \\ p & q & r & \cdot & k_0 & k \\ l_0 & m_0 & n_0 & k_0 & \cdot & \cdot \\ l & m & n & k & \cdot & \cdot \end{vmatrix} = 0,$$

which is the equation for the planes through the tangent lines of the conic σ. Denote this by $\Omega = 0$, the coefficients of l^2, m^2, n^2, k^2 herein being denoted by $-A$, $-B$, $-C$, $-D$. Then, if (x_0, y_0, z_0, t_0) be the coordinates of P, the general quadric touching the planes from P which touch the conic σ is, in tangential coordinates

$$(lx' + my' + nz' + kt')(lx_0 + my_0 + nz_0 + pt_0) + \Omega = 0,$$

where x', y', z', t' are arbitrary. In order that this may touch the planes BCD, CAD, ABD, ABC, it is necessary and sufficient that $x' = A/x_0$, $y' = B/y_0$, $z' = C/z_0$, $t' = D/t_0$. The quadric then has the form

$$(lx_0 + my_0 + nz_0 + pt_0)(lAx_0^{-1} + \ldots + kDt_0^{-1}) + \Omega = 0,$$

and is definite. In order that this should touch the plane ϖ, since Ω vanishes when l, m, n, p are replaced by l_0, m_0, n_0, p_0, we require that (x_0, y_0, z_0, t_0) should lie on the four-nodal cubic surface given by

$$\frac{l_0 A}{x} + \frac{m_0 B}{y} + \frac{n_0 C}{z} + \frac{k_0 D}{t} = 0.$$

If we proceed similarly for the plane problem, the term in D does not enter, and A, B, C are replaced respectively by

$$2fm_0 n_0, \quad 2gn_0 l_0, \quad 2hl_0 m_0,$$

so that the last condition merely reproduces the equation of the original conic.

[December, 1923. Since the preceding paper was written, in October, Mr Richmond has written, quite independently, a paper, "On an extension of Wallace's pedal property of the circumcircle" (see below), which is in close connexion with the last remark made above. He calls attention to a theorem due to Mantel, that if the perpendiculars, defined in regard to an Absolute conic, ω, from a point, E, to the planes, α, β, γ, δ, containing four points A, B, C, D, meet these in points of a plane, then the same is true of the perpendiculars from any one of these five points to the planes containing the other four. And adds the interesting theorem that the necessary and sufficient condition that this relation should hold for any five points, A, B, C, D, E, of a given cubic curve, which meets the plane of ω in the points P, Q, R, is that the conic ω should touch QR, RP, PQ; extending this to space of any number of dimensions. It seems of interest to remark:

(1) The condition that the perpendiculars, in regard to an Absolute conic, ω, drawn from a point, E, to the planes, α, β, γ, δ, containing four points A, B, C, D, should meet these planes in points of a plane, is that the quadric touching the planes α, β, γ, δ, and having $E\omega$ as enveloping cone, should touch the plane of ω.

(2) Let σ be the conic, in the plane of ω, in regard to which the joining lines and planes of A, B, C, D, E determine a self-polar system. Then, the enveloping cone from E to any quadric which touches the four planes α, β, γ, δ and also touches the plane of ω, meets the plane of ω in a conic which is inpolar to σ.

Thus the necessary and sufficient condition that the perpendiculars, in regard to ω, from E to α, β, γ, δ, should meet these in points of a plane, is that ω should be inpolar to σ. This establishes Mantel's theorem.

(3) If A, B, C, D, E be any set of five points of a fixed cubic curve, which meets the plane of ω in P, Q, R, the aggregate of the conics σ, defined as in (2), for all sets A, B, ..., E, is that of the conics in the plane of ω in regard to which P, Q, R are a self-polar triad. But a conic, ω, in the plane PQR, which is inpolar to all conics in regard to which P, Q, R is a self-polar triad, must touch QR, RP, PQ. And conversely.]

On an extension of Wallace's pedal property of the circumcircle. By Mr H. W. Richmond, King's College.

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1. Mr J. P. Gabbatt has discussed in the most recent Part* of the *Proceedings* of this Society the Pedal locus of a simplex in hyperspace. It is, however, possible to regard the pedal property of the circumcircle somewhat differently and so to seek other extensions. Given a circle, *any* three points on it are vertices of an inscribed triangle, and the feet of the perpendiculars on the sides from *any* fourth point of the circle are collinear. Is there any curve in space on which an analogous property holds for any five points, viz. that the feet of the perpendiculars from any one upon the faces of the tetrahedron formed by the other four are coplanar?

It will be shown that curves of order n exist in Euclidean space of n dimensions on which *any* $n+2$ points have such a property; but that the curves cannot be real if n is odd.

We will first consider ordinary space.

2. A tetrahedron $ABCD$ defines a system of tetrahedral coordinates (x, y, z, w). For metrical properties in the space we have to make use of the plane at infinity, whose equation is

$$x + y + z + w = 0 \qquad \text{......(1)},$$

and in it a locus of the second class; Ω, the Absolute whose tangential equation is written

$$\Omega(l, m, n, r) \equiv al^2 + bm^2 + cn^2 + dr^2 + 2fmn + 2gnl + 2hlm$$
$$+ 2f'lr + 2g'mr + 2h'nr = 0 \qquad \text{......(2)}.$$

The coefficients $a, b \ldots f \ldots$ can be expressed in terms of the areas of the faces of the tetrahedron and the cosines of its dihedral angles, but to do this is unnecessary. All that need be noted is the set of conditions

$$a + h + g + f' = b + f + h + g' = c + g + f + h' = d + f' + g' + h' = 0$$
$$\text{......(3)},$$

which imply that Ω is not a quadric but a conic lying in the plane at infinity.

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[*] *Proc. C.P.S.* vol. XXI, part VI, p. 763. I am indebted to Mr Gabbatt for the two references at the end of his first paragraph.

3. The tangential equation

$$k\Omega = (lx + my + nz + rw)(lx' + my' + nz' + rw') \quad \ldots\ldots(4),$$

represents a quadric of revolution with foci (x, y, z, w) (x', y', z', w'); for it expresses the fact that the product of the perpendiculars from these two points upon any tangent plane is constant. Should

$$x' + y' + z' + w' = 0 \qquad \ldots\ldots(5),$$

the second focus is at infinity, the surface is a paraboloid of revolution and the feet of the perpendiculars from the focus (x, y, z, w) upon all tangent planes lie in a plane, the tangent plane at the vertex of the paraboloid.

Given any point E (X, Y, Z, W), a quadric of revolution with focus E touches the faces of $ABCD$, the position of the second focus E', (X', Y', Z', W') being such that when X, Y, Z, W, X', Y', Z', W', are substituted for $x, y, z, w, x', y', z', w'$ in (4) the terms in l^2, m^2, n^2, r^2 go out. Thus

$$XX' : YY' : ZZ' : WW' :: a : b : c : d.$$

Hence by (5) the condition that the feet of the perpendiculars from E (X, Y, Z, W) on the faces of $ABCD$ should be coplanar is that (X', Y', Z', W') the coordinates of E' should satisfy (5), i.e.

$$a/X + b/Y + c/Z + d/W = 0^* \qquad \ldots\ldots(6).$$

This cubic surface I shall call the *Cayley Locus* of $ABCD$.

4. Equation (6) is simple, it is exactly similar in form to that of the circumcircle in two dimensions, and can be extended in the same form to Euclidean space of n dimensions. But it gives no hint of a most important fact, that the relation between A, B, C, D, E is symmetrical, i.e. that, if E lies on the Cayley Locus of $ABCD$, then A lies on the Cayley Locus of $BCDE$, B on the Cayley Locus of $ACDE$, etc. This may be proved† as follows: supposing A, B, C, D, E to be any five points in space, five other points A', B', C', D', E' can be determined as was done for E' in

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* See Gabbatt, *l.c.* 1st paragraph for references.

† See W. Mantel, *Wiskundige Opgaven*, 1899–1902, p. 396, No. 199. I have to thank Mr J. H. Grace for drawing my attention to this theorem and Mantel's beautiful proof, and Mr F. P. White for the reference which I had failed to find. Theorem (i) holds for Euclidean and non-Euclidean geometry, (ii) only in Euclidean; both can be extended to n dimensions.

Another proof is due to Mr Grace. Five quadrics can be constructed, one with A as centre self-polar to $BCDE$, one with B as centre self-polar to $ACDE$, etc.; these five are always similar and similarly situated. For if the first quadric is transformed by a homogeneous strain into a sphere, $BCDE$ will be strained into a tetrahedron of which A is the orthocentre. But if so B is the orthocentre of $ACDE$, and the second quadric must be changed into a sphere by the same homogeneous strain as the first. The condition for coplanarity of the feet of the perpendiculars is that one (and therefore each) of the quadrics should have its linear invariants equal to zero, or pass to infinity in three mutually perpendicular directions.

§ 3; D' is the second focus of a quadric of revolution having D for one focus and touching the faces of $ABCE$; and so on. Now the two planes BCA, BCE through the line BC, which touch a quadric of revolution with foci D and D', are equally inclined to the planes BCD, BCD' through this line and the foci; thus the angle between the planes BCD' and BCA is equal to the angle between BCD and BCE. But the angle between BCE' and BCA on similar reasoning must be equal to the same angle. Hence the plane ABC is equally inclined to the planes BCD' and BCE': and also is equally inclined to the planes CAD' and CAE' and to ABD' and ABE'. Therefore (i) D' and E' are symmetrically placed on opposite sides of ABC, and (ii) if for a special position of the points $ABCDE$, E' is at infinity, so also is D'. If the quadric of revolution with focus E, used in § 3, is a paraboloid, so are the four others with foci at A, B, C and D.

To obtain the equation of the Cayley Locus of $ABCE$, we make (4) touch the faces of this tetrahedron, viz.

$$w = 0, \quad Wx - Xw = 0, \quad Wy - Yw = 0, \quad Wz - Zw = 0,$$

i.e. (4) must be satisfied by the values

$$(0, 0, 0, 1)\,(W, -X, 0, 0)\,(W, 0, -Y, 0)\,(W, 0, 0, -Z)$$

of l, m, n, r. Therefore $\qquad kd = ww',$

$$k\,(aW^2 - 2f'XW + dX^2) = (Wx - Xw)\,(Wx' - Xw'),$$
$$k\,(bW^2 - 2g'YW + dY^2) = (Wy - Yw)\,(Wy' - Yw'),$$
$$k\,(cW^2 - 2h'ZW + dZ^2) = (Wz - Zw)\,(Wz' - Zw'),$$

and on elimination of x', y', z', w', k by help of (5) we have the required result

$$\frac{aW^2 - 2f'XW + dX^2}{Wx - Xw} + \ldots + \ldots = \frac{d\,(X + Y + Z + W)}{w} = 0,$$

or, after reduction

$$\frac{aWw - f'\,(Wx + Xw) + dXx}{Wx - Xw} + \ldots + \ldots = 0 \quad \ldots\ldots(7).$$

The symmetry of this result between the large and small letters proves that if $P\,(x, y, z, w)$ lies on the Cayley Locus of $ABCE$, then $E\,(X, Y, Z, W)$ lies on the Cayley Locus of $ABCP$. Or again, the condition that D lies on the Cayley Locus of $ABCE$ just found is

$$\frac{aW - f'X}{X} + \frac{bW - g'Y}{Y} + \frac{cW - h'Z}{Z} = 0,$$

or $$\frac{aW}{X} + \frac{bW}{Y} + \frac{cW}{Z} = f' + g' + h' = -d,$$

or
$$\frac{a}{X}+\frac{b}{Y}+\frac{c}{Z}+\frac{d}{W}=0,$$
which is by (6) the condition that E lies on the Cayley Locus of $ABCD$. Mantel's result is confirmed.

5. The Cayley Locus of the tetrahedron $ABCD$ is a cubic surface having double points at the vertices A, B, C, D, and passing through the six edges $AB, \ldots$; E is a point on the surface. The Cayley Locus of $ABCE$ has double points at A, B, C, E, and passes through D. The intersection of the two surfaces is therefore of order nine, having double points at D and E, and quadruple points at A, B, C. But the lines BC, CA, AB are common to the two surfaces and can be discarded. What is left is of order six, and has double points at A, B, C, D, E. [Moreover, since BC, CA, AB form the complete intersection of the surfaces with a plane, the residual intersection lies on a quadric.] But a proper sextic in space cannot have five double points, for its projection from one of them on to a plane is a plane quartic with four double points, which must be a composite curve. The sextic with five double points must be made up of two twisted cubics, each passing through A, B, C, D, E.

It is not clear to me at this stage how it can be seen that the Cayley Loci of $ABDE, ACDE, BCDE$ also pass through these two curves, though a proof by analysis as indicated at the end of this section is not difficult. If, however, any point F is taken on one of the cubics, then since the Cayley Locus of $ABCD$ goes through E and F, and the Cayley Locus of $ABCE$ through D and F, it follows that the Cayley Locus of $ABCF$ goes through D and E. Now the Cayley Loci of $ABCE$ and $ABCF$ intersect in two cubic curves, both of which pass through A, B, C, E, F and one of which passes through D. The latter is the cubic on which we chose F, since no other cubic can pass through six points A, B, C, D, E, F. We have in fact discovered a twisted cubic through ABC, such that if any point H is taken on the curve the feet of the perpendiculars from any point K of the curve on the faces of $ABCH$ are coplanar; for the Cayley Locus of $ABCH$ contains the entire curve.

[For an algebraic proof that the composite sextic curve common to the Cayley Loci of $ABCD$ and $ABCE$ lies also on the Cayley Loci of $ABDE, ACDE, BCDE$, we may obtain the equation of the quadric on which the curve lies, and satisfy ourselves that it is symmetrical in the coordinates and constants; for the curve is the complete intersection of this quadric with the Cayley Locus of $ABCD$
$$a/x+b/y+c/z+d/w=0 \qquad \ldots\ldots(8).$$
We have to combine (7), (8) and (6) to obtain a quadric equation, symmetrical if possible in the small and great letters. There is no

particular difficulty; we everywhere replace $dxyz$ from (8) and $dXYZ$ from (6) by expressions divisible by w and W. When this has been done, w and W divide out and we find for the quadric

$$\begin{aligned}
&- a\,[Ww\,(Yz + Zy) + Yy\,(Zw + Wz) + Zz\,(Wy + Yw)] \\
&- b\ [\text{similar}] - c\,[\text{similar}] \\
&- d\,[Xx\,(Yz + Zy) + Yy\,(Zx + Xz) + Zz\,(Xy + Yx)] \\
&+ (f + f')\,(Yz + Zy)\,(Wx + Xw) \\
&+ (g + g')\,(Zx + Xz)\,(Wy + Yw) \\
&+ (h + h')\,(Xy + Yx)\,(Wz + Zw)
\end{aligned}$$

which is quite symmetrical.]

6. In particular H may move to an infinitely distant point P on the cubic curve, so that the tetrahedron $ABCP$ becomes an oblique prism on the triangular base ABC, the edges AP, BP, CP being parallel. The feet of the perpendiculars from any point Q on the faces PBC, PCA, PAB lie in a plane through Q normal to AP, BP, CP; the perpendicular from Q on ABC is inclined to this plane. Thus the four feet are coplanar only (i) when Q lies in ABC, or (ii) when the feet of the three former perpendiculars are collinear, which is the case if Q lies on the circular cylinder through AP, BP, CP. The points A, B, C, D, E and the whole cubic curve lie on a right circular cylinder whose axis is directed towards P. Further, we can now argue that D lies on the Cayley Locus of $ABEP$ and therefore the Cayley Locus of $ABDE$ must pass through P; it thus has the four double points A, B, D, E, and two simple points C and P situated on our twisted cubic—ten points common to a cubic surface and a cubic curve. The Cayley Locus of $ABDE$ contains the whole curve, as also do those of $ACDE$, and $BCDE$.

The circular cylinder with axis directed to P is the limit of a cone standing on the curve whose vertex moves off to infinity along the curve: the line joining P to any point of the curve is a generating line of the cylinder. Now the curve passes to infinity in two other directions Q and R, and the lines PQ, PR, lying wholly at infinity must lie on the cylinder. But a right circular cylinder with axis directed to P cuts the plane at infinity in two lines through P which touch the imaginary circle at infinity. In order that the curve should lie on such a right circular cylinder it is necessary and sufficient that PQ and PR should touch the Absolute. And the same reasoning can be applied to the cylinders through the curve whose axes are directed to Q and R, and shows that QR must also touch the Absolute.

A real twisted cubic passes to infinity either to three real points P, Q, R, or to one real point P and two conjugate imaginary points

Q, R. In either case the line QR is real and cannot touch the Absolute. The Cayley Loci of $ABCD$ and $ABCE$ in § 5 have no real intersection except the isolated points A, B, C, D, E. In spite of this, we have been led to certain cubic curves in space which have properties resembling those enunciated in § 1, and have obtained some necessary conditions. It will be best now to investigate *ab initio* under what conditions a twisted cubic, or more generally a curve of order n in Euclidean space of n dimensions, has the properties enunciated in § 1.

7. In Euclidean space of n dimensions a curve of order n (necessarily normal and rational) meets the plane at infinity in n points, $X_1, X_2, X_3, \dots X_n$. Take in addition any other point of the curve X_0, and we have a basis for a system of homogeneous-coordinates $x_0, x_1, \dots x_n$, the plane at infinity being $x_0 = 0$.

The coordinates of points on a curve of order n are rational functions of a parameter θ. If the values of θ at the points $X_1, X_2, \dots X_n$ are $\alpha_1, \alpha_2, \dots \alpha_n$, if θ is infinite at X_0 the curve is defined by a system of n equations

$$x_r/x_0 = A_r/(\theta - \alpha_r) \dots \quad (r = 1, 2, 3, \dots n) \quad \dots\dots(9).$$

The equation of a 'plane' will be written $\Sigma(u_r x_r) = 0$, and the equation of the 'plane' through n points of the curve whose parameters are known, say $\lambda_1, \lambda_2, \dots \lambda_n$, is easily found. For if $\Sigma(u_r x_r) = 0$ is its equation, we have by (9)

$$u_0 + \sum_{r=1}^{n} \frac{u_r A_r}{\theta - \alpha_r} \equiv u_0 \frac{(\theta - \lambda_1)(\theta - \lambda_2)\dots(\theta - \lambda_n)}{(\theta - \alpha_1)(\theta - \alpha_2)\dots(\theta - \alpha_n)},$$

for the left-hand member is to vanish when $\theta = \lambda_1, \lambda_2, \dots \lambda_n$, and the factor u_0 on the right is correct if $\theta = \infty$. Multiply by $\theta - \alpha_r$, and let θ take the value α_r, then

$$u_r A_r = u_0 \frac{(\alpha_r - \lambda_1)(\alpha_r - \lambda_2)\dots(\alpha_r - \lambda_n)}{(\alpha_r - \alpha_1)(\alpha_r - \alpha_2)\dots(\alpha_r - \alpha_n)}.$$

If a set of $n + 1$ points were taken on the curve, having parameters $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_{n+1}$, and $n + 1$ planes were made to pass through all but one, the values of u_r in the different planes would (on the supposition that u_0 is unity in each) be inversely proportional to $\alpha_r - \lambda_s$, λ_s being the parameter of the omitted point.

8. The space we are considering being Euclidean, two planes are perpendicular if they are conjugate with respect to the Absolute, a quadric lying in the plane at infinity $x_0 = 0$. The tangential equation of the Absolute will be taken as

$$\Omega(u_1 u_2 \dots) \equiv \Omega \equiv c_{11} u_1^2 + 2c_{12} u_1 u_2 + \dots = 0.$$

In other words two planes

$$\sum_{0}^{n} (u_r x_r) = 0, \quad \sum_{0}^{n} (v_r x_r) = 0,$$

are perpendicular if a relation holds of the form

$$c_{11} u_1 v_1 + c_{12} (u_1 v_2 + v_1 u_2) + \dots = 0.$$

The condition that the second plane should be perpendicular to the first is that its equation should be satisfied by

$$x_0 = 0, \quad x_1 = (c_{11} u_1 + c_{12} u_2 + \dots + c_{1r} u_r \dots), \quad x_2 = (c_{12} u_1 + c_{22} u_2 + \dots),$$
$$\dots, \quad x_n = (c_{1n} u_1 + c_{2n} u_2 + \dots).$$

If now a line passing through a point $(z_0, z_1, z_2, \dots)$ is perpendicular to the plane $\Sigma (ux) = 0$, the coordinates of any point on the line can be expressed as

$$x_0 = z_0, \quad x_1 = z_1 + \mu (c_{11} u_1 + c_{12} u_2 + \dots + c_{1r} u_r \dots),$$
$$x_s = z_s + \mu \left(\sum_{r=1}^{n} c_{rs} u_r \right) \dots.$$

The coordinates of the foot of the perpendicular are found by giving μ such a value as will make the point lie in the plane $\Sigma (ux) = 0$; in fact μ is given by

$$0 = \Sigma (uz) + \mu \Omega (u_1 u_2 \dots).$$

The foot of the perpendicular lies in a given plane

$$U_0 x_0 + U_1 x_1 + \dots = 0,$$

if $0 = \Sigma (Uz) + \mu \{ U_1 (c_{11} u_1 + c_{12} u_2 \dots) + \dots + U_s \sum_{1}^{n} (c_{rs} u_r) + \dots \}$.

Elimination of μ gives the condition that the foot of the perpendicular from a point (z) upon the plane $\Sigma (ux) = 0$ should lie in a plane $\Sigma (Ux) = 0$. If we are not concerned with the plane in which the feet lie we may write the condition

$$k\Omega (u_1 u_2 \dots) + (u_0 z_0 + u_1 z_1 + \dots + u_n z_n)(u_1 Z_1 + u_2 Z_2 + \dots + u_n Z_n) \quad \dots\dots(10).$$

9. The equation of the Cayley Locus of $n + 1$ given planes is now easily found. We have to eliminate $k, z_1, z_2, \dots z_n$ from $n + 1$ equations derived from (10) by giving $u_0, u_1, \dots u_n$ their values appropriate to the $n + 1$ given planes. The equation is the vanishing of a determinant of $n + 1$ rows and columns, of which it is sufficient to quote one row—for all are precisely similar:

$$\left. \frac{\Omega (u_1 u_2 \dots u_n)}{u_0 z_0 + u_1 z_1 + \dots + u_n z_n}, \; u_1, \; u_2, \; u_3, \; \dots, \; u_n \; \right| .$$

In the problem we are discussing, the planes pass through n out of $n+1$ points, $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ taken on the curve (9). For these planes, it was pointed out at the end of § 7, the entries read downwards in the column of values of u_r are

$$\frac{1}{\alpha_r - \lambda_1}, \frac{1}{\alpha_r - \lambda_2}, \frac{1}{\alpha_r - \lambda_3}, \ldots, \frac{1}{\alpha_r - \lambda_{n+1}}$$

(a common factor may be removed from all the entries in a column).

As regards the first column, we want z to be any point on the curve (9), and we saw in the course of § 7 that the denominators reduce to $\Pi\,(\theta - \lambda) \div \Pi\,(\theta - \alpha)$, all the α's and all but one of the λ's being included. The factors $(\theta - \alpha)$ disappear, being common to all the entries in the first column, the n factors $\theta - \lambda$ are equivalent to a single factor $\theta - \lambda$ (the missing one) in the numerator. In fact the determinant is so simplified that the following is a typical row

$$|(\theta - \lambda_r)\,\Omega. \quad (\alpha_1 - \lambda_r)^{-1}, \quad (\alpha_2 - \lambda_r)^{-1}, \quad (\alpha_3 - \lambda_r)^{-1}, \ldots, (\alpha_n - \lambda_r)^{-1}|.$$

It remains to consider Ω. It includes two kinds of terms—squared terms such as $c_{11}u_1^2$, and product terms such as $2c_{12}u_1u_2$; and we may consider each term separately. We are not concerned with the values of the terms but with the question whether or no they vanish. For a product term u_1u_2, the values in the successive columns, read downwards, are proportional to

$$(\alpha_1 - \lambda_r)^{-1}, \quad (\alpha_2 - \lambda_r)^{-1}, \quad (r = 1, 2, 3, \ldots, n+1),$$

and the successive entries in the first column are

$$2c_{12}\frac{\theta - \lambda_r}{(\alpha_1 - \lambda_r)(\alpha_2 - \lambda_r)} \quad \text{or} \quad 2c_{12}\frac{\theta - \alpha_1}{\alpha_2 - \alpha_1}\frac{1}{\alpha_1 - \lambda_r} + 2c_{12}\frac{\theta - \alpha_2}{\alpha_1 - \alpha_2}\frac{1}{\alpha_2 - \lambda_r}.$$

Each entry in the first column is a sum of fixed multiples of entries in the next two columns. The terms multiplied by c_{12} (and all other product terms) therefore vanish. The squared terms $c_{11}u_1^2$, etc. clearly do not vanish. Hence the curve has the property required if, and only if, the Absolute contains no terms in $u_1^2, u_2^2, \ldots u_n^2$, i.e. is touched by the planes $x_1 = 0$, $x_2 = 0$, … $x_n = 0$.

Finally we have the theorem:

In Euclidean space of n dimensions, a curve of order n has the property that the feet of the perpendiculars from any point of the curve upon the $n+1$ faces of any $(n+1)$-hedron whose vertices lie on the curve are coplanar if, and only if, the n points in which the curve meets the plane at infinity form an n-hedron in that plane whose n faces touch the Absolute.

On the real twisted cubic curves which are geodesic upon a quadric surface. By Mr H. W. Richmond, King's College.

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1. Geodesic curves, even of simple surfaces such as quadrics*, are almost always transcendental: occasionally they are algebraic: still more rarely do they belong to simple and familiar types. The form of the differential equation referred to proves conclusively that even such integrals as are expressible by elementary functions must be quite exceptional.

For a paraboloid $x^2/a + y^2/b = 2z$ ....(1),
confocal coordinates can still be used, and the differential equation is simplified by the omission of the factors $c+p$, $c+q$ in the denominators. Should the geodesic also pass through an umbilic, the constant of integration θ is equal to a or b. It will be convenient to use the name *umbilical geodesic* on all occasions when $\theta = a$ or b, even when the quadric is ruled and has no real umbilics, and to state that

The differential equation of the umbilical geodesics of a paraboloid can be integrated in terms of simple functions.

The differential equation actually is (if $\theta = a$)

$$\frac{p^{\frac{1}{2}}\,dp}{(a+p)\,(b+p)^{\frac{1}{2}}} = \frac{q^{\frac{1}{2}}\,dq}{(a+q)\,(b+q)^{\frac{1}{2}}},$$

and its integral is found, if we substitute

$$(b+p)/p = u^2, \quad (b+q)/q = v^2,$$

to be
$$\left(\frac{1-u}{1+u}\right)^h \left(\frac{1-v}{1+v}\right)^h = C\left(\frac{h-u}{h+u}\right)\left(\frac{h-v}{h+v}\right),$$

where $h^2 = (a-b)/a$, and C is a constant. Hence we may make the further statement:

The umbilical geodesics of the paraboloid (1) *are algebraic curves, provided that $(a-b)/a$ is the square of a rational number.*

By considering how the simplest results of this theorem could be reached, taking $C = 1$ and $h = 2$ or $\frac{1}{2}$, I noticed several years ago that:

(1) On
$$x^2/4 + y^2/3 = 2cz,$$
$$3x^2 \pm 32\sqrt{3}xy = 96c^2$$

gives a geodesic, a rational curve of order four.

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* See Salmon, *Analytical Geometry of Three Dimensions*, 1912 edition, vol. I, §§ 397–420 for a general account of geodesics on quadrics, and specially the footnote to § 412 for the differential equation in terms of confocal coordinates.

(2) On
$$x^2 - y^2/3 = 2cz,$$
$$6x^2 \pm 2\sqrt{3}xy = 3c^2$$
gives a geodesic, a twisted cubic curve.

I propose now to consider under what conditions a twisted cubic can be geodesic upon a quadric. Imaginary curves are passed over, among them curves of zero length, which are geodesic on all surfaces if judged by the criterion of the differential equation. Real curves and surfaces alone are discussed.

2. When two points X and W are taken on a given cubic curve, a quadric surface is determined by the conditions that it shall contain the cubic curve and the tangents at X and W. With a tetrahedron of reference $XYZW$ (XY, WZ being tangents, and XYZ, WZY osculating planes at X and W), the equations of the curve and surface are respectively

$$X : Y : Z : W :: \theta^3 : \theta^2 : \theta : 1 \qquad \ldots\ldots(2),$$
$$XW = YZ \qquad \ldots\ldots(3).$$

On the other hand, every cubic that lies on a quadric touches two generators of one system. Equations (2) and (3) include all cases of a cubic curve which lies on a proper quadric (not a cone). For reality, X and W may be either two real or two conjugate imaginary points: in the latter case a new notation will be desirable.

3. The osculating plane of (2) and the tangent plane of (3) at the point θ are

$$X - 3\theta Y + 3\theta^2 Z - \theta^3 W = 0 \qquad \ldots\ldots(4),$$
$$X - \theta Y - \theta^2 Z + \theta^3 W = 0 \qquad \ldots\ldots(5).$$

For (2) to be geodesic on (3), the planes (4) and (5) must be perpendicular for all values of θ, i.e. must be conjugate with respect to a certain locus of class two, whose equation is denoted by $\Omega = 0$, called the Absolute. But the loci of class two with respect to which (4) and (5) are conjugate are easy to find. For

$$(a, b, c, d, f, g, h, f', g', h' \between L, M, N, P)^2 = 0,$$

includes all loci of class two; and (4) and (5) are conjugate if

$$a + 3b\theta^2 - 3c\theta^4 - d\theta^6 + 2g\theta^2 - 4h\theta - 2g'\theta^4 + 4h'\theta^5 = 0;$$

which holds for all values of θ if and only if

$$a = 0, \quad h = 0, \quad h' = 0, \quad d = 0, \quad 2g + 3b = 0, \quad 2g' + 3c = 0,$$

f and f' being subject to no conditions. The planes (4) and (5) are conjugate with respect to all surfaces of class two included in

$$b(M^2 - 3LN) + c(N^2 - 3MR) + 2fMN + 2f'LR = 0,$$

and if any one of these is the Absolute, $\Omega = 0$, (2) is geodesic on (3). With a change of notation, (2) is geodesic on (3) if for any values of the constants $\alpha, \beta, \gamma, \delta$

$$\Omega + \alpha LR \equiv \beta (M^2 - 3LN) + \gamma (N^2 - 3MR) + \delta (MN - 9LR).$$

The right-hand member equated to zero gives one of the surfaces of class two touched by all the osculating planes of (2); the left-hand gives a surface of resolution with foci X, W. Hence

In order that (2) *should be geodesic on* (3) *it is necessary and sufficient that all the osculating planes of* (2) *should touch a quadric of revolution with foci X and W.*

From the form of Ω, viz.

$$\Omega = \beta (M^2 - 3LN) + \gamma (N^2 - 3MR) + \delta (MN - 9LR) - \alpha LR \qquad \ldots\ldots(6),$$

in which the terms in L^2, LM, NR, R^2 are lacking, we see that XYZ, WZY, the osculating planes at X and W, touch Ω, the points of contact lying respectively on XY and WZ, the tangents to the cubic curve at X and W. These points, lines and planes must be imaginary. We see further that the osculating planes of (2)—six in number—which touch Ω coincide three by three in the planes XYZ and WZY.

4. We are considering ordinary space, and must express the fact that the discriminant of Ω vanishes: Ω is degenerate, and represents a conic (the imaginary circle) in a plane (the plane at infinity). Supposing the equation of the plane to be

$$AX + BY + CZ + DW = 0,$$

the differentials of Ω with respect to L, M, N, R must vanish when L, M, N, R are replaced by A, B, C, D: hence

$$3\beta C + (9\delta + \alpha) D = 0, \quad 3\gamma B + (9\delta + \alpha) A = 0,$$
$$2\beta B - 3\gamma D + \delta C = 0, \quad 2\gamma C - 3\beta A + \delta B = 0,$$

and, eliminating $\alpha, \beta, \gamma, \delta$

$$AC^3 = DB^3.$$

We may put $\qquad A = kB^3, \quad D = kC^3,$

$$\beta = kBC^3, \quad \gamma = kCB^3, \quad 9\delta + \alpha = -3BC,$$
$$\delta = 3k^2B^3C^3 - 2kB^2C^2.$$

But these can be simplified without loss of generality. For equations (2), (3), (4), (5) are unchanged if we replace θ, X, Y, Z by $n\theta$, n^3X, n^2Y, nZ· and thus by choosing $n = C/B$, we can make the coefficients B and C in the plane at infinity equal: in fact we may put $B = C = 1$. Thus finally we have the following equations, applicable to any cubic curve which is geodesic on a quadric:

Cubic curve $X : Y : Z : W :: \theta^3 : \theta^2 : \theta : 1$ ......(7).

Quadric surface $XW = YZ$ ......(8).

Plane at infinity $kX + Y + Z + kW = 0$ ......(9).

Absolute Conic, or imaginary circle at infinity

$$\Omega \equiv k(M^2 - 3LN + N^2 - 3MR) + (3k^2 - 2k)MN + 3LR$$
$$\equiv k(M - N)^2 + 3(kM - L)(kN - R) \qquad \text{......(10).}$$

We infer that, in addition to the paraboloid quoted in § 1, there is a simple infinity of instances in which a cubic is geodesic on a quadric (since k can have any value), and these equations enable us to discover their properties. We find that pairs of points of the curve of parameters θ and $1/\theta$ have important properties. If the former (P) has coordinates (X, Y, Z, W), the latter (P') has (W, Z, Y, X), and any point on the chord PP' has coordinates

$$X + rW, \quad Y + rZ, \quad Z + rY, \quad W + rX,$$

or

$$\theta^3 + r, \quad \theta^2 + r\theta, \quad \theta + r\theta^2, \quad 1 + r\theta^3.$$

When $r = 1$, we have a point lying on $X = W$, $Y = Z$; when $r = -1$, we have a point at infinity. Hence *all chords PP' are met and bisected by the line $X = W$, $Y = Z$.* The points at infinity on the chords lie on the line $X + W = 0$, $Y + Z = 0$; *hence all chords PP' are parallel to a certain plane*, or to the family of parallel planes $X + W + s(Y + Z) = 0$. The chords PP' generate a quadric (necessarily a paraboloid) whose equation is

$$XZ - Y^2 = YW - Z^2.$$

Lastly, the form of Ω provides the condition of perpendicularity of two planes, and on investigation shows that the parallel planes are perpendicular to any plane $X - W + t(Y - Z) = 0$, i.e. are perpendicular to the line $X = W$, $Y = Z$. *The chords PP' of the cubic are bisected perpendicularly by the line $X = W$, $Y = Z$*: i.e. the curve has this line as an axis of symmetry, meeting it perpendicularly at the point where $\theta = 1$.

The cubic curve meets the plane at infinity in three points

$$k(\theta^3 + 1) + (\theta^2 + \theta) = 0,$$
$$\theta = -1, \text{ or } k\theta^2 - k\theta + k + \theta = 0 \qquad \text{......(11).}$$

The two planes which pass through the axis of symmetry and touch Ω are

$$FX + GY - GZ - FW = 0,$$

where

$$4kG^2 = 3(kG - F)^2 \qquad \text{......(12).}$$

5. It is now not difficult to express these results first in a real system of coordinates, and then in a system of ordinary Cartesian coordinates. For the first step, since X and W are conjugate imaginaries, and also Y and Z, write

$$X : Y : Z : W :: \xi + i\omega : \eta + i\zeta : \eta - i\zeta : \xi - i\omega.$$

Also replace θ by $(1 + i\phi)/(1 - i\phi)$, so that to the values 0, ∞, 1, -1 of θ correspond values i, $-i$, 0, ∞ of ϕ. Equations (7), (8), (9), (11), (12) are transformed into

Cubic curve

$$\xi : \eta : \zeta : \omega :: 1 - 3\phi^2 : 1 + \phi^2 : \phi + \phi^3 : 3\phi - \phi^3 \qquad \text{......(7}a\text{)}.$$

Quadric surface $$\xi^2 + \omega^2 = \eta^2 + \zeta^2 \qquad \text{......(8}a\text{)}.$$

Plane at infinity $$k\xi + \eta = 0 \qquad \text{......(9}a\text{)}.$$

Real values of ξ, η, ζ, ω, ϕ now give us real points. The axis of symmetry is $\zeta = 0$, $\omega = 0$, and the chords which it bisects at right angles join points whose parameters are $+\phi$ and $-\phi$. These chords generate a paraboloid

$$2\eta\zeta = \omega\eta - \xi\zeta.$$

The curve meets the plane at infinity when $\phi = \infty$ or

$$k + 1 = (3k - 1)\,\phi^2 \qquad \text{......(11}a\text{)},$$

three real points unless k lie between 1/3 and -1.

The two planes which pass through the axis of symmetry and touch Ω, $F\omega + G\zeta = 0$, are again given by

$$4kG^2 = 3\,(kG - F)^2 \qquad \text{......(12}a\text{)}.$$

Since these must be imaginary, we learn that the results obtained are not capable of interpretation in ordinary space unless k is negative.

To transform (10) into an equation in λ, μ, ν, ρ, where

$$\lambda\xi + \mu\eta + \nu\zeta + \rho\omega$$

is the equivalent of

$$LX + MY + NZ + RW,$$

we see that

$$L + R = \lambda, \quad M + N = \mu, \quad iM - iN = \nu, \quad iL - iR = \rho,$$

$$L : M : N : R :: \lambda - i\rho : \mu - i\nu : \mu + i\nu : \lambda + i\rho,$$

$$\Omega \equiv -4k\nu^2 + 3\,(k\mu - \lambda)^2 + 3\,(k\nu - \rho)^2 \quad \text{......(10}a\text{)},$$

which again shows that k must not be positive, since Ω must not be touched by any real plane.

The four planes $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\omega = 0$, have been regarded as forming a tetrahedron of reference (whose vertices may be called ξ, η, ζ, ω). The planes $\zeta = 0$, $\omega = 0$ intersect in the line $\xi\eta$, the axis of symmetry of the cubic curve: they are not perpendicular to one another. The planes $\xi = 0$, $\eta = 0$ are both perpendicular to the axis $\xi\eta$, i.e. are parallel to one another, so that the vertices ζ and ω are at infinity, lying on the line common to all planes perpendicular to $\xi\eta$.

If we take an origin O at a convenient point of $\xi\eta$, and take $\xi\eta$ as axis of z, and any pair of perpendicular planes through $\xi\eta$ as coordinate planes zOx, zOy, we have a frame of rectangular Cartesian axes. Now two planes through $\xi\eta$, $\nu_1\zeta + \rho_1\omega = 0$, $\nu_2\zeta + \rho_2\omega = 0$ are perpendicular if

$$-4k\nu_1\nu_2 + 3(k\nu_1 - \rho_1)(k\nu_2 - \rho_2) = 0,$$

and the simplest solution appears to be $\nu_1 = 0$, $k\nu_2 - \rho_2 = 0$; the planes $\omega = 0$, and $\zeta + k\omega = 0$ are perpendicular. The actual perpendiculars from a point $(\xi', \eta', \zeta', \omega')$ upon various planes

$$\lambda\xi + \mu\eta + \nu\zeta + \rho\omega = 0$$

are proportional to

$$(\lambda\xi' + \mu\eta' + \nu\zeta' + \rho\omega')/\{-4k\nu^2 + 3(k\mu - \lambda)^2 + 3(k\nu - \rho)^2\}^{\frac{1}{2}},$$

so that if
$$\omega = 0 \text{ is the plane } yOz$$
$$\zeta + k\omega = 0 \text{ is the plane } xOz,$$

we have
$$x = h\omega/\sqrt{3}, \quad y = h(\zeta + k\omega)/\sqrt{(-4k)},$$

and, if xOy is the same plane as $\xi = 0$,

$$z = h\xi/\sqrt{3},$$

the constant h defining the scale of the figure.

To avoid the surds, write

$$\sqrt{(-k)} = p\sqrt{3},$$

or
$$k = -3p^2,$$

and replace h by another constant a: then the Cartesian coordinates x, y, z are such that

$$x : y : z : a :: \omega : (\zeta - 3p^2\omega)/(2p) : \xi : (\eta - 3p^2\xi)/(2p),$$

where
$$\xi : \eta : \zeta : \omega :: 1 - 3\phi^2 : 1 + \phi^2 : \phi + \phi^3 : 3\phi - \phi^3.$$

In this if we write $\phi = \tan\psi$ we see

$$\xi : \eta : \zeta : \omega :: \cos 3\psi : \cos\psi : \sin\psi : \sin 3\psi,$$

and so obtain the theorem.

If a twisted cubic curve is geodesic on a quadric surface, the equations of the curve can be written

$$\frac{x}{2p\sin 3\psi} = \frac{y}{\sin\psi - 3p^2\sin 3\psi} = \frac{z}{2p\cos 3\psi} = \frac{a}{\cos\psi - 3p^2\cos 3\psi},$$

and that of the quadric surface on which it is a geodesic

$$x^2 + z^2 = p^2(2y + 3px)^2 + p^2(2a + 3pz)^2,$$

where p may have any value.

The quadric is a hyperboloid of one sheet unless $3p^2 = 1$, when it is a hyperbolic paraboloid (the same as that noticed at the end of § 1). Oz is a principal axis and is met by the surface in two real points. Through one of these the cubic passes, cutting the axis of z perpendicularly. The planes $y =$ constant are planes of circular section. The cubic curve always has Oz for an axis of symmetry: it always passes to infinity at one place where $\psi = \pi/2$, and z is finite; it may or may not pass to infinity in two other real directions, according as $3p^2 \gtrless 1$[*].

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[*] I believe it to be the fact that a quadric cone cannot have on it a real twisted cubic which is geodesic, and a proper quadric cannot have on it a quartic curve of genus one (i.e. elliptic) which is geodesic.

On the Number of Lines which meet Four Regions in Hyper-Space.* By R. VAIDYANATHASWAMY. (Communicated by Mr H. W. TURNBULL, Trinity College.)

[*Received* 26 October 1923.]

The problem of finding the number of r-dimensional regions which are situated in a space S_n of n dimensions, and which satisfy a suitable number of conditions of certain assigned types (called "ground-conditions") has been investigated by Schubert†. A special class of such problems arises when the r-dimensional region is merely required to intersect k regions P_λ of r_λ dimensions $(\lambda = 1, 2 \dots k)$ situated in general position in S_n, where for the finiteness of the sought number we must have

$$r_1 + r_2 + \dots + r_k = (k - r - 1)(n - r).$$

These latter may be called "Transversal-Problems" and the required number of r-dimensional regions may be denoted by

$$[n, (r_1, r_2 \dots r_k), r].$$

The only Transversal Line-problems (that is, transversal-problems in which $r = 1$) which have been solved for general values of n are those in which

$$(1)\quad r_1 = r_2 = \dots = r_k = n - 2; \quad k = 2n - 2,$$

$$(2)\quad r_1 = r_2 = \dots = r_{k-1} = n - 2; \quad r_k = k - n,$$

$$(3)\quad r_1 = r_2 = \dots = r_k = n - 3; \quad k = n - 1.$$

It is the object of this communication to show that a simple and elegant result can be stated for the case $k = 4$.

The symbolism which I have used is in some ways more convenient for transversal-problems than the symbolism of Schubert. The proofs are based upon the Principle of Conservation of Number‡ ("Prinzip der Erhaltung der Anzahl") which is equally the foundation of Schubert's methods. The validity of this principle for the problem of finding the number of lines satisfying assigned ground-conditions has been established in a purely algebraic manner by Max Caspar§.

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* *Region* is a convenient term for "*flat subspace*" of a flat space of n dimensions. It is used by Whitehead, *Universal Algebra.*

† *Math. Annalen*, Bd. 26, 38.

‡ For an account of this principle see Schubert, *Kalkul der Abzahlenden Geometrie*, or preferably Zeuthen, "Abzahlenden Methoden" (*Ency. Math. Wiss.*).

§ *Math. Annalen*, Bd. 59.

I. There are $\infty^{n-1+n-1}$ lines in S_n and of these, ∞^{n-1+r} meet an arbitrary r-dimensional region. Thus a line in S_n which meets a region of r dimensions satisfies thereby $n-1-r$ conditions. Hence for the finiteness of $[n, (r_1 r_2 r_3 r_4), 1]$—which represents the number of lines meeting four base regions P_λ of r_λ dimensions ($\lambda = 1, 2, 3, 4$)—we must have

$$r_1 + r_2 + r_3 + r_4 = 2n - 2 \qquad \text{......(1)}.$$

To evaluate $[n, (r_1 r_2 r_3 r_4), 1]$ when this equality is satisfied, consider first the special case in which $r_1 + r_2 = r_3 + r_4 = n - 1$. Let us examine what happens to the transversals in this case when we make P_1 and P_2 intersect in a point p and P_3 and P_4 in a point q. From the Principle of Conservation of Number the number of transversals will either remain unaltered after such intersection or will become infinite. Now since $r_1 + r_2 = n - 1$ and P_1, P_2 intersect in a point, it follows that they are contained in an S_{n-1}; similarly P_3, P_4 are contained in an S'_{n-1}. Hence all transversals of the four regions other than the line pq must be contained in both S_{n-1} and S'_{n-1} and therefore in the S_{n-2} which is their intersection; in fact they will be identical with the transversals of the four regions P_λ' of $r_\lambda - 1$ dimensions in which this S_{n-2} cuts the regions P_λ ($\lambda = 1, 2, 3, 4$). Since the regions P_λ' are in general position equally with the regions P_λ and since

$$\Sigma (r_\lambda - 1) = 2(n-2) - 2,$$

it follows from (1) that the number of transversals do not become infinite. Hence the particular mode of intersection of the base regions which we have chosen has no effect on the number of transversals. Consequently

$$[n, (r_1 r_2 r_3 r_4), 1] = 1 + [n-2, (r_1 - 1, r_2 - 1, r_3 - 1, r_4 - 1), 1].$$

Since the relation (1) holds also for the term on the right, this process of reduction could be repeated, giving finally

$$[n, (r_1 r_2 r_3 r_4), 1] = r_1 + [n - 2r_1, (0, r_2 - r_1, r_3 - r_1, r_4 - r_1), 1],$$

on the supposition that r_1 is the least of the numbers r. To evaluate the term on the right, we notice that $r_2 - r_1$ being equal to $n - 2r_1 - 1$ might be cancelled out as it imposes no effective restriction on the transversal line. Thus the term represents the number of lines in S_{n-2r_1} which pass through a point and meet two regions of dimensions $r_3 - r_1$, $r_4 - r_1$ respectively. Since

$$r_3 - r_1 + r_4 - r_1 = n - 2r_1 - 1,$$

this number is unity and we have finally the result that if r_1 be the least of the numbers r,

$$[n, (r_1, r_2, r_3, r_4), 1] = r_1 + 1,$$

when $$r_1 + r_2 = r_3 + r_4 = n - 1.$$

II. Consider now the general case in which r_1, r_2, r_3, r_4 are any four numbers subject to the relation (1). Supposing that they are in ascending order of magnitude, we may write

$$r_1 + r_2 = n - 1 - \lambda,\ r_3 + r_4 = n - 1 + \lambda;\ \ \lambda = \tfrac{1}{2}(r_3 + r_4 - r_1 - r_2).$$

Since $r_1 + r_2 = n - 1 - \lambda$, P_1 and P_2 are contained in an $S_{n-\lambda}$ which must therefore contain all the possible transversals. Since this $S_{n-\lambda}$ cuts the other two base regions in regions of dimensions $r_3 - \lambda$, $r_4 - \lambda$, we have immediately

$$[n, (r_1 r_2 r_3 r_4), 1] = [n - \lambda, (r_1, r_2, r_3 - \lambda, r_4 - \lambda), 1].$$

But since $r_1 + r_2 = n - \lambda - 1$ the right-hand term is of the type already evaluated and is equal to the smaller of the numbers $r_1 + 1$, $r_3 - \lambda + 1 = n - r_4$. For the general case we have therefore finally the result:

$$[n, (r_1 r_2 r_3 r_4), 1] = \text{smaller of } r_1 + 1, n - r_4 \quad (r_1 < r_2 < r_3 < r_4).$$

If we define the *point-wise* and *plane-wise* orders of a region in S_n to be respectively the number of independent points and the number of independent S_{n-1}'s necessary to determine the region, then the point-wise and plane-wise orders of a region of r dimensions would be $r + 1$ and $n - r$ respectively. The theorem obtained may then be stated as follows:

The number of lines meeting four regions in S_n—if finite—is equal to the least of the orders point-wise and plane-wise of the four regions.

III. If r_4 is put equal to $n - 1$ (so that the plane-wise order of P_4 is unity) then P_4 may be cancelled from the list of base regions as it imposes no effective restriction on the transversal line. The theorem then reduces to

$$[n, (r_1, r_2, r_3), 1] = 1;\ r_1 + r_2 + r_3 = n - 1 \quad \ldots\ldots(2),$$

a result which may also be obtained directly. If we suppose that P_1, P_2, P_3 are osculating regions of a rational normal curve in S_n, then, by using the binary coordinate-system, (2) could be transformed into the following theorem relating to Co-Jacobian pencils of binary forms:

One and only one non-singular pencil of binary n-ics is determined by a Jacobian $(2n - 2)$-ic with three distinct linear factors—the multiplicity of no one of these factors being greater than $n - 1$.

IV. Putting $n = 5$, $r_1 = r_2 = r_3 = r_4 = 2$ in the theorem, we have the result:

There are precisely three lines which meet four planes in space of five dimensions.

This result is of particular interest as it can be verified as follows by Line-geometry.

Represent linear complexes in S_3 by points in S_5. The singular complexes, or—what is the same thing—the lines of S_3 will then correspond to points on a quadric Γ in S_5. A plane in S_5 intersects Γ in a conic which corresponds to one regulus of a quadric in S_3; a pair of mutually polar planes P_1, P_2 of Γ intersect it in a pair of conics which correspond to the two complementary reguli of a quadric Q in S_3. If q_1q_2 be a chord of Γ which intersects P_1 and P_2, then q_1, q_2 correspond to mutually polar lines of the quadric Q in S_3. Hence if P_3, P_4 be another pair of polar planes of Γ corresponding to the quadric Q' in S_3, there will be as many common transversals of P_1, P_2, P_3, P_4 as there are pairs of common polar lines of Q and Q' in S_3. But the only common polar line-pairs of two quadrics in S_3 are the pairs of opposite edges of their common self-polar tetrahedron. Thus P_1, P_2, P_3, P_4 have exactly three transversals. The same is true of *any* four regions, for since a quadric in S_5 involves 20 constants, any two pairs of planes are always polar pairs with respect to a quadric.

V. Supposing now that $r_4 = 1$ and none of r_1, r_2, r_3 is equal to 0 or $n - 1$, we have from the main theorem

$$[n, (r_1, r_2, r_3, r_4), 1] = 2; \quad r_1 + r_2 + r_3 = 2n - 3, r_4 = 1.$$

Now it is easily seen that the regions P_1, P_2, P_3 have ∞^{n-2} transversals. The locus Q of these transversals contains therefore ∞^{n-1} points and must be a quadric since it cuts an arbitrary line P_4 in two points. Now since

$$r_1 + r_2 + r_3 = 2n - 3; \quad r_1, r_2, r_3 < n - 1,$$

it follows that the sum of any two of the numbers r is equal to or greater than $n - 1$ and therefore a transversal can always be drawn from any point in any one of the regions P_1, P_2, P_3 to the other two. Thus the quadric Q contains P_1, P_2, P_3, and further, any quadric containing P_1, P_2, P_3 must contain their transversals and is therefore identical with Q. We have thus the theorem:

Any three regions in S_n *of dimensions* r_1, r_2, r_3, *such that* $r_1 + r_2 + r_3 = 2n - 3$, *are contained in a unique quadric* Q.

This is an extension of the theorem that three lines in three dimensions are contained in a unique quadric. It may be observed that the theorem may be supposed to hold also for the boundary case $r_3 = n - 1$; for here Q may be considered to have degenerated into P_3 and the S_{n-1} determined by P_1 and P_2.

The quadric Q *has a singular region of* $n - 2$ *dimensions in the boundary case* $r_3 = n - 1$ *and a singular region of* $n - 4$ *dimensions in every other case.*

The first part is obvious as Q splits up into two $(n-1)$-dimensional regions in the boundary case. To prove the second part, let $r_1 r_2 r_3$ be in ascending order of magnitude. Projecting from the $(r_2 + r_3 - n)$-

dimensional region common to P_2 and P_3, Q projects into a quadric Q' in a space of $r_1 + 2$ dimensions determined by three regions P_1', P_2', P_3' of the respective dimensions r_1, $n - 1 - r_3$, $n - 1 - r_2$. If P_1' and P_3' do not intersect, then $r_2 = n - 2$ and therefore $r_3 = n - 2$, $r_1 = 1$ and the quadric Q' becomes a non-singular quadric in three dimensions. Thus the singular region of Q is the common region of P_2 and P_3 and is of $r_2 + r_3 - n = n - 4$ dimensions. If P_1' and P_3' intersect, then projecting again from their common region of $n - 3 - r_2$ dimensions, Q' projects into a quadric Q'' in a space of $n - r_3 + 1$ dimensions determined by three regions P_1'', P_2'', P_3'' of the respective dimensions

$$n - 1 - r_3,\ n - 1 - r_3,\ 1.$$

If P_1'' and P_2'' do not intersect, then $r_3 = n - 2$, Q'' is a non-singular quadric in three dimensions, and the singular region of Q is of $r_2 + r_3 - n + n - 3 - r_2 + 1 = n - 4$ dimensions. If P_1'' and P_2'' intersect. then projecting from their common region of $n - 3 - r_3$ dimensions, Q'' projects into a non-singular quadric determined by three lines in three dimensions. Hence the singular region of Q is of $r_2 + r_3 - n + n - 3 - r_2 + n - 3 - r_3 + 2 = n - 4$ dimensions.

It is clear that in every case the singular region of Q is the join of the regions of intersection of P_1, P_2, P_3 two and two.

The Mechanism of the Rusting of Iron. By Mr U. R. Evans, King's College.

[*Read* 4 February 1924.]

In a previous paper* it has been shown that the corrosion of zinc, cadmium, iron and lead is connected with electric currents set up through unequal distribution of oxygen; the currents have been measured and compared with the weight of metal corroded away. The "unaerated" areas are anodic, and thus the attack tends to become concentrated upon the points relatively inaccessible to oxygen. When a drop of salt water is placed on an iron sheet, the central part of the drop suffers anodic corrosion, giving rise to ferrous salts, whilst the marginal ring over which oxygen has ready access to the metal becomes cathodic, alkali being produced there. Where the ferrous salt and alkali come into contact, ferrous hydroxide is precipitated, which oxidises to ferric hydroxide; after a few hours we get a membrane of brown ferric hydroxide extending right over the drop, but surrounded by a ring of clear alkaline liquid. All the actual corrosion of the metal is confined to the area within the membrane; the metal below the external ring of clear liquid, to which oxygen has access, suffers no attack.

Fresh experiments have now shown that if, at this stage, the membrane is washed away by a current of water, and if the whole iron sheet is wetted with a thin film of water or salt solution and exposed to the air, further rusting takes place readily over the area originally enclosed within the membrane, whilst the ring outside which was immune before still remains almost clear of rust. The area outside the drop, which was previously unwetted, becomes locally rusted, the rust tending to follow the grooves left by the emery when cleaning the metal; these grooves represent the parts least accessible to oxygen. The experiments show that the susceptibility or immunity towards corrosion determined by the unequal distribution of oxygen when the metal is first wetted persists even when the distribution is made approximately uniform.

The ferroxyl indicator of Cushman and Walker can be used to illustrate the distribution of cathodic and anodic areas produced by unequal aeration; naturally the agar-agar must be omitted. Drops of potassium chloride solution containing phenolphthalein and a trace of ferricyanide become blue in the interior, and pink around the margin, whilst a white precipitate, turning brown, appears where the pink and blue regions meet. If, however, an unduly large quantity of ferricyanide (0·2 %) is used, the results are misleading, since the precipitate of Turnbull's Blue itself interferes with the free passage of oxygen. If, for instance, the glass tube used for placing the liquid on the sheet is allowed to touch the metal at the edge of the drop, the momentary obstruction to the passage of oxygen to this point, produces a blue spot there, and this blue spot will continue to grow even when the tube is withdrawn.

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* *J. Inst. Met.* **30** (1923), 239.

Much work has been devoted by different experimenters to studying the action of salts on the velocity of rusting by so-called "pure water." Some of the results published are very contradictory. This may be explained if we remember that there are two stages in the corrosion process:

(*a*) The *transport* of oxygen from the air through the liquid to the metal, either by diffusion or convection.

(*b*) The *consumption* of this oxygen, directly or indirectly, in the corrosion of the metal.

Often the addition of a salt will slow down stage (*a*) by reducing the solubility of oxygen, whilst it will favour stage (*b*). If in a given series of experiments the specimens are very deeply immersed, we measure the effect of salt on the transport of oxygen; if they are covered merely with a layer of liquid which is constantly saturated with oxygen, we measure the true effect of the salt upon the corrosion process proper. In actual practice most workers have adopted intermediate conditions and measured a combination of the two effects; their results naturally vary according to the conditions observed. For instance, Friend* has recently stated that a 3 % solution of sodium chloride rusts iron more slowly than "distilled water"; he carried out his experiments with specimens suspended by hooks in the liquids. Different conditions give, however, quite different results. If drops of 3 % salt solution and a similar number of drops of distilled water are placed on two weighed specimens of iron, the amount of corrosion produced is about ten times as great beneath the salt solution as beneath the distilled water.

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* *Trans. Chem. Soc.* **123** (1923), 2996.

Angular Momentum and Electron Impact. By P. M. S. Blackett, King's College.

[*Received* 17 December 1923.]

1. There seems to be little doubt that the detailed mechanism of ionisation or excitation by electron impact will not admit of description, even to a first approximation, by the system of classical mechanics. However the experimental work of Franck and Horton and their collaborators, has shown that, within the limits of the error of experiment, an electron is able to impart the whole or part of its kinetic energy to the atom it excites. When its initial energy is greater than the excitation energy, the electron, instead of being brought to rest, retains the excess as kinetic energy.

This shows that the total energy of the system, consisting of one atom and one impacting electron, is, within the limits of the experimental error, the same before and after the collision. In this paper an investigation will be made of the assumption that there is exact conservation of both energy and linear and angular momentum. It will be shown that it is possible to derive from this assumption an upper limit to the probability of excitation by electron impact, for electron energies just greater than the excitation energy. One other assumption only will be made. This concerns certain necessary limitations on the nature of the resultant field of the atom. The effect of the quantum orientation of the atoms in a magnetic field will be shown to impose important restrictions on the conditions for excitation. The predicted probabilities are shown to be in rough agreement with some results of Franck.

2. We will first investigate the conditions under which an electron of velocity v_0 and energy E_0 can switch an atom from a stationary state of energy W_1 to another of energy W_2, when $E_0 = W_2 - W_1$. Here the small kinetic energy of the atom after the collision is ignored.

The process is supposed to take place without emission or absorption of radiation and in a space free from electric or magnetic fields. The atom itself is to have no resultant charge, so that its field may fall off rapidly with the distance. In order to simplify the discussion it will be assumed that the force between the atom and an electron vanishes entirely at a radial distance greater than some arbitrary distance d_0. Clearly d_0 must be taken to be of the order of, but somewhat larger than, the kinetic theory radius of the atom.

The Quantum Theory demands that each stationary state must be characterised by an integral, or half integral, number j, defining the total angular momentum $jh/2\pi$ of the atom.

The atom, of mass M, is supposed to be initially at rest with its centre of gravity at the origin of a set of rectangular coordinates x, y, z (Fig. 1). The total angular momentum of the atom is $jh/2\pi$ before collision and $j'h/2\pi$ after collision. The direction cosines of the initial and final axes of angular momentum are p, q, r and p', q', r'. Then without loss of generality the electron, of mass m, can move initially along the line $y = \rho$, $x = 0$, so that the y and z components of its angular momentum are both zero. The distance from the centre of the atom on to the initial path of the electron, which will be called the striking radius, is therefore ρ. After the collision the electron is at rest relative to the atom at a distance from the atom greater than d_0, while the atom and electron together will be in motion with total momentum mv_0 parallel to Oz. Since the atom must in general be expected to be displaced from the origin during the collision, its linear momentum will have a moment about the origin. In view of the small value of the ratio m/M and the assumption of a maximum radius d_0 of the atomic field, it is easy to see that this moment can be neglected in the analysis that follows.

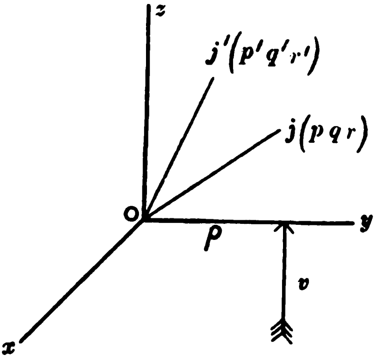

Fig. 1

As we are here dealing with the limiting case of an electron of initial velocity v_0, we will attach the suffix 0 to the quantities p', q', r' and ρ. Later when we consider a more general case we will drop this suffix.

The expressions for the energy and angular momentum, neglecting terms of the order of m/M compared with unity, are

$$mv_0^2 + 2(W_2 - W_1) = 0 \qquad \text{......(1)}.$$

$$mv_0\rho_0 + (jp - j'p_0')\,h/2\pi = 0 \qquad \text{......(2)},$$

$$\left.\begin{aligned} jq &= j'q_0' \\ jr &= j'r_0' \end{aligned}\right\} \qquad \text{......(3)}.$$

From (3) and the geometrical relations between the direction cosines we have

$$p_0' = \pm \left[1 - \frac{j^2}{j'^2}(1 - p^2)\right]^{\frac{1}{2}} \qquad \text{......(4).}$$

From (2) and (4) we obtain

$$\rho_0 = \frac{h}{2\pi m v_0}\left[-jp \pm \{j'^2 - j^2(1 - p^2)\}^{\frac{1}{2}}\right] \qquad \text{......(5).}$$

Equations (3) and (4) give the final direction of the axis of the atom in terms of the initial direction and the initial and final quantum numbers. The corresponding and unique values of ρ_0 for which excitation can occur are given by (5). The probability of such excitation is clearly zero.

It will be noticed that ρ_0 depends only on that component of the initial angular momentum of the atom which is parallel to that of the electron.

When $j' > j$, p_0' is real for all values of p. For any initial orientation there is therefore always some value ρ_0 for which excitation can occur. But when $j' < j$, p_0' is only real when $p > 1 - (j'/j)^2$. Only atoms whose initial orientation satisfies this relation can then be excited. This case will only arise when the atom is not initially in its normal state, or when, if such atoms exist, the normal state has a value of j greater than that of some other possible state.

3. We will now show that when electrons of energy greater than E_0 are considered, the probability of excitation is no longer zero. Suppose that the electron has a velocity v slightly greater than v_0 and that we represent its energy by

$$E = E_0(1 + \alpha^2),$$

where α^2 is a fraction small compared with unity, yet large compared with m/M. Since the energy lost on collision is still E_0, the final velocity of the electron instead of being zero is now αv_0.

Consider the resulting change in the angular momentum relations. The initial x component will now be $mv\rho$; the other components remain zero. If d is the perpendicular distance from the origin on to the final path of the electron, the final angular momentum of the electron is amv_0d. Let the direction cosines of this angular momentum be p'', q'', r''. Then instead of (2) and (3) we have to the same approximation,

$$mv\rho + (jp - j'p')\,h/2\pi - amv_0p''d = 0 \qquad \text{......(6),}$$

$$\left.\begin{aligned}(jq - j'q')\,h/2\pi - amv_0q''d = 0\\ (jr - j'r')\,h/2\pi - amv_0r''d = 0\end{aligned}\right\} \qquad \text{......(7).}$$

The final orientation of the atom is now denoted by p', q', r' instead of p_0', q_0', r_0', to distinguish this new case in which $\alpha \neq 0$.

From (7), introducing p_0' from (4), and neglecting terms containing α^2, we get

$$p'^2 = p_0'^2 + \alpha \,.\, 4\pi m v_0 d\,(q'q'' + r'r'')/hj'.$$

Provided $p_0'^2 \gg \alpha \,.\, 4\pi m v_0 d/hj'$, we can expand in ascending powers of α, and then neglect in the result terms containing its square and higher powers. Thus we find

$$p' = p_0' + \alpha \,.\, 2\pi m v_0 d\,(q'q'' + r'r'')/hp_0'j' \qquad \ldots\ldots(8).$$

Using this result in (6) and introducing ρ_0 from (5) we find, to the same approximation,

$$\rho = \rho_0 + \alpha\epsilon d/p_0'.$$

In this ϵ has been written for $(p_0'p'' + q'q'' + r'r'')$. Since p_0' and p' cannot differ greatly, we can see that ϵ is approximately equal to the cosine of the angle ζ between the final angular momenta of the atom and the electron. We must therefore have $-1 < \epsilon < +1$. Though d like ζ is undetermined, both depending on the unknown detailed mechanism of collision, we know that $d \not> d_0$. Thus ρ must lie between the values ρ' and ρ'' given by the positive and negative signs in the equation

$$\rho = \rho_0 \pm \alpha d_0/p_0' \qquad \ldots\ldots(9).$$

Defining $\delta\rho$ as the possible range $|\,\rho' - \rho''\,|$ of ρ we get

$$\delta\rho = |\,2\alpha d_0/p_0'\,| \qquad \ldots\ldots(10).$$

For a non-zero value of α there is therefore a *range* of values of ρ for which excitation can occur. A non-zero probability of excitation is thus obtained. The part of the atomic disc which will be effective as an excitation target will be an annular ring of mean radius ρ_0 and radial breadth $\delta\rho$.

When $p_0'^2 \not> \alpha \,.\, 4\pi m v_0 d/hj'$ the analysis must be altered. It is not worth while to consider this case, except to mention that when p_0' is very small, it can be shown that ρ is then of the order $d_0\alpha^{\frac{1}{2}}$. We will, however, calculate the probability of excitation as if (9) were valid for all values of p_0', since doing so will not introduce an appreciable error. It may be pointed out that $p_0'^2$ is always greater than $1 - (j/j')^2$ so that it can only become small when $j = j'$.

4. In order to calculate the area of the excitation target, it is convenient to refer the system to a new set of axes X, Y, Z (Fig. 2). The centre of the atom is again at the origin but its initial axis OC is fixed in the ZY plane, making an angle ψ with OZ. An electron

whose initial path is parallel to OZ and whose striking radius ρ makes an angle ϕ with OX, will excite an atom if ρ lies between ρ' and ρ''. If we denote by θ the angle between OC and a line OB in the XZ plane at right angles to the striking radius, we have

$$p = \cos\theta = -\sin\psi\cos\phi.$$

The area of the target which will be denoted by $G(\alpha)$ is

$$\int_0^{2\pi} |\rho_0|\,\delta\rho\, d\phi,$$

which gives on introducing the values of ρ_0 and $\delta\rho$, and integrating,

$$G(\alpha) = 2ad_0 j' h/mv_0 \qquad \text{......(11).}$$

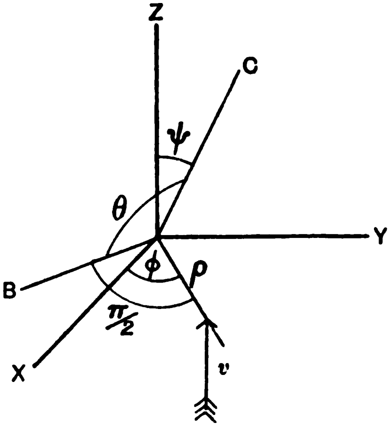

Fig. 2

The target area is therefore independent of the orientation ψ of the atom, relative to the initial path of the electron; it is therefore the same for all atoms. Expressing this area as a fraction $F(\alpha)$* of the whole atomic disc $\pi\sigma^2$ we obtain

$$F(\alpha) = 2ad_0 hj'/\pi mv_0\sigma^2 \qquad \text{......(12).}$$

No actual measurements of this quantity appear to have been made, but it is possible to make a rough comparison with experiment by proceeding as follows:

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* This fraction $F(\alpha)$ has been called by Hertz the Efficiency of Inelastic Collision (*Proc. Amst. Acad.* March 25, 1922). Klein and Rosseland (*Zeit. für Phys.*, 4, p. 46, 1921) defined a quantity $S_1^2(\epsilon)$ so that the probability that an atom would be switched in unit time from a state 1 to a state 2, is equal to $S_1^2(\epsilon)$ multiplied by the number of atoms in state 1 and by the number of electrons of energy ϵ. Since $G(\alpha)$ is merely an upper limit, we have the relation

$$S_1^2(\epsilon) < vG(\alpha),$$

in which v is the velocity of the electron of energy $\epsilon = E_0(1+\alpha^2)$.

We put F in (12) equal to unity and so obtain the value a_m for which the whole area of the atom becomes effective. Then $a_m^2E_0$ will be the smallest increase of electron energy, above the excitation point, necessary to give the maximum excitation,—that is, so that every collision may be effective.

Consider the excitation of the normal helium atom. The two resonance potentials 19·7 and 20·5 volts correspond to the switch of an atom from the normal $1S$ state to the ortho-state $2s$ and the par-state $2S$, respectively. The quantum numbers* j to be attributed to the three states are 1, 2 and 1. We will take σ to be $1{\cdot}1 \times 10^{-8}$ cms. and d_0 to be 2σ. This last assumption is arbitrary but will serve as a first approximation. Then for the $1S \rightarrow 2s$ switch we have $j' = 2$, whence $a_m = 0{\cdot}15$. For the $1S \rightarrow 2S$ switch, $j' = 1$, and $a_m = 0{\cdot}31$. The corresponding values of the energy range $a_m^2E_0$ are 0·5 and 1·9 volts.

If we turn to the curves given by Franck† and Knipping of the photo-electric current produced by the radiation resulting from the excitation of the $1S \rightarrow 2S$ and $1S \rightarrow 2s$ transitions of helium, we find that the main rise of current for the latter occurs within a range of electron energy of less than a volt. Thus the value predicted by the theory seems to be of the right order; this is all that could be expected in view of the uncertainty of the data and the extrapolation of (12) to such large values of a. It would be tempting to compare the greater range of voltage predicted for the former switch with the slower rise of the experimental curves above 20·5 volts. It is however doubtful how far this is legitimate since the number of electrons available for the excitation of this switch will not be independent of the frequency of occurrence of the lower voltage switch.

Again, in the curves given by Franck‡ and Einsporn, the whole rise of current due to the excitation of the mercury 1850 line, $1S - 2P$, occurs within a range of less than half a volt. If we put $\sigma = 2{\cdot}5 \times 10^{-8}$ cms., $d_0 = 2\sigma$, and $j' = 2$, which is the value attributed by Bohr to the singlet P terms, we get $a_m = 0{\cdot}20$ and $a_m^2E_0 = 0{\cdot}28$ volts. This again is in rough agreement with the measured value. It is possible, however, that the rise of current is controlled chiefly by the electron velocity distribution. If this is so the agreement shown above must be discounted.

Emphasis must again be laid on the fact that equation (12) applies only when a^2 is small compared with unity, and therefore refers strictly only to the slope of the curve measuring the degree of excitation *immediately* above the excitation point.

It is of interest to note that the form of the annular target of an atom in a given orientation is a circle not in general concentric with the atom. For (5) can be written in the form

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* Bohr, *Ann. der Physik*, 71, p. 286, 1923.

† Franck and Knipping, *Zeit. für Phys.*, 1, p. 321, 1920.

‡ Franck and Einsporn, *Zeit. für Phys.*, 2, p. 18, 1920.

$$\rho_0 = c \cos\phi \pm [(a^2 - c^2) + c^2 \cos^2\phi]^{\frac{1}{2}},$$

where
$$a = (j'^2 - j^2 \cos^2\psi)^{\frac{1}{2}}\, h/2\pi m v_0,$$

and
$$c = hj \sin\psi/2\pi m v_0.$$

This is the polar equation of a circle of radius a and whose centre is at a distance c from the atomic centre. When $a > 0$ the target is therefore a circular ring of varying radial width $\delta\rho$. Some of these circles are shown in Fig. 3, for the $1S \rightarrow 2s$ switch of helium for which j and j' are equal to 1 and 2 respectively. The numbers against the circles are the values of ψ to which the circle corresponds. The line OL is the projection of the initial axis on the plane of the disc. The dotted circle has a radius, equal on the same scale, to the kinetic theory radius of the atom. It is possible that the outer parts of the target are not actually effective. Especially will this be the case when j' is large.

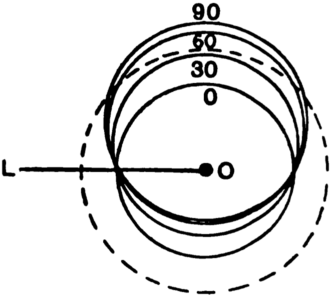

Fig. 3

5. We will now return to a more detailed consideration of the meaning of d_0. To do this we have to consider how the electron leaves the field of the atom after the collision. Suppose that $\alpha = 0$, and that the atomic field can be considered as central and represented by a force μr^{-s} on the electron. The electron will then leave the atom in a "critical" orbit of zero total energy. If χ is the angle between its velocity u and the radius vector r, its orbit will be defined by the two equations

$$mu^2 + 2\int_{\infty}^{r} \mu r^{-s}\, dr = 0,$$

and
$$mur \sin\chi = H,$$

where H is the constant angular momentum of the electron about the centre of force. Provided $s \neq 1$ these give

$$r^{3-s} = \frac{1}{2m}\frac{H^2}{\sin^2\chi}\frac{s-1}{\mu} \qquad \text{......(13).}$$

When $s = 2$ (or more generally $s < 3$) the orbit extends to infinity for any value of H. The maximum value H_2 that H can take is given by putting $\chi = \pi/2$ when r has its initial value r_0, which must be taken to be of the order of the atomic radius. Taking the intensity of the field as that of a singly charged ion at a distance

r_0, we find for this case of $s = 2$, that the maximum value H_2 is of the order of the whole angular momentum of an atom ($h/2\pi$). This means that in a field varying as the inverse square of the distance, an electron receding to infinity with a velocity tending to zero, retains an angular momentum comparable with that of the atom. This is why the foregoing analysis will not apply to the case of ionisation. Equations (2) and (3) do not hold and there appears to be no reason to suppose, from this point of view of angular momentum alone, that there are any necessary restrictions on the values of ρ_0. The whole area of the disc may become effective immediately the electron energy reaches the ionisation value.

When excitation is considered the field falls off very rapidly with the distance. Possibly s may have a value as great as 8. From (13) when $s > 3$, r can only become large when H is small. An electron will certainly escape from an atom if its orbit extends to a distance R comparable with the mean distances apart of the atoms. For $r > R$ we must have $H_s^2 < 2\mu m R^{3-s}/(s-1)$. If this field, varying as r^{-s}, has the same intensity at r_0 as the inverse square field already discussed, we have

$$\frac{H_s}{H_2} < \frac{1}{(s-1)^{\frac{1}{2}}} \left(\frac{r_0}{R}\right)^{\frac{s-3}{2}}$$

Since at all pressures in use $R \gg r_0$, H_s will be negligibly small compared with the angular momentum of the atom, provided s is greater than about 4. In cases of excitation of neutral atoms we are therefore entitled to neglect the final moment of momentum of the impacting electron, when its energy is zero.

When its energy is not zero, the orbit is not a critical one and cannot apparently be simply calculated for s greater than 3. But it does seem clear that even now there will be a definite maximum to the possible final angular momentum of an electron of given final energy. The simple assumption that has been made in section 2, to the effect that the field stops abruptly at a distance d_0 from the atom, expresses this expectation. For outside this distance, the path of the electron will be straight and its maximum moment of momentum will be amv_0d_0. If the calculation could be performed d_0 could be evaluated in terms of μ, s, r_0, R and the total energy α^2E_0 of the orbit. In this paper it has been taken arbitrarily as a constant equal to twice the kinetic theory radius of the atom.

6. So far we have been concerned with the excitation of atoms in a field free space. The study of the Zeeman effect has shown that to an atom in a magnetic field must be attributed a fourth quantum number defining the projection of the total angular momentum on the direction of the field. If λ be the angle between

the j axis of the atom and the field direction we have a relation of the type

$$\cos\lambda = m/j \qquad \ldots\ldots(14).$$

There seems to be some difference of opinion as to which of the $2j + 1$ values of m allowed by (14) actually occur in a given type of atom. However, neglecting such restrictions there will be $2j + 1$ cones*, having the field direction as axis, on which both the initial and final axes of the atoms must lie. A switch by electron impact from one orientation to another is then only possible when the final orientation specified by (3) and (4) is such as to fulfil one of these conditions.

We cannot however retain any of the analysis of section 2 by which these equations were obtained unless we can consider the system of atom and electron as a closed system. This will not be so when there is an external field. Strictly it is only within the whole system of atom, electron, and field generating bodies, that momentum will be conserved. But if the period of the Larmor rotation due to the field is very large compared with the time of collision, which will be satisfied in fields of moderate strength, it is then legitimate to neglect to a first approximation the influence of the field during the collision. We can thus apply directly the conditions already found for the excitation radius, with the additional restrictions as to the initial and final orientations. No attempt will be made to work out fully the probability of excitation in a field of force, owing to the somewhat complicated nature of the problem; but a discussion will be given which will illustrate the nature of the procedure.

7. If a unit sphere is drawn with its centre at the origin of the original axes x, y, z, then the initial and final orientations can be represented by two points P and P' on its surface. (Fig. 4.) Then when $\alpha = 0$ we see from (3) that P_0' must lie on the great circle through P and Ox: further, the ratio of the sines of the arcs xP and xP_0' must be j'/j. The point P_0' will have another possible position corresponding to the negative sign for p_0' in (4), but with the same values of q_0' and r_0'. If another point M on the surface of the sphere represents the direction of the field, we must also have P and P_0' lying on one or other of the set of small circles, whose common pole is M, and whose radii are given by the possible values of $\sin\lambda$. Only in exceptional circumstances can these conditions be satisfied.

When $\alpha > 0$, equations (7) take the place of (3). The condition for P' now is that it must lie within a small area about the point P_0', the extent of which is defined by the magnitude of the terms in (7) containing α. In general the effective target of an atom will reduce to certain isolated parts of that annular ring which was obtained in section 2 by placing no restrictions on the possible final orientations.

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* The line ($m = j$) and the plane ($m = 0$) are included as limiting cases of cones.

As an example of the singular spatial relations that are necessary for excitation, the switch $1S \rightarrow 2s$ of the helium atom will be studied. The initial and final quantum numbers are again 1 and 2. The diamagnetism of helium requires the initial axis to be at right angles to the field ($m = 0$). It is probable that ortho-helium is paramagnetic and so allows m to take non-zero values. As our purpose is only to illustrate the phenomenon, remembering the doubt already mentioned as to the exact values to be ascribed to m, we will assume that m can take the values ± 1 and ± 2. Then

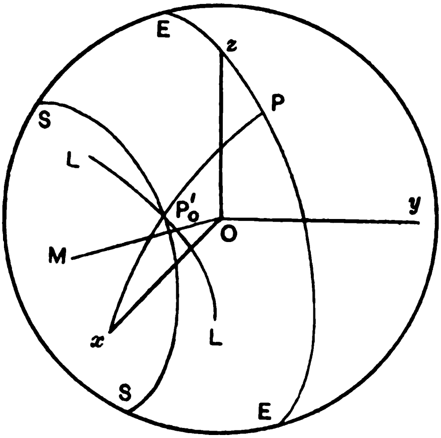

Fig. 4

in Fig. 4, P must lie on the equator EE of M: while P_0' must lie either at M, or on one of the two concentric small circles SS of latitude 30°. Equations (3) restrict P_0' also to that curve LL, which is such that sine (arc xP) divided by sine (arc xP_0') is equal to j'/j. The point where this locus passes through M or cuts one of the small circles SS (of which one only is drawn), will give the final orientations of the only possible switches.

Such considerations as these suggest that the effect of a magnetic field will not only influence the excitation of any switch but will influence different types in different ways. No attempt will be made to discuss the question of the strength of field in which the phenomenon should occur. While it is difficult to see how the results for the field free space can be in error, it is quite conceivable that the spatial quantum conditions have been interpreted too literally. The probabilities that have been calculated in section 3 have been for a field free space, but it is questionable whether the normal conditions inside such an apparatus as was used by Franck can be considered as such. It is hoped to settle by experiment this and other conclusions that can be derived from this analysis.

8. The principles of the conservation of angular momentum and energy have been assumed to hold during excitation by electron impact. For excitation to occur under these conditions the undeflected path of an electron, whose energy is only slightly greater than the excitation energy, must cross the atomic disc within a certain limited target. The area of this target is a function of the velocity, and expresses the upper limit to the probability of excitation. This area will be very drastically restricted if the atoms are orientated in a magnetic field both before and after collision.

These limitations on the type of a successful electron impact are in an important sense analogous to the limitations on the type of spontaneous emission of radiation found by Rubinowicz*, since they are derived from the same assumption of conservation of energy and angular momentum. Just as Rubinowicz's Selection Principle falls within the wider scope of Bohr's Correspondence Principle, so it may be anticipated that the limitations imposed on the excitation conditions, and the probabilities obtained from them, must be derivable from any generalised form of the Correspondence Principle that may yet be formulated to cover the process of excitation by electron impact.

I wish to express my gratitude to Mr R. H. Fowler of Trinity College for his valuable criticism and advice during this work.

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* Rubinowicz, *Phys. Zeit.*, 19, p. 441, 1918.

*Hankel Transforms**. By E. C. TITCHMARSH. (Communicated by Mr G. H. HARDY.)

[*Received* 26 January 1924.]

I have to correct the proof of the assertion on p. 471 of my paper that if $f(t)$ is integrable over $(0, a)$ and $\nu \geqq -\frac{1}{2}$, then

$$\lim_{N\to\infty} \int_0^a t^{\frac{1}{2}} f(t)\, dt \int_N^\infty J_{\nu+1}(ux)\, J_\nu(ut)\, du = 0.$$

This follows from Lebesgue's convergence theorem if

$$t^{\frac{1}{2}} \int_N^\infty J_{\nu+1}(ux)\, J_\nu(ut)\, du = t^{-\frac{1}{2}} \int_{Nt}^\infty J_{\nu+1}\left(\frac{ux}{t}\right) J_\nu(u)\, du$$

is bounded for $0 < t < a$, $N > N_0$. The result is immediate if $\nu = -\frac{1}{2}$, so it may be assumed that $\nu > -\frac{1}{2}$. Since the argument of the first Bessel function remains greater than a constant we may use the asymptotic form

$$J_{\nu+1}(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}\nu\pi - \frac{3\pi}{4}\right) + O\left(z^{-\frac{3}{2}}\right).$$

Since $|J_\nu(u)| < Au^{-\frac{1}{2}}$ for all values of ν, the error term is

$$O\left\{t^{-\frac{1}{2}} \int_{Nt}^\infty (u/t)^{-\frac{3}{2}} u^{-\frac{1}{2}}\, du\right\} = O\left(t \int_{Nt}^\infty u^{-2}\, du\right) = O(1).$$

The main term is a constant multiple of

$$\int_{Nt}^\infty \cos\left(\frac{ux}{t} - \frac{1}{2}\nu\pi - \frac{3\pi}{4}\right) J_\nu(u)\, u^{-\frac{1}{2}}\, du.$$

Any part of the integral for which $u < k$ is

$$O\left(\int_{Nt}^k u^{\nu-\frac{1}{2}}\, du\right) = O(1).$$

In the remainder we may use the asymptotic formula for the other Bessel function. This gives an error term

$$O\left(\int_k^\infty u^{-2}\, du\right) = O(1),$$

and the main term

$$\int_k^\infty \cos\left(\frac{ux}{t} - \frac{1}{2}\nu\pi - \frac{3\pi}{4}\right) \cos\left(u - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) \frac{du}{u}$$
$$= \frac{1}{2}\int_k^\infty \sin u\left(\frac{x}{t} - 1\right) \frac{du}{u} - \frac{1}{2}\int_k^\infty \cos\left\{u\left(\frac{x}{t} + 1\right) - \nu\pi\right\} \frac{du}{u},$$

which is easily seen to be bounded. This completes the proof.

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* These *Proceedings*, vol. XXI, pp. 463–473.

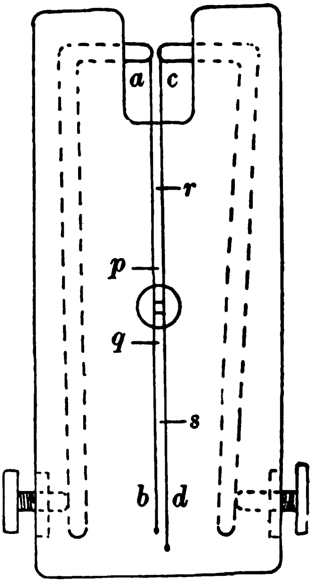
On a modified electrostatic oscillograph. By Dr J. A. CROWTHER, St John's College.

[*Read* 3 December 1923.]

The instrument to be described is a modification of an electrostatic oscillograph originally invented by Prof. Taylor Jones. In the original instrument a thin phosphor-bronze wire is stretched parallel to a flat metal plate towards which it is attracted when a difference of potential is established between them. A small mirror is fastened by one edge to the middle of the wire, and the other edge of the mirror is pivoted by bringing up to it a flat cork against which it presses. The motion of the wire is recorded by reflecting a beam of light from the mirror on to a rotating drum.

The free period of the wire is proportional to $\sqrt{\frac{T}{m}}$ where T is the tension and m the mass per unit length of the wire. The maximum value of this ratio, when breaking stress is reached. is obviously a constant for a given material. In my own experience, using phosphor-bronze strips, and mirrors about 2 mm. square, the maximum frequency attained was about 5000 per second. This was insufficient to enable the instrument to record accurately some of the faster variations which occurred in the circuits on which it was to be employed. Moreover, the attempt to use the wire near its breaking point lead to frequent break-downs in the apparatus.

As the deflecting force was very considerable, with the potential differences to be employed (30 to 100 kilovolts) it appeared that a stronger form of control might be adopted with advantage, and the apparatus improved mechanically at the same time. The actual form of the apparatus now used is indicated in the figure.



Two "strings" *ab* and *cd* are mounted parallel to each other and about 1½ mm. apart, just above the surface of a brass plate. The one, *ab*, is a phosphor-bronze wire of ·006 inch diameter, and is bridged at two points, p, q, near the centre at a distance of

½ cm. apart. This wire serves as the pivot on whch the mirror turns. The other "string" is a very flexible strip of phosphor-bronze, such as is used for suspending galvanometer coils, and bends easily under the electrostatic forces. It is bridged at two points, *r*, *s*, about 1½ cm. apart. The lower ends of the strings are soldered to the brass plate; the upper ends to two levers by which the tension of the wires can be adjusted. A small piece of silver leaf is soldered across the two wires half-way between the bridges, and the mirror is mounted on this with shellac varnish. The mirror is mounted on the side of the leaf facing the brass support, and is viewed through a hole in the brass. The silver leaf, being in metallic contact with the wires, shields the mirror from the action of the charged plate.

The plate is mounted in an ebonite box, and a second brass plate is mounted parallel to it in the same vessel, the distance between the plates being variable at will. A distance of 3 cm. is suitable for potentials of from 60 to 100 kilovolts. On applying such a potential difference the flexible strip is attracted and bends, giving a small rotation to the mirror. The restoring force is provided partly by the tension in the strip, but principally by the torsion in the fixed wire. The maximum displacement of the strip is about $\frac{1}{10}$th mm. The field may be regarded as sensibly constant over this range, and the deflections are thus proportional to the square of the potential difference between the plates.

Damping is secured by filling the ebonite box with castor oil. The viscosity of the oil decreases rapidly as the temperature rises. At temperatures above 80° F. the damping was obviously insufficient, while below 55° F. the instrument appeared to be somewhat overdamped. The dielectric strength of the oil was sufficiently high to obviate the use of a solid dielectric between the plates. Its chief defect was a tendency to turn a rich brown under the electrostatic strain. In this state it absorbed most of the actinic rays from the beam of light used to illuminate the mirror, thus necessitating the refilling of the instrument with fresh oil from time to time.

It may be mentioned that both the shielding of the mirror, and the continuous liquid dielectric seemed to be necessary for consistent readings, probably owing to induction effects.

The instrument actually constructed had a natural frequency of about 10,000 per second, and gave a deflection of 15 mm. at a distance of 30 cm. from the mirror for a potential difference of 100 kilovolts between the plates.

In conclusion, I wish to express my warmest thanks to Dr P. Kapitza for much practical assistance and advice during the designing and construction of the oscillograph.

An oscillographic study of the production of X-rays by a Coolidge tube. By Dr J. A. Crowther, St John's College.

(Plates I and II.)

[*Read* 3 December 1923.]

Radiologists have often had cause to note certain apparent anomalies in the behaviour of their apparatus. It is quite well known that different X-ray tubes, excited by different kinds of high tension apparatus, yield X-radiation of markedly varying quality and quantity even under conditions which, as measured by spark gap and milliammeter (the usual measuring instruments of the radiologist), are apparently identical. These anomalies seemed to offer an interesting field for investigation, and one which might not be without value on the practical side.

For a complete understanding of the working of an X-ray apparatus we require to know the instantaneous values (*a*) of the potential applied to the tube, (*b*) of the current passing through it, (*c*) of the quality and (*d*) of the quantity of the X-radiation emitted. A considerable number of oscillograms of the current through an X-ray tube have been published, and a few oscillograms of the potential have recently been taken by Taylor Jones with his electrostatic oscillograph. In the present experiments I have attempted to obtain a *simultaneous* record of all four factors on the same plate, under the normal working conditions of an X-ray plant.

The records of the potential were obtained by means of the electrostatic oscillograph described in an earlier paper. The current record was given by a current oscillograph of the Duddell type. Beams of light, reflected from the mirrors of the two instruments were focused, at exactly the same height, on the film of a photographic plate which could fall freely in a vertical slide. The X-ray tube was mounted, in the usual leaden box, on the opposite side of the vertical slide, and a narrow slit, about $\frac{1}{4}$ mm. wide, in a thick lead plate allowed a narrow horizontal pencil of X-rays to impinge on the plate at the same horizontal level as that of the two oscillograph spots. On allowing the plate to fall the spots of light traced out the current and potential wave forms, while the intensity of the X-rays at each moment during the discharge could be estimated from the blackening of the photographic film. As a rough measure of the penetrating power of the rays an aluminium wedge tapering downwards from 3 mm. thickness to zero was fastened across the middle of the slit. Some estimate could thus be made of the quality of the radiation at different points during the discharge.

The photographic film of the plate was turned to face the X-ray tube, and an intensifying screen was used to increase the effect. It is possible that the tube may emit radiation too soft to affect a photographic plate under these conditions, and the absence of any darkening on the plate does not necessarily indicate a complete absence of emission. The method, however, records all the radiation of any radiographic value, and in practice the presence of radiation was always recorded when the exciting potential was over 30,000 volts. This corresponds to a wave-length of about 0·4 A.U.

Experiments have been made both with "gas-filled" X-ray tubes and with the Coolidge tube. As the phenomena are distinctly simpler in the case of the Coolidge tube it will be convenient to deal with them first. For producing the high potentials necessary to excite X-radiation both high tension transformers, and ordinary induction coils have been used.

As is well known a Coolidge tube will rectify its own current, to a certain extent. Fig. 1 shows the behaviour of a medium focus standard tube when supplied with alternating current from a large transformer without rectification. The average current through the tube, as measured by a milliammeter, was 11 ma., and the peak voltage, as measured by the sparking distance between a pair of 3″ spheres, was 80 kilovolts, assuming that the values given by Owen for this particular size of spheres are correct. It may be noticed in the first place that the voltage as measured by a spark gap gives the maximum voltage during the period when the current, if any, is passing in the *reverse* direction through the tube, and that the peak voltage corresponding to the emission of X-rays is considerably less (about 0·7 in this instance). A spark gap cannot, therefore, be used to estimate the quality of the rays when the tube is used with unrectified current.

The current through the tube is not completely unidirectional, no doubt owing to the intense heating of the focal spot by the stream of electrons. Attention may also be called to the very marked distortion produced by the tube in the wave form of the transformer. The inverse half of the current cycle is not merely suppressed; it is partly reversed, so that direct current flows through the tube for considerably longer than half the period. This is also illustrated by the voltage curve, the high "inverse" peaks being considerably narrower than the low "direct" peaks. (It must be remembered that as the deflection electrostatic oscillograph varies as the square of the voltage the direction of the deflection is independent of the sign.) This behaviour appears to be characteristic of a hot cathode valve in the secondary circuit of a high tension transformer whether the valve is used at potentials far above the saturation point, as in the case of the Coolidge tube, or below saturation, as in the case of the Kenotron rectifier. Fig. 2 shows the current and potential graphs for the same transformer when passing a current through a water resistance of about 5 megohms in series with a Kenotron. There is no inverse current, but an appreciably direct current flows through the system for practically three-quarters of the complete cycle. In the absence of the Kenotron the current through the water resistance was strictly sinusoidal. It is hoped to make a further communication on this point shortly.

Fig. 3 represents the typical discharge through a Coolidge tube with the transformer current rectified by the Kenotron. The current rises fairly sharply and reaches saturation for a potential too small to be perceptible on the voltage curve. The curve is not quite flat topped, the current increasing slightly as the voltage rises. A similar effect has been recorded in the case of ordinary "wireless" valves, and has been ascribed to the dragging out of the hot wire of additional electrons by the strong electrostatic field. The band of X-rays is much narrower than the current wave, and the rays are obviously more intense and more penetrating as the potential is higher. As the current is approximately constant during the emission this may be regarded as illustrating the well-known law that the intensity of the radiation is proportional to the square of the applied potential. Although the density of the various parts of the X-ray bands has not been measured by any accurate method, it may be stated that they appear to show that the intensity of the emission at any instant is proportional to the product of the instantaneous value of the current into the square of the corresponding potential.

If the filament current is kept constant the oscillograph records of the current through the tube are identical through the whole range of voltages which are employed in radiology, and this result was verified by the readings of the milliammeter which were also independent of the exciting potential. The current through the Coolidge tube is thus a saturation current, and saturation is produced by potentials which are very small in comparison with the peak potentials generally employed in X-ray production. The potential required to produce saturation, using a filament current of 4·1 amperes, was approximately 12,000 volts.

This fact is well brought out when we turn to the induction coil as our source of potential. Both the potential and the current wave forms are quite different from those for the transformer, and, for moderate filament currents, consist of a succession of narrow peaks. This is illustrated in Fig. 4 and Fig. 5, the former representing the case of a coil discharge when rectified by a Kenotron, the latter when there is no rectification. These plates were taken with the same filament current as Fig. 3. It will be seen that the maximum current ordinate is practically the same in all three. A close examination of the originals shows that the peaks produced by the sudden application of potential by the coil, are a few per cent. higher than the top of the current curve which corresponds to the gradual application of potential by the transformer. An exactly similar result is obtained when a potential is suddenly applied to an ordinary triode valve and has been ascribed by Appleton (*Phil. Mag.* 1921) to a chilling of the filament owing to the increased emission of electrons when a current flows through the valve. An increase in peak voltage produces a slight increase in the peak current, probably owing to the same cause as that which produces the rounding of the tops of the transformer curves. The effect is so small as to be imperceptible in the reproductions.

Returning to Fig. 3 it will be noticed that the width of the X-ray band is considerably less than that of the current wave. It corresponds in fact to the period during which the potential curve is above 30 kilovolts. A considerable portion of the current through the tube is thus wasted from a practical point of view. The supposed over-heating of the tube by this waste current, however, is not so serious as has sometimes been supposed, as a comparison of the current and potential graphs shows that the potential behind it is very small. If a rotary rectifier is used instead of a Kenotron this waste current can be cut out by shortening the contact pieces so that the circuit is only made for, say, two-thirds of the half cycle. Very delightful theoretical curves have been printed by some manufacturers showing the manifold advantages of this method. In these, both the current and the potential curves are supposed to rise instantaneously, on contact being established, to about one-half of their peak values, and useful radiation to be emitted during the whole time the current is allowed to flow. These anticipations are not completely borne out by the oscillograph records (Fig. 6). The rise in the potential curve is by no means instantaneous, though somewhat more rapid than when the mechanical rectifier is not employed. As a result the band of X-rays is narrower than the current wave, just as it is with the valve rectification, and as nearly as can be judged in about the same ratio. The average hardness of the rays is somewhat increased, however, as it is the softer rays at the edges of the band which are cut off. The failure of the method to achieve its full aim may be due in part to sparking in the rectifier.

Attention may be called to the high frequency oscillations imposed upon the main current, as shown both by the current and potential curves in Fig. 6. Their frequency is about 4200 per second, and they may be observed even in those portions of the current curve where the secondary circuit is completely broken. In this part of the cycle they must be due to capacity. They have not been observed in any of the curves taken with the transformer in the absence of the mechanical rectifier.

The ability of the Coolidge tube to transmit oscillations is shown in Fig. 5, which shows an unrectified coil discharge through the Coolidge tube. It will be noticed that the rise of the small inverse potential is always accompanied by a small inverse peak in the current graph. This is followed by a train of slightly damped oscillations, which continue until the next wave of direct current. These oscillations appear to be completely suppressed if a Kenotron is placed in series with the tube (Fig. 4).

In the induction coil discharge (Figs. 4, 5) the potential rises almost instantaneously to its maximum value, and falls again to zero almost as quickly. The X-rays are, therefore, emitted in a series of very narrow bands. The intensity of these bands appears to be identical with the intensity produced by the transformer discharge at the same instantaneous current and potential. Owing to the narrowness of the bands, however, the output of radiation per second is much less with the coil than with the transformer. The average current through the tube, for the same filament current, is also correspondingly small. To pass the same current through the tube by means of a coil it is necessary to heat the filament to a much higher temperature than is required for a transformer.

The current waves in Figs. 4 and 5 are appreciably wider than the potential curves. This is due to the fact that a very small potential, too small to be recognized on the potential curve, will suffice to pass a very appreciable current through a Coolidge tube.

The wave form of the secondary potential of an induction coil is not constant, but varies with the effective resistance in the circuit. As the resistance is reduced the time for which the direct potential is produced is prolonged. The effect can readily be demonstrated by taking oscillograms of the discharge through different water resistances and has been discussed at length by Taylor Jones. If the resistance is sufficiently low, direct potential persists until it is wiped out by the inverse potential excited at the next "make" of the primary current. Now the effective resistance of the Coolidge tube decreases as the filament becomes hotter. The potential wave-form of the discharge through the tube therefore changes considerably as the filament current is increased, and a point is eventually reached when a considerable current flows through the tube during the whole period between a "break" and the succeeding "make." This is shown in Fig. 7. As, however, the width of the potential wave near its peak is almost unaltered, the width of the X-ray band remains substantially the same, and

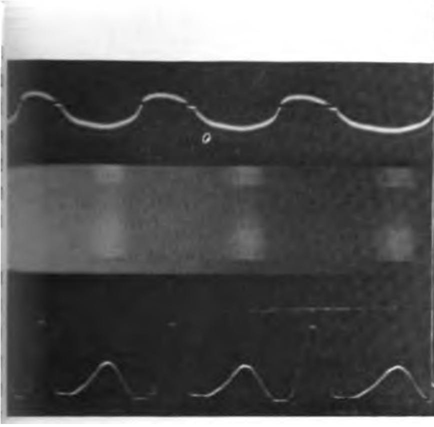


Fig. 1.

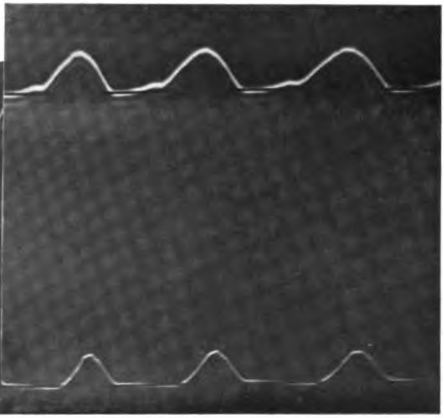


Fig. 2.

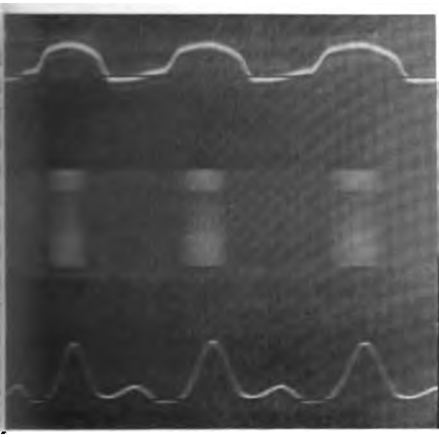


Fig. 3.

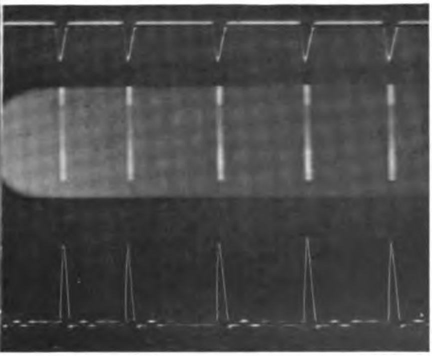


Fig. 4.

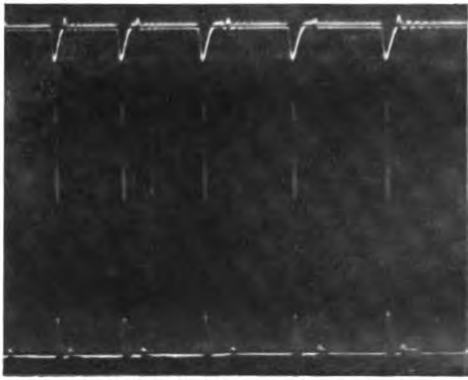


Fig. 5.

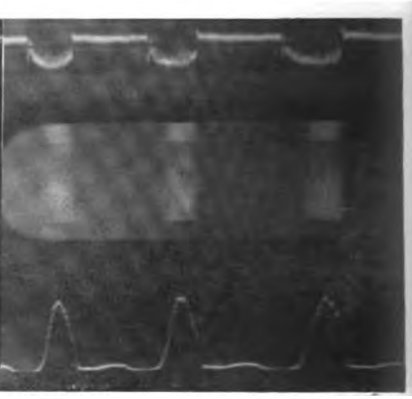


Fig. 6.

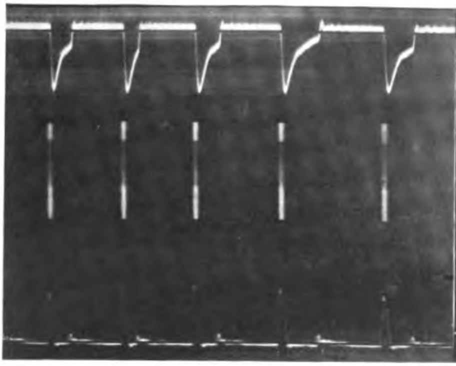


Fig. 7.

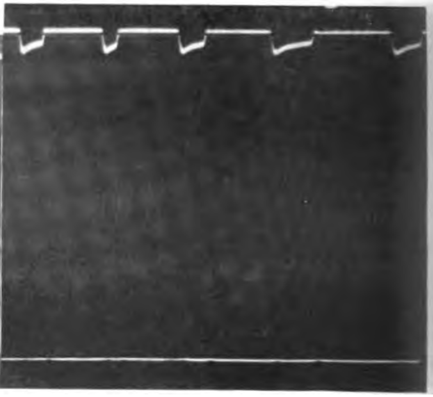


Fig. 8.

most of the current passes through the tube without exciting any useful X-radiation. The average current through the tube thus affords no measure of the intensity of the X-radiation. This want of proportionality between the current and the output has been demonstrated by Owen by the ionisation method (Owen, *Journal Röntgen Society*, 1923). Figs. 4 and 7 provide the explanation.

A rather amusing illustration of the same phenomenon is afforded if the Coolidge tube is run with a moderate filament current, and the alternative spark gap is arranged so that a spark passes across the gap at each discharge. It is found that when sparking occurs the average current through the tube, as measured by the milliammeter, actually increases, sometimes by as much as 75 per cent., although at the same time the emission of X-rays becomes practically negligible. The spark gap provides a low resistance path for the secondary current and thus causes a prolongation of the direct potential. The small potential drop across the spark gap is sufficient to produce a very considerable current through the Coolidge tube, though much too small to produce any useful X-radiation. The effect is illustrated by the oscillograms in Fig. 8.

From this study of the Coolidge tube we may draw two general conclusions, (*a*) that the Coolidge tube and an induction coil make a very unsuitable combination, and (*b*) that the very common practice of attempting to measure the quantity and quality of X-radiation by means of a milliammeter and an alternative spark gap is liable to result in very grave errors.

DESCRIPTION OF PLATES I AND II

The upper curve in each case is the current curve; the lower the potential curve. The intervening bands are due to X-radiation; the absorbing aluminium wedge affecting the middle of the band. The direction of increasing time is from left to right. The whole time represented by each plate is $\frac{1}{50}$th second.

FIG. 1. Unrectified transformer discharge through Coolidge tube. Filament current 4·11 amp.; discharge current 11 ma. Peak voltage (inverse) 98 kilovolts.

FIG. 2. Transformer discharge through water resistance, rectified by Kenotron. Average current, 10 ma.

FIG. 3. Transformer discharge rectified by Kenotron. Coolidge tube. Filament current 4·10 amp.; discharge 10·5 ma. Peak voltage 73 kilovolts.

FIG. 4. Coil discharge rectified by Kenotron. Coolidge tube. Filament current 4·10 amp.; discharge 2·6 ma. Peak potential 100 kilovolts.

FIG. 5. Coil discharge, unrectified. Coolidge tube. Filament current 4·10 amp.; discharge 2·0 ma. Peak potential 75 kilovolts.

FIG. 6. Transformer with mechanical rectifier. Coolidge tube. Filament current 4·0 amp.; discharge 6 ma. Peak potential 94 kilovolts.

FIG. 7. Coil discharge, unrectified. Coolidge tube. Filament current 4·25 amp.; discharge 8 ma. Peak potential 80 kilovolts.

FIG. 8. Coil discharge. Coolidge tube with parallel spark gap sparking. Filament current 4·1 amp.; discharge 4·0 ma.

An early formulation by Stokes of the theories of the rotatory polarizations of light. By Sir Joseph Larmor, Lucasian Professor.

[*Received* 21 January 1924.]

(*Abstract.*)

The two letters now communicated are from G. G. Stokes to W. Thomson, of dates Dec. 12–13, 1848, three years after Faraday's great magneto-optic discovery. They formulated already the permissible types for general equations of propagation, virtually on the basis of the very modern criterion of covariance,—relative to all changes of the spatial frame of reference in the case of active fluids, but having regard to the fixed direction of the extraneous magnetic field in the Faraday case. Their form was elucidated in each case by correlation with a remarkable and significant type of rotational stress in a propagating medium.

The fundamental discovery of the rotation of the plane of polarization in magnetised glass was made by Faraday in 1845. The following letters from G. G. Stokes to W. Thomson* at Glasgow are of date three years later, and are marked by Lord Kelvin as having both been answered on Jan. 27, 1849. They are remarkable. as already exhibiting the formal equations of a complete theory, for waves in all directions, of both types of rotatory polarization. natural and magnetic, with striking physical interpretation on the lines of the elastic aether-theory of the time. His "setting to work inductively" can only mean the construction by the criterion of invariance of the appropriate set of equations—in fact procedure along the most modern formal lines, as is conspicuous in the formulation of the stresses. The papers of Kelvin on dynamical analogues to the gyrostatic features of the magnetic rotation appeared ten years later. The cognate general formulae, constructed on the molecular aspect of the electric theory of light, may be found in Larmor, *Aether and Matter* (1900), ch. XII and final appendix; also with extension to atomic crystalline lattices in *Roy. Soc. Proc.* 1920.

The natural axial rotatory polarization by quartz had been discovered by Arago in 1811, followed up by Biot; and soon after, the isotropic effect for active liquids: the rotation was elucidated as due to the interference of waves of circular displacement and of permanent type by Fresnel a few years later. The type of these waves in a direction oblique to the axis of the crystal was examined by Airy in *Cambridge Transactions*, 1831; also by MacCullagh in

---

* Lord Kelvin graduated in the Mathematical Tripos of 1845, and soon after became Professor at Glasgow: Sir George Stokes became Lucasian Professor in 1849.

1836 (*Collected Papers*, pp. 63–74), who appears to have been the first to assign dynamical equations for the type of propagation along the axis of z. The analogous dynamical equation of propagation for the Faraday magnetic rotation was given for the same special case by Airy in *Phil. Mag.*, 1846, thus immediately after its discovery: cf. Maxwell, *Electricity and Magnetism*, vol. II. § 830. The present equations connect together all directions of propagation.

The paper in the *Annales de Chimie* referred to by Stokes seems to be an experimental study of the magnetic rotation by Bertin. The preceding volume of the *Annales* however contains the experimental memoir by H. de Senarmont on the modes of conduction of heat in the various types of crystals, with some attempt at comparison with their optical properties. This perhaps gave occasion to Stokes' theoretical memoir of 1851 on crystalline conduction (in *Scientific Papers*, vol. III), which gave the complete equations for that problem, on the same general lines as the more intricate equations of the rotational optical theory as now published.

The letters illustrate Stokes' well-known early familiarity with the idea of vorticity (MacCullagh's vector of intrinsic curl), cf. *Papers*, vol. II, p. 48: which is also implicit in the great memoir on the Dynamical Theory of Diffraction in the same volume (1849), and in the formulation in 1845 of the equations of viscous fluid motion, vol. I, p. 81, and became the foundation of Helmholtz's great theory of vortex motion in 1858.

The early recognition that an elastic medium modified by spiral molecular structures dispersed through it behaves as if they were smoothed out into three principal spiralities in rectangular directions, as sufficiently indicated at the end of the second letter, arrests attention. But the analysis is here restricted from the beginning to equivalent stress distributions in an isotropic medium: the cognate reductions for a crystal would naturally be conducted through the function expressing the Action (or the Energy) as in the references *supra*, or in the memoir of J. G. Leathem on 'Magneto-Optic Reflection,' *Phil. Trans.*, 1897. This subsidiary stress, expressed in terms of the gradients of the strain, of course adds on to the main stress expressed in terms of the strain itself, which propagates the waves. Here it depends, as Stokes remarks, on the gradient of rotation, or torsion, in the medium: twist of the isotropic medium round any direction produces an extra tension or thrust along that direction, and another related one transverse to it, and *vice versa*. It alters the waves of permanent type to the two spiral forms. Other strain gradients in addition to twists may be present, but they and their stresses would be optically negligible: it is the rotation of the plane of vibration that shows conspicuously under an analyzer.

This last remark carries the inference that the natural rotational quality of any crystalline medium for wave propagation must arise from a term in the strain-energy which is linear in the rotation and in the strain, so that the permissible types may thence be explored by the usual Action analysis. Cf. on the electric side *Aether and Matter*, § 133, also MacCullagh's speculation as quoted in *Phil. Trans.*, 1894.

PEMBROKE COLLEGE, CAMBRIDGE
December 12*th*, 1848

DEAR THOMSON,

I have hitherto put off writing my paper on Waves, and I fear I shall have no time for it now on account of the Senate-House Examination. However you have one No. of the "Notes on Hydrodynamics" for the next No. of the *Journal*.

I read my paper "On the formation of the central spot of Newton's Rings beyond the critical angle" at the Philos. [Society] last night. If they print it, as I suppose they will, I hope pretty soon to be able to send you a copy.

An article I saw in the *Annales de Chimie* for May (?) last set me to work at the circular polarization of magnetised glass. Setting to work inductively I have arrived at the following equations for the motion of the luminiferous ether—I was going to refer to them, but I have mislaid the paper. I am however nearly sure they were $\left(\xi, \eta, \zeta\right.$ being the displacements, and ∇ meaning $\left.\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$,

$$\frac{d^2\xi}{dt^2} = v^2\nabla\xi + A\frac{d^2}{dt\,dz}\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right),$$

$$\frac{d^2\eta}{dt^2} = v^2\nabla\eta + A\frac{d^2}{dt\,dz}\left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right),$$

$$\frac{d^2\zeta}{dt^2} = v^2\nabla\zeta + A\frac{d^2}{dt\,dz}\left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right),$$

z being in the direction of the line of magnetic force. In these equations ξ, η, ζ are not arbitrary, but connected by the condition of incompressibility

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0.$$

The additional terms multiplied by A correspond to a possible system of pressures and tensions, the expressions for which I cannot exactly write down from memory.

If we suppose that the optical effects of these magnetic forces are superposed, it follows that the rotation of the plane of polarization is proportional to the difference of the potentials at the points where the ray enters and quits the glass.

I have a notion as to a mechanical arrangement which would introduce the d/dt in the A terms; in fact which might be conceived to introduce the terms themselves, but I have got nothing very definite on this score. It may be remarked that

$$\frac{d}{dt}\left(\frac{d\eta}{dz}-\frac{d\zeta}{dy}\right)=-2\omega''', \text{ etc.},$$

ω''' being an angular velocity.

This same method applied to syrop of sugar, etc., gives*

$$\frac{d^2\xi}{dt^2}=v^2\nabla\xi+A\frac{d}{dx}\left(\frac{d\eta}{dz}-\frac{d\zeta}{dy}\right),$$

$$\frac{d^2\eta}{dt^2}=v^2\nabla\eta+A\frac{d}{dy}\left(\frac{d\zeta}{dx}-\frac{d\xi}{dz}\right),$$

$$\frac{d^2\zeta}{dt^2}=v^2\nabla\zeta+A\frac{d}{dz}\left(\frac{d\xi}{dy}-\frac{d\eta}{dx}\right).$$

The A terms here again correspond to a very simple system of pressures and tensions; and the whole turns out to be explicable on the supposition that a torsion produces a pressure or tension in the direction of the axis of torsion, as would be the case for instance with a spiral spring, or a medium composed of an immense number of spiral springs arranged in all directions.

I think you arrived at this conclusion, or something very like it. If so, please let me know, any time before February, that if you have anything to bring forward, I may not get before you, or if you don't bring it forward that I may acknowledge it if I write a paper on the subject.

Perhaps you will be able to come to some definite notion as to a mechanical arrangement which would produce the magnetic terms in the first set of equations.

Yours very truly,

G. G. Stokes.

---

* These are not covariant: they are corrected in the letter following.

PEMBROKE COLLEGE, CAMBRIDGE
December 13th, 1848

DEAR THOMSON,

I have found my papers, and copy the results.

Magnetised glass.

[Equations same as above.]

To the A terms correspond the pressures and tangential forces

$$p_x = p_y = 0,$$

$$p_z = 2A\left(\frac{du}{dy} - \frac{dv}{dx}\right) = 4A\omega'''^*,$$

$$t_{xz} = t_{zx} = A\left(\frac{dv}{dz} - \frac{dw}{dy}\right) = 2A\omega',$$

$$t_{yz} = t_{zy} = A\left(\frac{dw}{dx} - \frac{du}{dz}\right) = 2A\omega'',$$

$$t_{xy} = t_{yx} = 0 \quad \left(u = \frac{d\xi}{dt},\ v = \frac{d\eta}{dt},\ w = \frac{d\zeta}{dt}\right).$$

Syrop of Sugar, etc.

$$\frac{d^2\xi}{dt^2} = v^2\nabla\xi + \left(C\frac{d^2}{dt^2} + D\nabla\right)\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right),$$

$$\frac{d^2\eta}{dt^2} = v^2\nabla\eta + \left(C\frac{d^2}{dt^2} + D\nabla\right)\left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right),$$

$$\frac{d^2\zeta}{dt^2} = v^2\nabla\zeta + \left(C\frac{d^2}{dt^2} + D\nabla\right)\left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right),$$

C and D being arbitrary coefficients. We may make two hypotheses, either that the terms which really exist are

$$C\frac{d^2}{dt^2}\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right), \text{ etc., etc.,}$$

or that they are

$$D\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right), \text{ etc., etc.}$$

---

* The signs of $(\omega', \omega'', \omega''')$ here should be changed.

The latter seems most easily conceivable mechanically, and these terms would be introduced by the system of pressures and tangential forces [in which $A = -D$],

$$p_x = 2A\frac{d\theta_x}{dx},\quad p_y = 2A\frac{d\theta_y}{dy},\quad p_z = 2A\frac{d\theta_z}{dz},$$

$$t_{yz} = t_{zy} = A\left(\frac{d\theta_z}{dy} + \frac{d\theta_y}{dz}\right),$$

$$t_{zx} = t_{xz} = A\left(\frac{d\theta_z}{dx} + \frac{d\theta_x}{dz}\right),$$

$$t_{xy} = t_{yx} = A\left(\frac{d\theta_y}{dx} + \frac{d\theta_x}{dy}\right),$$

where θ_x, θ_y, θ_z are the rotations about x, y, z so that

$$\theta_x = \frac{1}{2}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right),\ \text{etc., etc.}$$

These pressures and tensions again arise from *three principal torsions* in rectangular directions (I call $d\theta_x/dx$ a torsion) each producing a pressure or tension in the direction of the axis of torsion*.

Yours very truly,

G. G. STOKES.

---

* For they are correlative to the stress in an isotropic incompressible elastic solid for which $(\theta_x, \theta_y, \theta_z)$ is the displacement.

CONTENTS.

PAGE

On certain nets of plane curves. By Mr F. P. White, St John's College 1

The conics through fives of six points. By Mr F. P. White, St John's College 11

On the stability of the periodic states of the triode oscillator. By Mr W. M. H. Greaves, St John's College 16

Involutions on a normal quartic curve in space of four dimensions. By Mr C. G. F. James, Trinity College 24

On errors of observation. By Dr W. Burnside, Pembroke College . 26

On the generalisation of a theorem of Steiner. By Prof. H. F. Baker . 28

On an extension of Wallace's pedal property of the circumcircle. By Mr H. W. Richmond, King's College 34

On the real twisted cubic curves which are geodesic upon a quadric surface. By Mr H. W. Richmond, King's College 42

On the number of lines which meet four regions in hyper-space. By R. Vaidyanathaswamy. (Communicated by Mr H. W. Turnbull, Trinity College) 49

The mechanism of the rusting of iron. By Mr U. R. Evans, King's College 54

Angular momentum and electron impact. By P. M. S. Blackett, King's College. (With Four Text-figures) 56

Hankel transforms. By E. C. Titchmarsh. (Communicated by Mr G. H. Hardy) 67

On a modified electrostatic oscillograph. By Dr J. A. Crowther, St John's College. (With One Text-figure) 68

An oscillographic study of the production of X-rays by a Coolidge tube. By Dr J. A. Crowther, St John's College. (With Plates I and II) 70

An early formulation by Stokes of the theories of the rotatory polarizations of light. By Sir Joseph Larmor, Lucasian Professor . . . 76

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PROCEEDINGS
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On Graeffe's Method for Complex Roots of Algebraic Equations.

By Mr S. BRODETSKY, Trinity College, and G. SMEAL.

[*Received* 14 January 1924.]

The only really useful practical method for solving numerical algebraic equations of higher orders, possessing complex roots, is that devised by C. H. Graeffe early in the nineteenth century. When an equation with real coefficients has only one or two pairs of complex roots, the Graeffe process leads to the evaluation of these roots without great labour. If, however, the equation has a number of pairs of complex roots there is considerable difficulty in completing the solution: the moduli of the roots are found easily, but the evaluation of the arguments often leads to long and wearisome calculations. The best method that has yet been suggested for overcoming this difficulty is that by C. Runge (*Praxis der Gleichungen*, Sammlung Schubert). It consists in making a change in the origin of the Argand diagram by shifting it to some other point on the real axis of the original Argand plane. The new moduli and the old moduli of the complex roots can then be used as bipolar coordinates for deducing the complex roots completely: this also checks the real roots.

There is one outstanding difficulty and this is to choose correctly *corresponding* moduli. Thus let A, A' be two points, and let P_1 be at distances r_1, r_1' from A, A'; P_2 at distances r_2, r_2' from A, A'; and so on. If we know only the distances $r_1, r_2, \ldots; r_1', r_2', \ldots$, but do not know which r corresponds to which r', then it may happen that we cannot find the positions of $P_1, P_2, \ldots$ without ambiguity.

Our object here is to modify Runge's suggestion in such a way that this outstanding difficulty disappears. The principle of our method is as follows. If in the above we make the distance AA' less than the smallest difference between two of the moduli $r_1, r_2, r_3, \ldots$, then no confusion can arise in the choice of corresponding pairs of moduli, and the solution can be completed without ambiguity. We therefore suppose that the distance AA' becomes infinitesimal, and develop a process for carrying out the main Graeffe calculation and this subsidiary calculation simultaneously. In this way we get, without great multiplication of labour, all the roots, real and complex, with the real roots verified automatically. Since equations of the eighth degree are of importance in practical aeronautics, and equations of the twelfth degree may need solving, it is clear that the arithmetical process to be described here should be of some value to practical computers.

Let $f(x) = 0$ be an algebraic equation. Put $x = x' - \epsilon$; then when ϵ is very small the equation becomes $f(x') - \epsilon f'(x') = 0$. If now we have the modulus r of any pair of complex roots of the original equation, and the modulus r' of the corresponding roots of the new equation, then since ϵ is small

$$r' - r = \epsilon \cos\theta,$$

where $\pm\theta$ are the arguments of the original pair of roots. In other words, we get

$$\cos\theta = \operatorname*{Lt}_{\epsilon\to 0} \frac{r'-r}{\epsilon}.$$

Real roots are verified by the resulting angle being 0 or π, and their signs are determined in this way.

The actual process would be as follows. Let the equation be

$$p_0 x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots = 0.$$

If we substitute $x' - \epsilon$ for x we get

$$p_0 x'^n - (p_1 + q_1\epsilon)\, x'^{n-1} + (p_2 + q_2\epsilon)\, x'^{n-2} - \ldots = 0,$$

where $q_1 = np_0$, $q_2 = (n-1)\,p_1$, ..., and powers of ϵ above the first are omitted. Multiply this equation by

$$p_0 x'^n + (p_1 + q_1\epsilon)\, x'^{n-1} + (p_2 + q_2\epsilon)\, x'^{n-2} + \ldots = 0.$$

Using detached coefficients, and presupposing alternate + and − signs, we get, for the squares of the roots of the equation in x', the new equation

$$\begin{aligned}&p_0^2,\quad p_1^2 - 2p_0p_2 + 2\epsilon\,(p_1q_1 - p_0q_2),\\ &p_2^2 - 2p_1p_3 + 2p_0p_4 + 2\epsilon\,(p_2q_2 - p_1q_3 - p_3q_1 + p_0q_4),\\ &p_3^2 - 2p_2p_4 + 2p_1p_5 - 2p_0p_6\\ &+ 2\epsilon\,(p_3q_3 - p_2q_4 - p_4q_2 + p_1q_5 + p_5q_1 - p_0q_6),\ \ldots = 0.\end{aligned}$$

We now form the equation whose roots are the squares of the roots of this new equation, and proceed as in the usual Graeffe algorithm.

In practice it is convenient to write at any stage the p's in one line and the q's below the corresponding p's, the ϵ being omitted altogether. As an easy illustration we take an equation occurring in the discussion of the lateral stability of an aeroplane, namely

$$x^4 + 9{\cdot}10x^3 + 5{\cdot}52x^2 + 11{\cdot}26x - 0{\cdot}960 = 0.$$

To four significant figures the work is as follows:

x	1	$-9\cdot10$	$5\cdot52$	$-11\cdot26$	$-0\cdot960$	
		4	$-2\cdot730^1$	$1\cdot104^1$	$-1\cdot126^1$	
x^2	1	$7\cdot177^1$	$-1\cdot7638^2$	$1\cdot3739^2$	$0\cdot9216$	
		$-1\cdot820^1$	$-3\cdot290^1$	$-1\cdot7673^2$	$2\cdot162^1$	
x^4	1	$5\cdot504^3$	$1\cdot1391^4$	$1\cdot9201^4$	$0\cdot8493$	
		$-2\cdot547^3$	$4\cdot202^4$	$-4\cdot088^4$	$3\cdot985^1$	
x^8	1	$3\cdot027^7$	$-8\cdot161^7$	$3\cdot677^8$	$0\cdot7213$	P
		$-2\cdot812^7$	$1\cdot505^9$	$-1\cdot571^9$	$6\cdot769^1$	Q
Q/P		$-0\cdot9290$	$-18\cdot44$	$-4\cdot273$	$93\cdot85$	

We stop at the eighth powers (third stage) where the separation into a linear, a quadratic, and a linear is obvious, so that we have a real root, a pair of complex roots, and then another real root. If the real roots are denoted by x_1, x_4 and the complex roots by $r_2e^{\pm i\theta_2}$, then $x_1^8 = 3\cdot027^7$ and $8x_1^7 = -2\cdot812^7$. The former gives $x_1 = \pm 8\cdot612$; the ratio of the latter to the former gives $x_1 = -8\cdot611$, verifying and determining the sign of the first root.

Now the coefficients of the quadratic equation for the complex roots are

$$3\cdot027^7 - 2\cdot812^7\epsilon, \quad -8\cdot161^7 + 1\cdot505^9\epsilon, \quad 3\cdot677^8 - 1\cdot571^9\epsilon;$$

call these $P_1 + Q_1\epsilon$, $P_2 + Q_2\epsilon$, $P_3 + Q_3\epsilon$. Divide by the first coefficient; we get, omitting ϵ^2, etc.

$$1, \quad \frac{P_2}{P_1}\left(1 + \frac{Q_2}{P_2} - \frac{Q_1}{P_1}.\epsilon\right), \quad \frac{P_3}{P_1}\left(1 + \frac{Q_3}{P_3} - \frac{Q_1}{P_1}.\epsilon\right)$$

as the coefficients in the quadratic equation whose roots are the eighth powers of $r_2e^{\pm i\theta_2}$. Hence

$$r_2^{16} = \frac{P_3}{P_1}, \quad 16\frac{r_2\cos\theta_2}{r_2^2} = \frac{Q_3}{P_3} - \frac{Q_1}{P_1}.$$

In our case we get therefore

$$r_2^{16} = \frac{3\cdot677^8}{3\cdot027^7}, \quad 16\frac{r_2\cos\theta_2}{r_2^2} = -4\cdot273 + 0\cdot929 = -3\cdot344.$$

The former gives $r_2 = 1\cdot169$, $r_2^2 = 1\cdot367$, so that from the latter $r_2\cos\theta_2 = -0\cdot285$. We deduce $\pm r_2 \sin\theta_2 = \pm 1\cdot134$, and the complex roots are therefore $-0\cdot285 \pm 1\cdot134i$.

Finally we have

$$x_4^8 = \frac{0\cdot7213}{3\cdot677^8}, \text{ and } \frac{8}{x_4} = \frac{Q_4}{P_4} - \frac{Q_3}{P_3} = 93\cdot85 + 4\cdot27 = 98\cdot12.$$

The former gives $x_4 = \pm 0\cdot0816$, and the latter gives $x_4 = 0\cdot0815$, verifying this root and giving its sign. We now have the complete solution (cf. Bairstow, *Applied Aerodynamics*, p. 479).

As an example of an equation in which the complex roots are a serious difficulty we now take one of the eighth degree, again one used by Bairstow for aeroplane stability. The equation is

$$x^8 + 20{\cdot}4x^7 + 151{\cdot}3x^6 + 490x^5 + 687x^4 + 719x^3 + 150x^2 + 109x + 6{\cdot}87 = 0.$$

The work is as follows:

x	1	$-2{\cdot}04^{1}$	$1{\cdot}513^{2}$	$-4{\cdot}90^{2}$	$6{\cdot}87^{2}$	$-7{\cdot}19^{2}$	$1{\cdot}50^{2}$	$-1{\cdot}09^{2}$	$6{\cdot}87$
		8	$-1{\cdot}428^{2}$	$9{\cdot}078^{2}$	$-2{\cdot}45^{3}$	$2{\cdot}748^{3}$	$-2{\cdot}157^{3}$	$3{\cdot}00^{2}$	$-1{\cdot}09^{2}$
x^2	1	$1{\cdot}136^{2}$	$4{\cdot}274^{3}$	$6{\cdot}125^{4}$	$-1{\cdot}917^{5}$	$4{\cdot}155^{5}$	$-1{\cdot}248^{5}$	$9{\cdot}820^{3}$	$4{\cdot}720^{1}$
		$-4{\cdot}08^{1}$	$-3{\cdot}233^{3}$	$-7{\cdot}137^{4}$	$-4{\cdot}963^{4}$	$-7{\cdot}099^{5}$	$1{\cdot}999^{5}$	$-3{\cdot}063^{3}$	$-1{\cdot}498^{3}$
x^4	1	$4{\cdot}348^{3}$	$3{\cdot}973^{6}$	$5{\cdot}485^{9}$	$-1{\cdot}522^{10}$	$1{\cdot}260^{11}$	$7{\cdot}397^{9}$	$1{\cdot}082^{8}$	$2{\cdot}228^{3}$
		$-2{\cdot}800^{3}$	$-6{\cdot}527^{6}$	$-9{\cdot}754^{9}$	$1{\cdot}678^{11}$	$-5{\cdot}274^{11}$	$-3{\cdot}284^{10}$	$-4{\cdot}529^{8}$	$-1{\cdot}414^{5}$
x^8	1	$1{\cdot}096^{7}$	$-3{\cdot}194^{13}$	$3{\cdot}021^{19}$	$-1{\cdot}150^{21}$	$1{\cdot}610^{22}$	$2{\cdot}745^{19}$	$1{\cdot}168^{16}$	$4{\cdot}964^{6}$
		$-1{\cdot}130^{7}$	$6{\cdot}401^{13}$	$-1{\cdot}085^{20}$	$3{\cdot}135^{21}$	$-1{\cdot}365^{23}$	$-2{\cdot}575^{20}$	$-9{\cdot}578^{16}$	$-6{\cdot}301^{6}$
x^{16}	1	$1{\cdot}840^{14}$	$3{\cdot}580^{26}$	$9{\cdot}126^{38}$					
		$-3{\cdot}757^{14}$	$-1{\cdot}028^{27}$	$-6{\cdot}556^{39}$					
x^{32}	1	$3{\cdot}314^{28}$	$-2{\cdot}077^{53}$	$8{\cdot}328^{77}$					
		$-1{\cdot}362^{29}$	$2{\cdot}362^{54}$	$-1{\cdot}197^{79}$					

We have clearly one real root, then three pairs of complex roots, and then another real root. Call the roots x_1, $r_2e^{\pm i\theta_2}$, $r_4e^{\pm i\theta_4}$, $r_6e^{\pm i\theta_6}$, x_8.

Our calculations give

$$x_1^{32} = 3{\cdot}314, \quad \frac{32}{x_1} = \frac{Q_1'}{P_1'} = -4{\cdot}110;$$

$$r_2^{64} = \frac{8{\cdot}328^{77}}{3{\cdot}314^{28}}, \quad 64\frac{r_2\cos\theta_2}{r_2^2} = \frac{Q_3'}{P_3'} - \frac{Q_1'}{P_1'} = -14{\cdot}377 + 4{\cdot}110 = -10{\cdot}267;$$

$$r_4^{16} = \frac{1{\cdot}610^{22}}{3{\cdot}021^{19}}, \quad 16\frac{r_4\cos\theta_4}{r_4^2} = \frac{Q_5}{P_5} - \frac{Q_3}{P_3} = -8{\cdot}479 + 3{\cdot}592 = -4{\cdot}887;$$

$$r_6^{16} = \frac{1{\cdot}168^{16}}{1{\cdot}610^{22}}, \quad 16\frac{r_6\cos\theta_6}{r_6^2} = \frac{Q_7}{P_7} - \frac{Q_5}{P_5} = -8{\cdot}200 + 8{\cdot}479 = 0{\cdot}279;$$

$$x_8^{8} = \frac{4{\cdot}964^{6}}{1{\cdot}168^{16}}, \quad \frac{8}{x_8} = \frac{Q_8}{P_8} - \frac{Q_7}{P_7} = -126{\cdot}94 + 8{\cdot}20 = -118{\cdot}74;$$

and we deduce

$$x_1 = -7{\cdot}786; \qquad r_2e^{\pm i\theta_2} = -5{\cdot}610 \pm 1{\cdot}873i;$$

$$r_4e^{\pm i\theta_4} = -0{\cdot}670 \pm 1{\cdot}320i; \quad r_6e^{\pm i\theta_6} = 0{\cdot}002979 \pm 0{\cdot}413i;$$

$$x_8 = -0{\cdot}06736,$$

the real roots x_1, x_8 being verified to three significant figures at least.

A comparison of this work with that given by Bairstow (*ibid.* pp. 553–8) will show that our process is far more straightforward and possesses the advantage of supplying verification of the real roots without reference to the original equation. But the method of this paper also possesses the great advantage that it can be applied to *any* algebraic equation, with *any number* of pairs of complex roots, whereas Bairstow's process is one that can hardly be applied to cases of more than three pairs of complex roots, and even in the case of three pairs it is really quite laborious and complicated. Our method needs no modification or development for four or five or any higher number of pairs of complex roots, and to illustrate this we take the case of an equation of the eighth degree having no real roots, i.e. four pairs of complex roots. The equation is

$$x^8 - x^7 - 11x^6 + 10x^5 + 56x^4 + 62x^3 + 247x^2 + 214x + 158 = 0.$$

The complete solution is as follows:

x	1	1	−11	−10	56	− 62	247	−214	158	
		8	7	−66	−50	224	−186	494	−214	
x^2	1	23	253	714	$-2{\cdot}794^{3}$	$-1{\cdot}606^{4}$	$5{\cdot}217^{4}$	$-3{\cdot}226^{4}$	$2{\cdot}496^{4}$	
		2	38	−736	254	$2{\cdot}920^{4}$	$2{\cdot}548^{4}$	$-4{\cdot}694^{4}$	$-6{\cdot}762^{4}$	
x^4	1	23	$2{\cdot}558^{4}$	$1{\cdot}080^{6}$	$5{\cdot}867^{7}$	$4{\cdot}908^{8}$	$1{\cdot}546^{9}$	$-1{\cdot}564^{9}$	$6{\cdot}230^{8}$	
		16	$5{\cdot}074^{4}$	$2{\cdot}608^{5}$	$-4{\cdot}775^{7}$	$-8{\cdot}092^{8}$	$3{\cdot}425^{9}$	$8{\cdot}812^{9}$	$-3{\cdot}376^{9}$	
x^8	1	$-5{\cdot}063^{4}$	$7{\cdot}220^{8}$	$-1{\cdot}816^{12}$	$2{\cdot}461^{15}$	$5{\cdot}607^{16}$	$3{\cdot}998^{18}$	$5{\cdot}198^{17}$	$3{\cdot}881^{17}$	
		$-1{\cdot}007^{5}$	$2{\cdot}454^{9}$	$-2{\cdot}976^{12}$	$-3{\cdot}779^{15}$	$-1{\cdot}030^{18}$	$-1{\cdot}047^{18}$	$-2{\cdot}139^{19}$	$-4{\cdot}206^{18}$	
x^{16}	1	$1{\cdot}119^{9}$	$3{\cdot}423^{17}$	$-2{\cdot}615^{23}$	$6{\cdot}266^{30}$	$-1{\cdot}654^{34}$	$1{\cdot}593^{37}$	$-2{\cdot}833^{36}$	$1{\cdot}506^{35}$	
		$5{\cdot}289^{9}$	$2{\cdot}869^{18}$	$4{\cdot}280^{24}$	$-2{\cdot}199^{31}$	$-8{\cdot}006^{34}$	$-4{\cdot}926^{36}$	$1{\cdot}221^{37}$	$-3{\cdot}265^{36}$	
x^{32}	1	$5{\cdot}676^{17}$	$1{\cdot}178^{35}$	$-4{\cdot}221^{48}$	$3{\cdot}925^{61}$	$7{\cdot}394^{67}$	$2{\cdot}537^{74}$	$3{\cdot}228^{72}$	$2{\cdot}268^{70}$	P
		$6{\cdot}099^{18}$	$1{\cdot}957^{36}$	$-2{\cdot}314^{49}$	$-2{\cdot}755^{62}$	$3{\cdot}411^{69}$	$-1{\cdot}570^{74}$	$3{\cdot}632^{73}$	$-9{\cdot}834^{71}$	Q
Q/P			16·611		−7·019		−0·6189		−43·359	
$\log P$			35·07121		61·59384		74·40431		70·35565	
$\log r^2$			1·09598		0·82883		0·40033		$\bar{1}{\cdot}87348$	
r^2			12·473		6·7426		2·5139		0·7473	
$2r\cos\theta$			6·475		−4·979		0·5028		−0·9981	
Roots:			$3{\cdot}24 \pm 1{\cdot}404i$;		$-2{\cdot}49 \pm 0{\cdot}735i$;		$0{\cdot}251 \pm 1{\cdot}566i$;		$-0{\cdot}499 \pm 0{\cdot}706i$	

The original equation can be factorised thus

$$(x^2 - 6{\cdot}47x + 12{\cdot}47)\,(x^2 + 4{\cdot}98x + 6{\cdot}74)\,(x^2 - 0{\cdot}503x + 2{\cdot}514)\,(x^2 + 0{\cdot}998x + 0{\cdot}747) = 0.$$

On the Problem of Four Particles. By Mr J. Brill, St John's College.

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The object of the present communication is to give an account of an investigation of the problem of four particles on the lines of my investigation of the problem of three particles, an account of which has been recently published by the Society.

The masses of the four particles are taken as m_1, m_2, m_3, m_4, and they are supposed to instantaneously occupy the positions defined by the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), the system of coordinates being so chosen that the origin coincides with the centre of gravity of the system, and the axes with the principal axes of inertia. We have thus the six conditions

$$\begin{aligned} m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 &= 0, \\ m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 &= 0, \\ m_1z_1 + m_2z_2 + m_3z_3 + m_4z_4 &= 0, \\ m_1y_1z_1 + m_2y_2z_2 + m_3y_3z_3 + m_4y_4z_4 &= 0, \\ m_1z_1x_1 + m_2z_2x_2 + m_3z_3x_3 + m_4z_4x_4 &= 0, \\ m_1x_1y_1 + m_2x_2y_2 + m_3x_3y_3 + m_4x_4y_4 &= 0. \end{aligned}$$

As in the former case, we take θ as the angle made with our axis of z by the invariable line of the system, and ϕ as the angle made by the plane containing the invariable line and the axis of z with the plane of (xz). Also ψ is taken for the angle made by the plane through the invariable line and the axis of z with some fixed plane through the invariable line.

As in the former case, the angular velocities of the axes about themselves are given by the equations

$$\theta_1 = \dot{\theta} \sin \phi + \dot{\psi} \sin \theta \cos \phi, \quad \theta_2 = - \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi,$$
$$\theta_3 = \dot{\psi} \cos \theta - \dot{\phi}.$$

In this case, however, we have

$$2T = 2E + P\theta_1^2 + Q\theta_2^2 + R\theta_3^2 + 2M_1\theta_1 + 2M_2\theta_2 + 2M_3\theta_3,$$

where P, Q, R are the principal moments of inertia, and

$$2E = m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2) + m_3(\dot{x}_3^2 + \dot{y}_3^2) + m_4(\dot{x}_4^2 + \dot{y}_4^2),$$

$$M_1 = m_1(y_1\dot{z}_1 - \dot{y}_1z_1) + m_2(y_2\dot{z}_2 - \dot{y}_2z_2) + m_3(y_3\dot{z}_3 - \dot{y}_3z_3) + m_4(y_4\dot{z}_4 - \dot{y}_4z_4),$$

$$M_2 = m_1(z_1\dot{x}_1 - \dot{z}_1x_1) + m_2(z_2\dot{x}_2 - \dot{z}_2x_2) + m_3(z_3\dot{x}_3 - \dot{z}_3x_3) + m_4(z_4\dot{x}_4 - \dot{z}_4x_4),$$

$$M_3 = m_1(x_1\dot{y}_1 - \dot{x}_1y_1) + m_2(x_2\dot{y}_2 - \dot{x}_2y_2) + m_3(x_3\dot{y}_3 - \dot{x}_3y_3) + m_4(x_4\dot{y}_4 - \dot{x}_4y_4).$$

Taking the three quantities α, β, γ as defined by the equations

$$\alpha^2 = m_1 x_1^2 + m_2 x_2^2 + m_3 x_3^2 + m_4 x_4^2,$$
$$\beta^2 = m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 + m_4 y_4^2,$$
$$\gamma^2 = m_1 z_1^2 + m_2 z_2^2 + m_3 z_3^2 + m_4 z_4^2,$$

so that $\quad P = \beta^2 + \gamma^2, \quad Q = \gamma^2 + \alpha^2, \quad R = \alpha^2 + \beta^2,$

we arrive at the set of relations

$$\dot{x}_r = \alpha\dot{\alpha}\frac{x_r}{\alpha^2} - \frac{1}{2}M_3\frac{y_r}{\beta^2} + \frac{1}{2}M_2\frac{z_r}{\gamma^2}, \quad \dot{y}_r = \beta\dot{\beta}\frac{y_r}{\beta^2} - \frac{1}{2}M_1\frac{z_r}{\gamma^2} + \frac{1}{2}M_3\frac{x_r}{\alpha^2},$$

$$\dot{z}_r = \gamma\dot{\gamma}\frac{z_r}{\gamma^2} - \frac{1}{2}M_2\frac{x_r}{\alpha^2} + \frac{1}{2}M_1\frac{y_r}{\beta^2}, \quad (r = 1, 2, 3, 4).$$

Thus we obtain

$$2E = \dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + \frac{1}{4}\left\{\frac{1}{\alpha^2}(M_2^2 + M_3^2) + \frac{1}{\beta^2}(M_3^2 + M_1^2) + \frac{1}{\gamma^2}(M_1^2 + M_2^2)\right\}.$$

Consequently the expression for $2T$ may be written in the form

$$\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + \frac{1}{4}\left\{M_1^2\frac{(\beta^2-\gamma^2)^2}{\beta^2\gamma^2(\beta^2+\gamma^2)} + M_2^2\frac{(\gamma^2-\alpha^2)^2}{\gamma^2\alpha^2(\gamma^2+\alpha^2)} + M_3^2\frac{(\alpha^2-\beta^2)^2}{\alpha^2\beta^2(\alpha^2+\beta^2)}\right\}$$
$$+ (\beta^2+\gamma^2)\left\{\theta_1 + \frac{M_1}{\beta^2+\gamma^2}\right\}^2 + (\gamma^2+\alpha^2)\left\{\theta_2 + \frac{M_2}{\gamma^2+\alpha^2}\right\}^2$$
$$+ (\alpha^2+\beta^2)\left\{\theta_3 + \frac{M_3}{\alpha^2+\beta^2}\right\}^2.$$

Using the symbol V to denote Hamilton's characteristic function, we have

$$\theta_1 + \frac{M_1}{\beta^2+\gamma^2} = \frac{1}{\beta^2+\gamma^2}\left\{\frac{\partial V}{\partial\theta}\sin\phi + \frac{\cos\phi}{\sin\theta}\left(\frac{\partial V}{\partial\phi}\cos\theta + \frac{\partial V}{\partial\psi}\right)\right\},$$

$$\theta_2 + \frac{M_2}{\gamma^2+\alpha^2} = -\frac{1}{\gamma^2+\alpha^2}\left\{\frac{\partial V}{\partial\theta}\cos\phi - \frac{\sin\phi}{\sin\theta}\left(\frac{\partial V}{\partial\phi}\cos\theta + \frac{\partial V}{\partial\psi}\right)\right\},$$

$$\theta_3 + \frac{M_3}{\alpha^2+\beta^2} = -\frac{1}{\alpha^2+\beta^2}\frac{\partial V}{\partial\phi} \qquad \ldots\ldots(1).$$

Further, we can prove that

$$\alpha^2(y_1 z_2)M_1 dt = -\beta^2\gamma^2(x_2 dx_1 - x_1 dx_2) + \gamma^2\alpha^2(y_2 dy_1 - y_1 dy_2)$$
$$+ \alpha^2\beta^2(z_2 dz_1 - z_1 dz_2),$$

$$\beta^2(z_1 x_2)M_2 dt = \beta^2\gamma^2(x_2 dx_1 - x_1 dx_2) - \gamma^2\alpha^2(y_2 dy_1 - y_1 dy_2)$$
$$+ \alpha^2\beta^2(z_2 dz_1 - z_1 dz_2),$$

$$\gamma^2(x_1 y_2)M_3 dt = \beta^2\gamma^2(x_2 dx_1 - x_1 dx_2) + \gamma^2\alpha^2(y_2 dy_1 - y_1 dy_2)$$
$$- \alpha^2\beta^2(z_2 dz_1 - z_1 dz_2).$$

Writing

$$u = m_1 (m_1 + m_3 + m_4) x_1^2 + m_2 (m_2 + m_3 + m_4) x_2^2 + 2m_1 m_2 x_1 x_2,$$
$$v = m_1 (m_1 + m_3 + m_4) y_1^2 + m_2 (m_2 + m_3 + m_4) y_2^2 + 2m_1 m_2 y_1 y_2,$$
$$w = m_1 (m_1 + m_3 + m_4) z_1^2 + m_2 (m_2 + m_3 + m_4) z_2^2 + 2m_1 m_2 z_1 z_2,$$

we obtain

$$\begin{aligned}(m_3 + m_4)\, \alpha^2\beta^2\gamma^2 &= u\beta^2\gamma^2 + m_1 m_2 (m_1 + m_2 + m_3 + m_4)\, \alpha^2 (y_1 z_2)^2 \\ &= v\gamma^2\alpha^2 + m_1 m_2 (m_1 + m_2 + m_3 + m_4)\, \beta^2 (z_1 x_2)^2 \\ &= w\alpha^2\beta^2 + m_1 m_2 (m_1 + m_2 + m_3 + m_4)\, \gamma^2 (x_1 y_2)^2 \qquad \text{......(2);}\end{aligned}$$

and also $\quad u\beta^2\gamma^2 + v\gamma^2\alpha^2 + w\alpha^2\beta^2 = 2 (m_3 + m_4)\, \alpha^2\beta^2\gamma^2.$

We will now put

$$\frac{k}{u}(x_2 dx_1 - x_1 dx_2) = \frac{d\xi}{1 + \xi^2}, \quad \frac{k}{v}(y_2 dy_1 - y_1 dy_2) = \frac{d\eta}{1 + \eta^2},$$
$$\frac{k}{w}(z_2 dz_1 - z_1 dz_2) = \frac{d\zeta}{1 + \zeta^2},$$

where $\quad k^2 = m_1 m_2 (m_3 + m_4)(m_1 + m_2 + m_3 + m_4).$

We can express the integral form of each of these assumptions in two ways. If we write

$$k\xi = m_1 (m_1 + m_3 + m_4) \frac{x_1}{x_2} + m_1 m_2,$$

$$k\xi' = m_2 (m_2 + m_3 + m_4) \frac{x_2}{x_1} + m_1 m_2,$$

we find that ξ and ξ' are connected by the relation

$$\xi\xi' - \frac{m_1 m_2}{k}(\xi + \xi') - 1 = 0,$$

or its equivalent

$$\tan^{-1}\xi + \tan^{-1}\xi' + \tan^{-1}(k/m_1 m_2) = 0.$$

We also obtain the three relations

$$\frac{u}{\alpha^2(1 + \xi^2)} + \frac{v}{\beta^2(1 + \eta^2)} + \frac{w}{\gamma^2(1 + \zeta^2)} = \frac{k^2}{m_1 m_2 (m_1 + m_2 + m_3 + m_4)},$$

$$\frac{u}{\alpha^2(1 + \xi'^2)} + \frac{v}{\beta^2(1 + \eta'^2)} + \frac{w}{\gamma^2(1 + \zeta'^2)} = \frac{k^2}{m_1 m_2 (m_1 + m_2 + m_3 + m_4)},$$

$$\frac{u}{\alpha^2(\xi + \xi')} + \frac{v}{\beta^2(\eta + \eta')} + \frac{w}{\gamma^2(\zeta + \zeta')} = -\frac{k}{m_1 + m_2 + m_3 + m_4}.$$

The first two of these relations arise from the fact that the particle m_r lies on the ellipsoid*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} + \frac{1}{m_1 + m_2 + m_3 + m_4} = \frac{1}{m_r},$$

and the third from the fact that the tetrahedron, at whose apices the particles are situated, is self-conjugate with respect to the imaginary ellipsoid*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} + \frac{1}{m_1 + m_2 + m_3 + m_4} = 0.$$

Solving the above three equations in u, v, w, and taking account of the relations connecting ξ', η', ζ' with ξ, η, ζ, we obtain

$$\frac{u}{\alpha^2} = (m_3 + m_4) \frac{(1 + \xi^2)(1 + \eta\zeta)}{(\xi - \eta)(\xi - \zeta)}.$$

From this we deduce, by means of equations (2), given above,

$$(y_1 z_2)^2 = \beta^2\gamma^2 \cdot \frac{m_3 + m_4}{m_1 m_2 (m_1 + m_2 + m_3 + m_4)} \cdot \frac{(1 + \zeta\xi)(1 + \xi\eta)}{(\zeta - \xi)(\xi - \eta)},$$

with two similar relations.

Taking account of these results, we eventually obtain three others of the type

$$\frac{1}{4}\left(\frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) M_1 + \theta_1$$

$$= \frac{1}{2\beta\gamma} \left\{\frac{(1 + \zeta\xi)(1 + \xi\eta)}{(\zeta - \xi)(\xi - \eta)}\right\}^{\frac{1}{2}} \left\{\frac{(\eta - \zeta)(\eta - \xi)}{1 + \zeta\xi} \frac{\partial V}{\partial \eta} + \frac{(\zeta - \xi)(\zeta - \eta)}{1 + \xi\eta} \frac{\partial V}{\partial \zeta}\right\}.$$

By subtraction of these equations from equations (1), we obtain values for M_1, M_2, M_3. Since we also have

$$\dot{\alpha} = \frac{\partial V}{\partial \alpha}, \quad \dot{\beta} = \frac{\partial V}{\partial \beta}, \quad \dot{\gamma} = \frac{\partial V}{\partial \gamma},$$

we have sufficient material for the expression of the characteristic equation.

If we consider the problem from the point of view of the determination of the geodesics of a ninefold, we see from the expression for $2T$ that the groundform will be degenerate, i.e. that certain invariants will be evanescent.

* **Supposed instantaneous.**

Canonical Forms of the Quaternary Cubic associated with arbitrary Quadrics. By Mr H. W. TURNBULL, Trinity College.

[*Received* 18 January 1924.]

§ 1. The general quaternary cubic

$$C_3 = a_x^3 = \Sigma a_{ijk} x_i x_j x_k, \quad (i, j, k = 1, 2, 3, 4),$$

can be expressed in various interesting canonical forms involving suitably chosen linear forms X_i, Y_i, A_i. Thus, referred to the pentahedron $X_1 X_2 X_3 X_4 X_5$ of Sylvester* the cubic becomes the sum of five cubes

$$a_1 X_1^3 + a_2 X_2^3 + a_3 X_3^3 + a_4 X_4^3 + a_5 X_5^3 \qquad \ldots\ldots(1)$$

with its Hessian in the form

$$[(a_1 X_1)^{-1} + (a_2 X_2)^{-1} + (a_3 X_3)^{-1} + (a_4 X_4)^{-1} + (a_5 X_5)^{-1}] \Pi a_i X_i \qquad \ldots\ldots(2),$$

where the coefficients a_i may if necessary be taken as equal to unity and the five linear forms X_i each contain four independent parameters, making a total of twenty parameters which is the number of coefficients a_{ijk} in the given cubic C_3.

Other possible forms† of the cubic are, for instance,

$$Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3 + Y_5^3 + Y_6^3, \text{ where } \Sigma Y_i \equiv 0 \quad \ldots(3),$$

and

$$ABC + A'B'C' \qquad \ldots\ldots(4),$$

which explicitly involve six linear forms and are suitable for the study of the twenty-seven straight lines on the cubic surface corresponding to the equation

$$C_3 = 0.$$

§ 2. The present paper proposes other possible forms most of which appear to be new, and all of which may be proved to exist by applying the remarkably powerful method established by the late Mr E. K. Wakeford‡.

This method proves the existence of such canonical forms, but provides no clue either to the number of ways of obtaining each type of form, or to the actual determination of the form. The result may be enunciated as follows:

If Q_i $(i = 1, 2, 3, 4, 5)$ *denote any five arbitrary quaternary quadrics, and if* X_j, K_j *denote properly chosen quaternary linear*

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* *Camb. Math. Journ.* 6 (1851), p. 198: Clebsch, *Crelle*, 59.

† Reye, *Journ. f. Math.* 78 (1874), p. 114.

‡ "On Canonical Forms," *Proc. Lon. Math. Soc.* Ser. 2, Vol. 18 (1920).

forms, then the general quaternary cubic C_3 may be expressed in any of the six canonical forms F_1, F_2, F_3, F_4, F_5, F_6, where

$$\left.\begin{aligned} F_1 &= X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 \\ F_2 &= X_1^3 + X_2^3 + X_3^3 + X_4^3 + K_1Q_1 \\ F_3 &= X_1^3 + X_2^3 + X_3^3 + K_1Q_1 + K_2Q_2 \\ F_4 &= X_1^3 + X_2^3 + K_1Q_1 + K_2Q_2 + K_3Q_3 \\ F_5 &= X_1^3 + K_1Q_1 + K_2Q_2 + K_3Q_3 + K_4Q_4 \\ F_6 &= K_1Q_1 + K_2Q_2 + K_3Q_3 + K_4Q_4 + K_5Q_5 \end{aligned}\right\} \quad \ldots\text{(A)}.$$

There is no direct connection between any of these sets (X, K); for example, X_1 stands for a different linear form in each case. Also each form F requires a separate proof for its justification, although one proof follows naturally from its predecessor in the order cited.

The proof in each case depends on two conditions:

(1) the number of parameters in F must at least equal the number twenty of coefficients in C_3;

(2) the Wakeford apolar cubic Φ must not exist.

The first condition obviously holds since each F has five linear undetermined forms X or K. The proof therefore turns on proving the non-existence of the Wakeford Φ, in each case, where

$$\Phi = \Sigma a_{ijk} u_i u_j u_k, \quad (i, j, k = 1, 2, 3, 4)$$

a cubic in variables (u) contragredient to (x); and *where** *each first polar of Φ, namely $\frac{\partial \Phi}{\partial u_i}$, is inpolar to each first polar of C_3, namely $\frac{\partial C_3}{\partial x_j}$.*

Since the proof that F_1 exists is already known, consider the next form F_2. Here, since each X is general if, as we suppose, C_3 is general, we may make the linear transformation

$$T: \begin{matrix} x_i \to X_i \\ u_i \to U_i \end{matrix} \quad \Sigma u_i x_i \to \Sigma U_i X_i, \quad (i = 1, 2, 3, 4),$$

and write

$$\Phi = \Sigma p_i U_i^3 + 3\Sigma q_{ij} U_i^2 U_j + 6\,(A_1 U_2 U_3 U_4 + A_2 U_1 U_3 U_4 \\ + A_3 U_1 U_2 U_4 + A_4 U_1 U_2 U_3),$$

where now the sixteen conditions—that $\frac{\partial \Phi}{\partial U_i}$ is inpolar to $\frac{\partial F_2}{\partial X_j}$ $(i, j = 1, 2, 3, 4)$—must hold. These conditions are satisfied if and only if all coefficients p, q vanish, as is easily verified.

Again, since $\frac{\partial \Phi}{\partial U_i}$ is also inpolar to $\frac{\partial F_2}{\partial K_1}$, in order to satisfy completely the Wakeford condition, then four more conditions hold, after dropping the terms containing p and q in Φ, namely

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* Cf. Wakeford, *loc. cit.*

$$\begin{aligned} A_2 s_{34} + A_3 s_{42} + A_4 s_{23} &= 0, \\ A_1 s_{34} + A_3 s_{41} + A_4 s_{13} &= 0, \\ A_1 s_{24} + A_2 s_{41} + A_4 s_{12} &= 0, \\ A_1 s_{23} + A_2 s_{31} + A_3 s_{12} &= 0, \end{aligned}$$

where the coefficients s_{ij} are those of the arbitrary quadric

$$Q_1 = \Sigma s_{ij} X_i X_j, \quad (i, j = 1, 2, 3, 4).$$

Whence
$$\Delta_2 \equiv \begin{vmatrix} \cdot & s_{34} & s_{42} & s_{23} \\ s_{34} & \cdot & s_{41} & s_{13} \\ s_{24} & s_{41} & \cdot & s_{12} \\ s_{23} & s_{31} & s_{12} & \cdot \end{vmatrix} = 0.$$

Now Δ_2 does not vanish identically when expanded. Also its elements are rational linear functions both of the coefficients e_{ij} of the transformation T and also of the arbitrary given quadric

$$Q_1 = \Sigma s_{ij} X_i X_j = \Sigma k_{ij} x_i x_j,$$

where the coefficients (k) are determined beforehand. Thus Δ_2 is a function

$$\Delta_2 (e, k)$$

of both sets e and k. Either the vanishing of Δ_2 implies a condition between the k's, which is contrary to the hypothesis that the k's are arbitrary; or, Δ_2 contains a factor involving all the k's which on removal still leaves a condition between the e's. This last also is contrary to hypothesis since it implies a condition between the coefficients of X_1, X_2, X_3, X_4 when expressed in terms of x_1, x_2, x_3, x_4 by the inverse of the transformation T.

Since Φ only exists if Δ_2 vanishes, and Δ_2 has been proved to be non-vanishing, therefore Φ does not exist. Thus the canonical form F_2 is proved to exist.

§ 3. A similar final determinant Δ_3, Δ_4, Δ_5 or Δ_6 emerges for the successive forms F_3, F_4, F_5, F_6. Thus for F_3 we choose a new set of variables X instead of x, where X_1, X_2, X_3 are given by the first three terms of F_3, while X_4 is any independent fourth coordinate. For Φ to be inpolar to $\frac{\partial F_3}{\partial X_i} = X_1^2$, X_2^2 or X_3^2 but *not* X_4^2 we now have

$$p_1 = p_2 = p_3 = p_{ij} = p_{i4} = 0, \quad (i, j = 1, 2, 3, \quad i \neq j).$$

The Wakeford function Φ is thus reduced to

$$p_4 U_4^2 + 3p_{41} U_1 U_4^2 + 3p_{42} U_2 U_4^2 + 3p_{43} U_3 U_4^2 + 6\Sigma A_i U_j U_k U_l,$$
$$(i, j, k, l = 1, 2, 3, 4 \text{ in some order}).$$

So the eight further conditions, for Φ to be inpolar to two quadrics

$$Q_1 = \Sigma r_{ij} X_i X_j, \quad Q_2 = \Sigma s_{ij} X_i X_j,$$

only hold if

$$\Delta_3 \equiv \left| \begin{array}{cccccccc} \cdot & s_{34} & s_{42} & s_{23} & s_{44} & \cdot & \cdot & \cdot \\ s_{34} & \cdot & s_{41} & s_{13} & \cdot & s_{44} & \cdot & \cdot \\ s_{24} & s_{41} & \cdot & s_{12} & \cdot & \cdot & s_{44} & \cdot \\ s_{23} & s_{31} & s_{12} & \cdot & s_{14} & s_{24} & s_{34} & s_{44} \\ \cdot & r_{34} & r_{42} & r_{23} & r_{44} & \cdot & \cdot & \cdot \\ r_{34} & \cdot & r_{41} & r_{13} & \cdot & r_{44} & \cdot & \cdot \\ r_{24} & r_{41} & \cdot & r_{12} & \cdot & \cdot & r_{44} & \cdot \\ r_{23} & r_{31} & r_{12} & \cdot & r_{14} & r_{24} & r_{34} & r_{44} \end{array} \right| = 0.$$

Since the coefficient of r_{44}^4 here is Δ_2 it follows that Δ_3 also does not vanish identically, and as before only vanishes if either the parameters of X_1, X_2, X_3, X_4, or the coefficients of Q_1 and Q_2 in their original forms, are restricted. Hence F_3 exists.

§ 4. Next for F_4, proceeding as before, Φ is inpolar to X_1^2 and X_2^2 but not to X_3^2 and X_4^2, so that

$$p_1 = p_2 = p_{12} = p_{13} = p_{14} = p_{21} = p_{23} = p_{24} = 0.$$

If r, s, t denote the coefficients of the X's in three quadrics Q_1, Q_2, Q_3, the corresponding final determinant Δ_4 is

$$\left| \begin{array}{cccccccccccc} \cdot & r_{34} & r_{42} & r_{23} & r_{33} & \cdot & \cdot & \cdot & r_{44} & \cdot & \cdot & \cdot \\ r_{34} & \cdot & r_{41} & r_{13} & \cdot & r_{33} & \cdot & \cdot & \cdot & r_{44} & \cdot & \cdot \\ r_{24} & r_{41} & \cdot & r_{12} & r_{13} & r_{23} & r_{33} & r_{34} & \cdot & \cdot & r_{44} & \cdot \\ r_{23} & r_{31} & r_{12} & \cdot & \cdot & \cdot & \cdot & r_{33} & r_{14} & r_{24} & r_{34} & r_{44} \\ \cdot & s_{34} & s_{42} & s_{23} & s_{33} & \cdot & \cdot & \cdot & s_{44} & \cdot & \cdot & \cdot \\ s_{34} & \cdot & \multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \vdots & & & & & & & & & & & \\ \cdot & t_{34} & t_{42} & t_{23} & t_{33} & \cdot & \cdot & \cdot & t_{44} & \cdot & \cdot & \cdot \\ \vdots & \multicolumn{11}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ t_{23} & t_{31} & t_{12} & \cdot & \cdot & \cdot & \cdot & t_{33} & t_{14} & t_{24} & t_{34} & t_{44} \end{array} \right|$$

which again does not vanish identically since the coefficient of t_{33}^4 here is Δ_3. Let Δ_4 be symbolized topographically by the array

$$\begin{bmatrix} (r_{ij}) & R_3 & R_4 \\ (s_{ij}) & S_3 & S_4 \\ (t_{ij}) & T_3 & T_4 \end{bmatrix},$$

where R_i denotes an array with zeros everywhere except in the leading diagonal, r_{ii}^4, and in the ith row, and where (s_{ij}) denotes the array of Δ_2. Then the corresponding determinants, Δ_5 for the case of F_5, and Δ_6 for the case of F_6, may be symbolized, when coefficients l, m of new quadrics Q_4 and Q_5 are added, as follows:

$$\Delta_5 = \begin{bmatrix} (r_{ij}) & R_2 & R_3 & R_4 \\ (s_{ij}) & S_2 & S_3 & S_4 \\ (t_{ij}) & T_2 & T_3 & T_4 \\ (l_{ij}) & L_2 & L_3 & L_4 \end{bmatrix}, \qquad \Delta_6 = \begin{bmatrix} (r_{ij}) & R_1 & R_2 & R_3 & R_4 \\ (s_{ij}) & S_1 & S_2 & S_3 & S_4 \\ (t_{ij}) & T_1 & T_2 & T_3 & T_4 \\ (l_{ij}) & L_1 & L_2 & L_3 & L_4 \\ (m_{ij}) & M_1 & M_2 & M_3 & M_4 \end{bmatrix},$$

representing a sixteen row and a twenty row determinant respectively. These are readily established by writing down the inpolar conditions in each case. Also none of the determinants vanishes identically. Their existence proves the existence of each corresponding F form, F_3, F_4, F_5 or F_6 precisely as Δ_2 proved the existence of F_2. Hence results (A) are all possible.

§ 5. *Alternative forms of* F_2. First we notice that

$$G_2 = Y_2 Y_3 Y_4 + Y_1 Y_3 Y_4 + Y_1 Y_2 Y_4 + Y_1 Y_2 Y_3 + KQ_1 \quad \ldots(\text{B})$$

is a possible canonical form analogous to F_2, where Y_1, Y_2, Y_3, Y_4, K are properly chosen linear forms with twenty parameters shared between them equally.

For a similar proof first makes the Wakeford form Φ inpolar to the four

$$\frac{\partial G_2}{\partial Y_i},$$

which implies that

$$\Phi = q_1 V_1^3 + q_2 V_2^3 + q_3 V_3^3 + q_4 V_4^3,$$

where (V) and (Y) are contragredient variables: and next Φ is only inpolar to

$$\frac{\partial G_2}{\partial K} = Q_1 = \Sigma s_{ij}' Y_i Y_j,$$

if
$$q_1 q_2 q_3 q_4 s_{11}' s_{22}' s_{33}' s_{44}' = 0,$$

which as before implies restrictions among independent coefficients or parameters. Thus G_2 exists. This gives an analytical justification for the geometrical result proved by Professor W. P. Milne*; for the four-term series

$$\Sigma Y_i Y_j Y_k$$

equated to zero represents a four-nodal cubic surface containing the common curve of the original cubic ($G_2 = 0$) and quadric ($Q_1 = 0$) surfaces.

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* Cf. "Sextactic Cones and Tritangent Planes," *Proc. Lon. Math. Soc.* Ser. 2, Vol. 21 (1922), p. 375.

§ 6. *The Complete Cubic* $f(\xi_1, \xi_2, \xi_3, \xi_4)$. These forms F_2 and G_2 lead to the conception of a more general form*, namely, that the general cubic may be written as

$$H_2 = f(Z_1, Z_2, Z_3, Z_4) + K''Q \qquad \text{......(C)},$$

where Z_1, Z_2, Z_3, Z_4, K'' are again properly chosen linear forms and where Q is any given quadric, and where the cubic form

$$f(\xi_1, \xi_2, \xi_3, \xi_4) = b_\xi^3$$

is *complete*, in a sense now to be defined.

Let (v) denote variables contragredient to (ξ), and let v_a^3 be the symbolic form of an undetermined cubic form Ψ. Suppose now that the following condition holds:

$$b_a^2 b_\xi v_a = 0 \quad \{\xi, v\},$$

symbolizing that for every value of the set (ξ) and the set (v) a first polar of f is outpolar to a first polar of Ψ. This gives in general sixteen conditions which determine Ψ as one of an ∞^3 linear system

$$\lambda_1\Psi_1 + \lambda_2\Psi_2 + \lambda_3\Psi_3 + \lambda_4\Psi_4 \qquad \text{......(5)}.$$

Let such a cubic f which has this property be called *complete*.

It follows that the same argument which established the form F_2 may be used also for H_2. In fact the Wakeford form Φ exists for the cubic $f(\xi_1, \xi_2, \xi_3, \xi_4)$ and is the form (5). Hence the Wakeford form Φ exists for the assumed form H_2 only if, in addition, every first polar of such a four-parameter (λ) linear set is inpolar to $\frac{\partial H_2}{\partial K''}$, or Q. This leads to four linear homogeneous equations for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, from which by elimination there follows a relation

$$g(e, q) = 0,$$

involving the parameters e of the linear transformation

$$x_i \rightarrow Z_i, \quad (i = 1, 2, 3, 4),$$

and the coefficients q_{ij} of the given quadric Q. As before this is contrary to the hypothesis; hence the form H_2 exists.

§ 7. As to the number of possible reductions all that is immediately known consists in the uniqueness of the first and last forms F_1 and F_6. The last is unique because it is linear in all its twenty parameters in the K's. The first is unique as has been proved elsewhere†. The intermediate cases present great algebraic difficulties.

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* I am indebted to Professor H. F. Baker for this extension of the original theorem, and also for suggesting the verification of G_2 (§ 5).

† H. W. Richmond, *Quarterly Journal of Mathematics* (1902).

§ 8. *Application to related forms.* The case which led me to the preceding investigation was that of the *plane tricircular sextic* whose equation may be written

$$a_z{}^3 a_{z'}{}'^3 \equiv a_{00} z^3 z'^3 + 3a_{01} z^3 z'^2 + \ldots + a_{33} = 0,$$

$$z = x + iy, \quad z' = x - iy,$$

where x, y are rectangular Cartesian coordinates. If this is called the general (3, 3) double binary form in homogeneous variables z_1, z_2, z_1', z_2', where

$$z_1 : z_2 = z \text{ and } z_1' : z_2' = z',$$

then it is in one-one correspondence with the complete intersection of a general cubic surface and a quadric Q_1. Accordingly we may write the cubic in the form

$$F_2 \equiv X_1^3 + X_2^3 + X_3^3 + X_4^3 \bmod Q_1,$$

where each X_i corresponds uniquely with a (1, 1) form

$$C_i = \alpha_i zz' + \beta_i z + \gamma_i z' + \delta_i.$$

Thus we have the theorem:

The general double binary (3, 3) *form is always expressible as the sum of four cubes*:

or

The general tricircular sextic can be expressed in terms of four circles, where if $C_1 = 0$, $C_2 = 0$, $C_3 = 0$, $C_4 = 0$ *are the equations of the circles the equation of the sextic is*

$$C_1^3 + C_2^3 + C_3^3 + C_4^3 = 0.$$

Again, from the G_2 canonical form we infer that the form

$$D_2 D_3 D_4 + D_1 D_3 D_4 + D_1 D_2 D_4 + D_1 D_2 D_3 = 0,$$

where $D_i = 0$ again represents a circle, is always possible for the general tricircular sextic.

§ 9. *Cubic surfaces through the eight common points of three quadrics.* From the form F_4 we derive the canonical form

$$G_4 = K_1 Q_1 + K_2 Q_2 + K_3 Q_3 \qquad \ldots\ldots(6)$$

for the cubic passing through the eight points common to three quadrics Q_1, Q_2, Q_3.

For if we write the cubic in the general form F_4 subject to the eight linear conditions that its coefficients must satisfy for the cubic to contain the common points of

$$Q_1 = Q_2 = Q_3 = 0,$$

then, for the corresponding eight sets of values of the variables (x_i),

$$X_1^3 + X_2^3$$

must vanish. Geometrically this requires *either* that these two terms vanish identically *or* that the eight points in question lie on three planes

$$X_1 + X_2 = 0, \quad X_1 + \omega X_2 = 0, \quad X_1 + \omega^2 X_2 = 0,$$

which have a common line. Seven of the points being arbitrary and no four being coplanar, since Q_1, Q_2, Q_3 are arbitrary, it follows that the points must lie in groups of 3, 3, 2 on these planes. If $u_\alpha, u_\beta, u_\gamma, u_\delta, u_\epsilon, u_\zeta, u_\eta, u_\theta$ denote these eight points, the condition for such a configuration is

$$(\alpha\beta.\gamma\delta\epsilon.\zeta\eta\theta) = 0,$$

that is $$(\alpha\gamma\delta\epsilon)(\beta\zeta\eta\theta) - (\beta\gamma\delta\epsilon)(\alpha\zeta\eta\theta) = 0,$$

where $(\alpha\gamma\delta\epsilon)$ is the determinant $|\alpha_1\gamma_2\delta_3\epsilon_4|$ of the four sets of point coordinates of the points $u_\alpha, u_\gamma, u_\delta, u_\epsilon$.

Writing for simplicity 12345678 for these eight Greek letters $\alpha\ldots\theta$ we may quote a known result* that the line (123, 567) intersects the join of the two points (34, 678) and (45, 812), which is true whatever permutation of 12...8 is taken. Hence if the eight points lie three on a plane π_1, three on π_2, and two on π_3, where π_1, π_2, π_3 have a common line, then letting 123, 567, 48 denote the points in this order, it follows that the four points

$$4,\ 8,\ (34,\ 678) \text{ and } (45,\ 812)$$

must be coplanar. Symbolically the coordinates x_i $(i = 1, 2, 3, 4)$ of these four points are

$$4_i,\ 8_i,\ (3678)\,4_i - (4678)\,3_i \text{ and } (4812)\,5_i - (5812)\,4_i$$

which are only coplanar if

$$(4835)(4678)(4812) = 0,$$

since (4485), for example, vanishes identically.

But this last result only holds if certain four of the original eight points are coplanar, which case is excluded. Hence the planes π_1, π_2, π_3 do not exist and the first alternative alone holds, so that the canonical form (6) is justified.

§ 10. As an example of the negative use of Wakeford's method it may be shown that the cubic form

$$XYZ + K_1Q_1 + K_2Q_2$$

is uncanonical although it contains twenty parameters if the linear forms X, Y, Z, K_1, K_2 are undetermined. For in this case the apolar Φ function can be proved to exist.

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* Cf. Baker, *Principles of Geometry*, Vol. III, p. 155 (Cambridge, 1923).

§ 11. It is interesting to note that in the F_2 case the Hessian of the special cubic form

$$X_1^3 + X_2^3 + X_3^3 + X_4^3$$

breaks up into four linear factors. The converse of this is also true, namely, that the Hessian as given in (2) breaks up into four linear factors if and only if one of the coefficients a_i vanishes.

It is proved by taking the general cubic as

$$Z_1^3 + Z_2^3 + Z_3^3 + Z_4^3 + (b_1Z_1 + b_2Z_2 + b_3Z_3 + b_4Z_4)^3$$

and its Hessian as

$$(b_1^2Z_2Z_3Z_4 + b_2^2Z_1Z_3Z_4 + b_3^2Z_1Z_2Z_4 + b_4^2Z_1Z_2Z_3) \times (b_1Z_1 + b_2Z_2 + b_3Z_3 + b_4Z_4) + Z_1Z_2Z_3Z_4,$$

which is to be resolved into four linear factors as

$$(\Sigma p_iZ_i)(\Sigma q_iZ_i)(\Sigma r_iZ_i)(\Sigma s_iZ_i).$$

In this identity of two forms for the Hessian, put $Z_4 = 0$. Then it follows that, say

$$s_1 = r_2 = q_3 = 1 \text{ and } s_2, s_3, r_1, r_3, q_1, q_2 \text{ vanish.}$$

Thence $\quad p_1 = b_1b_4^2, \quad p_2 = b_2b_4^2, \quad p_3 = b_3b_4^2.$

Substituting these values for p in the identity and then putting $Z_1 = 0$, the result readily follows.

On Protecting and Sensitising Colloidal Sols. By Mr Eric K. Rideal.

[*Read* 4 February 1924.]

Stable colloidal sols are always charged, and disperse systems in water appear in most cases to acquire a constant electrokinetic potential of 70 m.v. When the electrokinetic potential falls to 30 m.v. coagulation commences and the rate of coagulation is, as Hardy first pointed out, most rapid at the isoelectric point. Thus the question on what the stability of a colloidal system rests must ultimately be referred to the magnitude of the electrokinetic potential and the methods by which this is increased or decreased in solution.

The protective action of certain organic substances noted by Faraday was systematically investigated by Zsigmondy (*Kolloidchemie*, p. 175) who determined the gold numbers of various protective agents, the more important being the gums, gelatine, starch, lysalbinic and protalbinic acids and their salts. That the protective action is due to selective adsorption there can be little doubt since a parallelism between the two has been observed by Michaelis (*Zeit. f. Elektroch.* 17, 917, 1914), Kruyt and Duin (*Kolloid chemische Beih.* 5, 287, 1914), Freundlich (*Kapillarchemie*, p. 639) and others. The selective adsorption of a capillary active non-electrolyte does not, however, always result in protection; we find occasionally that an increased susceptibility to all electrolytes, sometimes an increased susceptibility to electrolytes of some particular valency, and even actual precipitation on the addition of the capillary active non-electrolyte results. Thus Freundlich and Rona (*Biochem. Zeit.* 81, 87, 1917) found that a ferric hydroxide sol was made more sensitive to precipitation by electrolytes on the addition of camphor, thymol and the urethanes. Kruyt and Duin (*loc. cit.*) noted that isoamyl alcohol increased the sensitivity of an arsenious sulphide sol to mono and trivalent ions but protected it for divalent ions, whilst Rona and György (*Biochem. Zeit.* 105, 133, 1920) obtained an actual precipitation of kaolin suspensions on the addition of camphor and thymol. Freundlich (*Kapillarchemie*, p. 637) adopts the hypothesis first advanced by W. Ostwald (*Grundriss der Kolloidchemie*, p. 441) to explain the increased sensitisation in which a lowering of the electrokinetic potential is ascribed to a change in the dielectric constant of the medium. If ζ be the electrokinetic potential, K the dielectric constant of the medium, r the radius of the colloid particle of charge e, and δ the thickness of the double layer, then

$$\zeta = \frac{e\delta}{Kr\,(r+\delta)}.$$

Evidently for a constant ζ value a decrease in K must be associated with a decrease in e involving a smaller adsorption of ions to reduce the electrokinetic potential to the critical coagulating value. Since the mobility of a colloid particle under the influence of a potential fall of V volts is given by the relationship

$$u = \frac{\zeta V K}{4\pi\eta},$$

a decrease in K will result in a decrease in u. Freundlich cites in favour of this hypothesis the following experimental values for a ferric hydroxide sol:

Pure sol	$u = 0{\cdot}4$	10^{-4} cms. per sec.
+ 5 millimols camphor	$u = 0{\cdot}2$	10^{-4} ,, ,,
+ 2·5 millimols thymol	$u = 0{\cdot}3$	10^{-4} ,, ,,

This hypothesis is evidently untenable since a concentration of 5 millimols of camphor per litre of water of S.I.C. of 81 corresponds to a dilution with only 1 part in 10,000 of a substance of S.I.C. of ca. 10 and a scarcely perceptible alteration in the S.I.C. would result. The fact, however, that the mobility is affected by such a small addition is an indication that if K be not altered, a reduction in the electrokinetic potential ζ must have taken place.

There are two considerations which must not be forgotten in any discussion on the mechanism of protection. In a number of cases the protective colloid is an amphoteric electrolyte, e.g. gelatine, casein, albumen. Gann (*Kolloid chemische Beihefte* 8, 252, 1916) noted that protective substances containing a basic nitrogen group exerted a precipitating action on an acid red gold sol determining the "Umschlag" numbers for a series of such colloids. Thus the protective action of a colloid may vary with the P_H of the medium.

A solution of Zsigmondy's red gold sol was prepared and to portions of 5 c.c. each in well steamed test-tubes, equal quantities of a very dilute gelatine solution were added, the gelatine being previously rendered alkaline or acid by the addition of hydrochloric acid or caustic soda. The addition of gelatine on the alkaline side of P_H 5 rendered the gold sol stable towards a 10 per cent. sodium chloride solution, i.e. the gold had become well protected. On the acid side of P_H 4 a definite colour change from the red to the violet occurred, indicating that the gold had been partly precipitated by the acid gelatine. Since the isoelectric point of gelatine is 4·7 it is evident that the addition of neutral gelatine and gelatinate anions protects the negatively charged gold sol, whilst the acid and kationic gelatine hydrochloride causes the mutual precipitation of oppositely charged colloids. It would, indeed, appear possible to determine the isoelectric points of other amino acids by an extension and amplification of this experiment.

Recently Kermack and Wright (*Biochem. J.* 17, 635, 1923) have published very similar observations on the action of gelatine on gum benzoin sols dispersed in media of varying P_H and were able to observe the isoelectric point of $P_H = 4{\cdot}7$ by the change in protective power. The failure of Gann to observe this change in precipitating and protecting power is due to the fact that he employed in all cases a red gold sol distinctly acid in reaction. Thus the addition of salts such as magnesium chloride to a protected sol may, owing to a change in the P_H of the medium, cause an alteration in the properties of such amphoteric protecting agents, which are evidently associated with a marked change in protective power.

In addition to the amphoteric amino acids a number of protective agents possess properties akin to the soaps in that there is in solution a ternary equilibrium of the type

$$\text{colloid} \rightleftarrows \text{ionic micelle} \rightleftarrows \text{ion}.$$

Zsigmondy had already noted that the protective action of soap for gold decreases with elevation of the temperature, and the experiments of Bircumshaw and Rideal (*Journ. Chem. Soc.* 1566, 123, 1922) on the protective action of soap in water alcohol mixtures indicated quite definitely that the ionic or molecular soap was not a protective agent and that protection was determined by the presence of colloidal soap in the solution. Owing to the existence of a dynamic equilibrium between the colloidal, micellar and ionic forms the protective action will be diminished on diminution of the colloidal form, i.e. by the addition of alkali, whilst an increase in protection results on the addition of neutral salts which increase the colloidal constituent.

Thus the addition of various salts to a protected colloid may exhibit anomalous results in protection or sensitising due to a change in either the nature or the quantity of the protective agent caused by the salt solution. Apart from these secondary effects, however, we would anticipate relatively marked changes in ionic susceptibility when we consider the mechanism of protection. In general the suspensoid colloids are highly susceptible to electrolytes. On addition of an electrolyte to a suspensoid both ions of the electrolyte are adsorbed but one ion, the precipitating ion, is adsorbed more strongly and it is this excess of one ion which lowers the electrokinetic potential and produces coagulation; thus the Hoffmeister or lyotropic series of precipitation by electrolytic kations of a negative colloid will vary if different anions are employed. We find, for example, that the electrokinetic potential of a platinum sol is reduced to below its critical value on the addition of 2·5 millimols of sodium chloride, but 130 millimols of caustic soda are required to effect the same reduction.

On addition of a protective colloid to colloidal gold, the adsorptive power of the gold for ions is reduced and it is only when the protective agent is strongly adsorbed and added in large quantities that the protected colloid will acquire all the properties of the protective agent, which will naturally possess a different Hoffmeister series to that of the gold and in general be less sensitive to electrolytes. But a feebly adsorbed protective agent or one added in small quantities does not reduce the adsorption power of the gold to negligible values. If we imagine a gold sol thus feebly protected immersed in a dilute solution of hydrochloric acid the gold itself preferentially adsorbs hydrogen ions and is precipitated. If the protective colloid cuts down the adsorption of the hydrogen and chlorine ions equally, then the gold will be rendered more stable, also if the protective colloid cuts down the adsorption of the hydrogen ions more than the chlorine ions enhanced stability will result, but if the colloid cuts down the chlorine ion adsorption more than the hydrogen ion the sol will be rendered less stable and thus sensitised. We find that colloidal silver is precipitated by the following concentration of kations, Ba : Na : H, 473 : 6·1 : 0·16, but mastic, a protective colloid, possesses the following values: H : Ba : Na, 100 : 40 : 1. Evidently the silver sol on the addition of mastic would be protected for hydrogen ions, slightly sensitised for sodium ions but made relatively more susceptive to precipitation by the divalent barium ions.

The Equation of State of a Gas. By J. E. JONES, 1851 Exhibition Senior Research Student, Trinity College, Cambridge. (Communicated by Mr R. H. FOWLER.)

[*Received* 12 February 1924.]

§ 1. Most of the methods which have been devised to give an equation of state of more generality than that of Van der Waals differ from the original method used by him in that they refer only to the uniform conditions in the interior of the gas while his had special reference to the conditions at the boundary. In fact, one of the corrections to the perfect gas law introduced by him is due entirely to the existence of a boundary field of force. The question arises as to what physical interpretation is to be given to the more general equation. The methods previously employed leave the interpretation obscure. In this note two new methods of obtaining the equation of state are given, one applicable to the interior of a gas, the other to the boundary. These methods seem to have certain advantages over those previously given in that they lend themselves to a very simple physical interpretation. While the pressure at the boundary of a gas is the same as that in the interior, the word *pressure* has a different meaning in the two cases. At the boundary, it is due entirely to the motion of the molecules, whereas in the interior part only is due to the motion of the molecules; this part is the same whatever the nature of the molecules and is in fact given by the perfect gas law. The remaining part is due to the stress set up by the existence of intermolecular fields and, although at any given point it is a fluctuating function of the time, it has everywhere within the gas the same statistical mean value. In this paper, it is referred to as the *statical* pressure to distinguish it from the more usually understood *dynamical* or *momentum* pressure.

§ 2. According to the method used by Van der Waals in obtaining his equation of state, the 'a' is a measure of the cohesive force brought into play at the boundary owing to the attractive fields of the molecules in the interior of the gas, while the 'b' represents the effective diminution of volume owing to the finite size of the molecules. The pressure calculated purports to be the pressure actually exerted on the walls of the containing vessel and so gives the rate of change of momentum of the molecules colliding with unit area of the boundary in unit time. It may accordingly be described as a *momentum* pressure. The term a/v^2 is on this view the diminution of momentum pressure owing to the loss of momentum of the molecules in moving through the attractive field at the boundary.

The simple physical arguments used by Van der Waals are suitable only for his special model of an attracting rigid sphere and his method does not lend itself readily to generalisation. The equations of state for molecular models of more generality have been derived by essentially different methods. There is, for instance. the method of Clausius* who established the well-known Virial Theorem. This gives a simple relation between the time average of the kinetic energy of the molecules in a certain volume of gas and the time average of the forces acting on those molecules. In this way he obtained an expression for the pressure in terms of the kinetic energy and the intermolecular forces. As already indicated the virial theorem was applied by Clausius only to the interior of a gas.

Another method has been developed by Keesom†, based on a determination of the entropy in a way first suggested by Boltzmann‡. This depends on a calculation of the most probable distribution of velocity and spatial coordinates of the molecules. From the expression for entropy a formula for pressure is easily deduced.

Still another treatment of the problem has recently been given by Core§ who, by finding the value of the molecular concentration at points where the potential due to the molecular fields is zero, is able virtually to calculate the effective diminution of volume on account of the field of force surrounding each molecule. It may be regarded as a generalisation of Van der Waals' 'b.'

All these methods lead, of course, to the same result, which is usually expressed in the form

$$pv = kNT + \frac{B}{v},$$

where p and v denote pressure and volume respectively, k is the usual Boltzmann constant, N the total number of molecules in the gas and T the temperature. The quantity B is a function of temperature depending on the molecular model and is referred to as the second virial coefficient. None of the methods makes clear what physical meaning is to be attached to B. It should be added here that the above equation is, like the equation of van der Waals, only a first approximation and true only for gases of moderately large dilution.

In this paper it is shown that the quantity $\frac{B}{v^2}$ in the interior of a gas is the stress across unit area due to the interaction of the gas on the two sides of the area, and exists apart from the motion of the molecules. If all the molecules of the gas were suddenly brought to rest, there would still be a stress at every point in the gas. For this reason, we refer to $\frac{B}{v^2}$ as a *statical* stress as distinct from the momentum or dynamic pressure a/v^2 of van der Waals' equation. By the consideration of a statistical distribution of molecules, it is here shown how a direct calculation of B can be made as a function of the temperature. The result agrees with that of the writers cited.

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* Clausius, *Phil. Mag.* August 1870; Jeans, *Dynamical Theory of Gases*, 1921, p. 129.

† Keesom, *Communications from the Phys. Lab., Leiden*, Supp. 24, 1912.

‡ Boltzmann, *Gastheorie*, II, § 61.

§ Core, *Phil. Mag.* August 1923, p. 256.

It is also shown here that the pressure at the boundary can be calculated even for molecular models of more generality than that of van der Waals. This is done by calculating the force on any selected molecule near the boundary due to an average statistical distribution of the molecules in the interior and introducing the expression so obtained into the Clausius Virial. The method of calculating the force is the same as that used by Fowler* in his calculation of the van der Waals' 'a' in terms of the attractive field surrounding a rigid, spherical molecule. The same expression for the pressure is obtained as before, but in this case $\frac{B}{v^2}$ represents the diminution of the momentum pressure at the boundary whereas in the interior it represents a statical stress.

Fowler's work, just referred to, has led Core† to inquire how the second virial coefficient B can best be divided into parts corresponding to the 'a' and 'b' of van der Waals' equation. In the light of the interpretation of B given in this paper, it is evident that 'a' and 'b' are respectively the contributions to the statical stress of the attractive and repulsive parts of the field surrounding a van der Waals' molecule. For it is to be remembered always that van der Waals' model of a molecule is a very special and localised kind. A rigid sphere is merely a repulsive field of infinite magnitude concentrated within a thin spherical shell, so that the repulsive and attractive parts of the van der Waals' model occupy separate compartments of space. It is not surprising therefore that their respective contributions to the statical stress should be separable. It is not to be expected that in other models the effects of the repulsive and attractive parts of the field can be so distinguished, and any division of B must accordingly be of an artificial character.

§ 3. In an ideal gas, the pressure is everywhere given by $p = k\nu T$ and is a momentum pressure, being due entirely to the motion of the molecules. In an imperfect gas, this pressure still exists and is independent of the nature of the molecules. It is the pressure which would be exerted by the impacts of the molecules on each unit area of a membrane in the interior of the gas, if the membrane were of such a character as to be able to transmit and be itself unaffected by the intermolecular fields. There is however a stress across the membrane owing to the interaction of the molecules on its two sides. The magnitude of this stress will depend on the arrangement (or complexion) of the molecules on each side of the membrane, and so will be a fluctuating function of the time, but the average value can clearly be calculated by assuming, as usual, a statistical distribution of the molecules. If the gas on one side were removed, the gas on the other side would shrink under the influence of the boundary field thus created. It could only be kept in the same state by introducing a system of forces across the membrane. These forces accordingly constitute the contribution to the pressure which we have referred to as the *statical* pressure. It will be denoted by p_s while the dynamical or *momentum* pressure will be denoted by p_m. The stress p_s will be of the same sign as p when the force between molecules is one of repulsion. We now proceed to a direct calculation of p_s.

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* Fowler, *Phil. Mag.* May 1922, p. 785.

† Core, *Phil. Mag.* March 1923, p. 622.

We suppose that the molecules are point centres of force, which repel each other according to the law $\phi(r)$. The distribution of the molecules about any selected molecule is then given by the Boltzmann formula

$$\nu e^{-2j\int_r^\infty \phi(r)\,dr},$$

where* $2j = 1/kT$ and ν is the average molecular concentration. It will be convenient to write

$$\pi(r) = -\int_r^\infty \phi(r)\,dr,$$

so that the distribution is given by $\nu e^{2j\pi(r)}$.

Consider the force due to the molecules contained in an infinite slab of thickness df on a selected molecule at a distance z from it. Using the above law of molecular distribution, it is easy to show† that the force is given by

$$dF = \frac{2\pi\nu z\,df}{2j}(e^{2j\pi(z)} - 1).$$

In order to calculate the stress across a membrane we suppose the slab to be at a distance f to the left while the selected molecule is at a distance g to the right. Hence the force on the selected molecule due to the whole of the gas to the left is given by

$$F = \frac{2\pi}{2j}\int_0^\infty \nu(g+f)\{e^{2j\pi(g+f)} - 1\}\,df$$

$$= \frac{\pi}{j}\int_g^\infty \nu u\{e^{2j\pi(u)} - 1\}\,du.$$

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* I adopt Fowler's suggestion (*Phil. Mag.* 43, p. 791, 1922) and use $2j$ here instead of the more usual $2h$.

† Fowler, *loc. cit.* p. 792.

It follows that the molecular force per unit area between the gas to the left and that to the right is given by

$$p_s = \int_0^\infty \nu F dg = \frac{\pi}{j}\int_0^\infty \nu dg \int_g^\infty \nu u \{e^{2j\pi(u)} - 1\}\, du,$$

or, if the average molecular concentration ν be everywhere the same,

$$p_s = \frac{\pi\nu^2}{j}\int_0^\infty dg \int_g^\infty u\, \{e^{2j\pi(u)} - 1\}\, du.$$

Integrating by parts, we have

$$p_s = \frac{\pi\nu^2}{j}\left[g\int_g^\infty u\, \{e^{2j\pi(u)} - 1\}\, du\right]_0^\infty + \frac{\pi\nu^2}{j}\int_0^\infty g^2\,(e^{2j\pi(g)} - 1)\, dg.$$

In order that p_s shall remain finite, the first term must vanish at the upper limit. This places a restriction on the kind of model permissible. For instance, if the force is according to an inverse power of the distance (r^{-s}), then $s \nless 4$. We may write finally

$$p_s = \frac{\pi\nu^2}{j}\int_0^\infty g^2\,(e^{2j\pi(g)} - 1)\, dg,$$

and the pressure is given by

$$\begin{aligned} p &= p_m + p_s \\ &= k\nu T + \frac{\pi\nu^2}{j}\int_0^\infty g^2\,(e^{2j\pi(g)} - 1)\, dg, \end{aligned}$$

or, in the usual form, $$pv = kNT + \frac{B}{v},$$

where $$B = \frac{\pi N^2}{j}\int_0^\infty g^2\,(e^{2j\pi(g)} - 1)\, dg.$$

Here N is the total number of molecules in the volume v. This result for B is the same as that obtained by other writers.

§ 4. The effect of the intermolecular fields on the pressure at the boundary can also be calculated. For this purpose we make use of the Clausius virial. The usual treatment by means of the virial (for the gas in the interior) involves, as Jeans* has pointed out, the assumption that the pressure is the same at every point of the containing vessel. Each molecule is virtually regarded as subject to the pressure p but there is an element of vagueness in this since p is a statistical effect, and Clausius' equation requires the forces on the individual molecules. The effect of the boundary is here introduced in a new way.

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* Jeans, *Dynamical Theory of Gases*, p. 130.

According to the theorem of Clausius, the kinetic energy of the molecules in any volume of gas is on the average during a long period of time given by

$$\tfrac{1}{2}\Sigma mc^2 = -\tfrac{1}{2}\Sigma(xX + yY + zZ).$$

In this equation X, Y, Z are the components of force acting on a molecule of the gas. We here apply this relation to a narrow strip of gas at the boundary. The only forces to be considered are the boundary forces and the intermolecular forces.

The force brought into play near the boundary will depend on the nature of the molecules of the gas and the nature of the boundary. It is usual to calculate its effect by assuming the molecules to be rigid spheres and the boundary a perfectly plane surface. We know as a fact that neither of these assumptions is correct. The molecules and the boundary both consist of electrical charges and the forces are of a complicated nature. However it is really simpler to consider the effect of a force which varies as the distance than one arising from impulses, for then the effect on neighbouring molecules is represented by a continuous function. We may legitimately assume that this force rapidly diminishes within distances comparable with the size of a molecule, so that we may safely regard the boundary as wholly plane.

Suppose that the force on a molecule due to the boundary at a distance ξ from it is represented by $\psi(\xi)$. Due to an element ds of the boundary, there will accordingly be a total force on the gas of

$$ds\int_0^\tau \nu\psi(\xi)\,d\xi,$$

where τ is a distance beyond which the boundary force may be neglected. We may suppose the strip of gas under consideration at the boundary is greater than τ, but still small on ordinary standards of measurement. We denote it by δ. Since action and reaction are equal and opposite, it follows that if p is the force per unit area on the boundary

$$p = \int_0^\tau \nu\psi(\xi)\,d\xi.$$

The molecular concentration ν fluctuates at any one point, so that p will likewise be a fluctuating function, but in practice only its mean value over a long period of time is measured and this can be calculated by considering the mean value of ν. We now calculate the contribution of these forces to the virial.

Let $dxdydz$ be an element of volume such that the value of the force $\psi(\xi)$ is sensibly uniform throughout it. Then the force on the molecules in this element due to the boundary is $\nu dxdydz\psi(\xi)$ and hence the contribution to the virial is

$$\tfrac{1}{2}\nu dxdydz\,\{xl + ym + zn\}\,\psi(\xi),$$

where l, m, n are the direction cosines of the outward drawn normal; the sign is positive because the force $\psi(\xi)$ is directed along the inward drawn normal.

It is more convenient to use coordinates other than x, y, z. Consider the element of volume enclosed by two surfaces each parallel to the boundary, $d\varpi$ apart, and a cylinder normal to the boundary of cross sectional area $d\sigma$. Then instead of $dxdydz$ we write $d\varpi d\sigma$ and instead of $xl + ym + zn$ we write ϖ, where ϖ is the perpendicular from the origin of coordinates to the tangent plane at $d\sigma$ of the inner parallel surface. The contribution to the virial can now be written

$$\frac{\nu}{2} \varpi \psi(\xi)\, d\varpi\, d\sigma,$$

and the integral of this throughout the narrow strip of gas of depth δ gives the total contribution to the virial. The integral with respect to σ is easily evaluated, for $\psi(\xi)$ and ν both remain constant, and so the contribution of the boundary force to the virial is

$$\tfrac{3}{2} \int \nu V \psi(\xi)\, d\xi,$$

where V is the volume enclosed by the surface σ.

The volume V is a function of ξ, but actually the distance at which the boundary forces are effective is so small on ordinary units of measurement, that the value of the above integral arises from parts of the gas which are very close indeed to the boundary, i.e. for small values of ξ. Hence when V varies appreciably, the other factor of the integrand $\psi(\xi)$ is negligibly small, and when $\psi(\xi)$ is appreciable, V may be regarded as constant and equal to the whole volume of the gas v. Hence approximately we may write

$$\begin{aligned} + \tfrac{3}{2} \int \nu V \psi(\xi)\, d\xi &= + \tfrac{3}{2} v \int \nu \psi(\xi)\, d\xi \\ &= + \tfrac{3}{2} pv, \end{aligned}$$

using a former result. This is the same result as that obtained by Clausius by a less precise method.

We next consider the effect of intermolecular forces at the boundary. From considerations of symmetry, we infer that the unbalanced force due to the molecules in the interior of the gas will act along the normal to the boundary, which again we may regard as plane. The force acting on a molecule at a distance ξ from the boundary due to an infinite plane slab of thickness $d\eta$ which is at a distance η from the boundary and parallel to it is*

$$dF = \frac{2\pi \nu u d\xi}{2j} \{e^{2j\pi(u)} - 1\},$$

where $u = \eta - \xi$.

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* Fowler, *loc. cit.*

We have supposed above that the thickness of the strip of gas under consideration is large compared with the distance at which the force due to the bounding wall is effective while still small on ordinary scales of measurements. Without interfering in any way with these conditions we may suppose that its thickness is also large compared with the distance at which the molecular force $\phi(r)$ is appreciable.

The effective force on a molecule at ξ is

$$F_\xi = \int_0^\infty \frac{2\pi\nu u}{2j}\{e^{2j\pi(u)} - 1\}\,du - \int_0^\xi \frac{2\pi\nu u}{2j}\{e^{2j\pi(u)} - 1\}\,du.$$

The first term represents the force due to the molecules further from the boundary than ξ, the second term that due to the molecules nearer to the boundary. Hence

$$\begin{aligned} F_\xi &= \int_\xi^\infty \frac{2\pi\nu u}{2j}\{e^{2j\pi(u)} - 1\}\,du \\ &= \chi(\xi), \text{ say.} \end{aligned}$$

As in the above work on the effect of boundary forces, we find the contribution to the virial to be

$$\begin{aligned} &-\frac{3}{2}\int_0^\delta \nu V\chi(\xi)\,d\xi \\ &= -\frac{3}{2}v\int_0^\delta \nu\chi(\xi)\,d\xi. \end{aligned}$$

By definition of δ, the force $\chi(\xi)$ is negligible for values of ξ greater than δ and we may accordingly replace the upper limit of the last integral by infinity, thus getting

$$-\frac{3}{2}\frac{\pi v}{j}\int_0^\infty \nu\,d\xi\int_\xi^\infty \nu u\{e^{2j\pi(u)} - 1\}\,du.$$

Fowler* has shown that in calculating the boundary field, correct to the first order, the variation of the molecular concentration in the neighbourhood of the boundary may be neglected. Regarding then ν as constant and integrating by parts, we find that the above expression becomes

$$-\frac{3}{2}\frac{\pi\nu v^2}{j}\int_0^\infty \xi^2\,(e^{2j\pi(\xi)} - 1)\,d\xi,$$

and so the virial relation

$$\tfrac{1}{2}\Sigma\, mc^2 = -\tfrac{1}{2}\Sigma\,(xX + yY + zZ)$$

leads to $$pv = NkT + \frac{\pi N^2}{vj}\int_0^\infty \xi^2\,(e^{2j\pi(\xi)} - 1)\,d\xi,$$

the same result as before.

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* Fowler, *loc. cit.* p. 794.

On the Differential Invariants of the Laguerre Group. By TADAHIKO KUBOTA. (Communicated by Professor H. F. BAKER.)

[*Received* 16 February 1924.]

1. In a paper "Beiträge zur Inversionsgeometrie," which will appear in the forthcoming volume of the *Science Reports of the Tôhoku Imperial University,* I have treated the problem of determining the necessary and sufficient condition in order that two plane curves which are given by natural equations should be transformable into each other by inversion. In connection with that paper I propose here to treat the analogous problem for Laguerre transformations:

Having given two plane curves, by natural equations, in which the functional relations are all supposed to be analytic, it is required to determine the necessary and sufficient condition in order that the two plane curves should be transformable into each other by a Laguerre transformation.

2. Let us consider two continuous series of ∞^1 oriented circles; it is required to find the necessary and sufficient condition in order that these two continuous series of oriented circles should be transformable into each other.

In treating this problem for those series of oriented circles defined by analytic relations we may suppose without any loss of generality that no circles of infinitely great radius are present. If this be not the case then we may divide the continuous series of oriented circles into portions such that in each portion the above condition is satisfied. The curve which is described by the centre of the oriented circle in this series we may suppose to be given by the equations

$$\xi = x(s), \quad \eta = y(s),$$

where x, y denote analytic functions of the curve length s; and the radius of the oriented circle is also given as an analytic function of the curve length s.

Let the point (ξ, η, ir) correspond to the oriented circle on the $\xi\eta$ plane having the point (ξ, η) as its centre and r as its radius. This correspondence is nothing but Lie's method of minimal projection. Then to the continuous series of oriented circles

$$\xi = x(s), \quad \eta = y(s), \quad r = r(s),$$

there corresponds a curve in space given by

$$\xi = x(s), \quad \eta = y(s), \quad \zeta = ir(s).$$

Similarly we may make correspond to the second series of oriented circles defined by

$$\xi = \bar{x}(\bar{s}), \quad \eta = \bar{y}(\bar{s}), \quad r = \bar{r}(\bar{s}),$$

the second curve in space given by

$$\xi = \bar{x}(\bar{s}),\quad \eta = \bar{y}(\bar{s}),\quad \zeta = i\bar{r}(\bar{s}).$$

It is already known that by the above correspondence there corresponds to a Laguerre transformation on the ζ plane a motion. or a motion compounded with a reflexion with respect to a plane. According to Prof. W. Blaschke's proposal[*] let us call the Laguerre transformation of the former class a *proper Laguerre transformation*. and that of the latter class an *improper Laguerre transformation*. In order that the two continuous series of ∞^1 oriented circles should be transformable into each other by a proper or an improper Laguerre transformation it is necessary and sufficient that the two curves in space corresponding to them should be transformable into each other by a motion, or by a motion compounded with a reflexion with respect to a plane.

In this article I will treat only the case where

$$x'(s)^2 + y'(s)^2 - r'(s)^2$$

does not vanish identically, while the other case will be treated in the next article.

In order that the two curves in space which are not minimal curves should be transformable into each other by a motion it is necessary and sufficient that at the corresponding points such that

$$\int\sqrt{x'(s)^2 + y'(s)^2 - r'(s)^2}\,.\,ds = \int\sqrt{\bar{x}'(\bar{s})^2 + \bar{y}'(\bar{s})^2 - \bar{r}'(\bar{s})^2}\,.\,d\bar{s}$$

holds, the two functions

$$f(s) = \{(x''^2 + y''^2 - r''^2)(x'^2 + y'^2 - r'^2) - (x'x'' + y'y'' - r'r'')^2\} \div (x'^2 + y'^2 - r'^2)^3$$

and

$$F(s) = \begin{vmatrix} x' & y' & r' \\ x'' & y'' & r'' \\ x''' & y''' & r''' \end{vmatrix} \div (x'^2 + y'^2 - r'^2)^3,$$

and the corresponding functions $\bar{f}(\bar{s})$ and $\bar{F}(\bar{s})$ for the second space curve should be respectively equal, that is

$$f(s) = \bar{f}(\bar{s}),\quad F(s) = \bar{F}(\bar{s})$$[†].

If we denote the radius of curvature of the curve of centres by $R(s)$, then

$$f(s) = \{(1 - r''^2R^2)(1 - r'^2) - r'^2r''^2\} \div R^2(1 - r'^2)^3,$$
$$F(s) = \{r' + r''RR' + r'''R^2\} \div R^3(1 - r'^2)^3.$$

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[*] W. Blaschke, "Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene," *Monatshefte für Math. u. Phys.* 21, 1910.

[†] G. Scheffers, *Einführung in die Theorie der Curven in der Ebene und im Raume*, 1901, S. 208.

Similarly we can form analogous functions $\bar{f}(\bar{s})$ and $\bar{F}(\bar{s})$ for the second series of oriented circles. Consequently we get the following result:

In order that the two continuous series of oriented circles should be transformable into each other by a proper Laguerre transformation it is necessary and sufficient that the relations

$$f(s) = \bar{f}(\bar{s}) \text{ and } F(s) = \bar{F}(\bar{s})$$

hold at the corresponding points such that

$$\int^{s} \sqrt{x'(s)^2 + y'(s)^2 - r'(s)^2}\, ds = \int^{\bar{s}} \sqrt{\bar{x}'(\bar{s})^2 + \bar{y}'(\bar{s})^2 - \bar{r}'(\bar{s})^2}\, d\bar{s}$$

is satisfied, where the lower limits of integration are to be properly chosen. In order that the two continuous series of oriented circles should be transformable into each other by an improper Laguerre transformation it is necessary and sufficient that the relations

$$f(s) = \bar{f}(\bar{s}) \text{ and } F(s) = -\bar{F}(\bar{s})$$

hold at the corresponding points such that

$$\int^{s} \sqrt{x'(s)^2 + y'(s)^2 - r'(s)^2}\, ds = \int^{\bar{s}} \sqrt{\bar{x}'(\bar{s})^2 + \bar{y}'(\bar{s})^2 - \bar{r}'(\bar{s})^2}\, d\bar{s}$$

is satisfied, where the lower limits of integration are to be properly chosen. If we denote the common value of

$$\int^{s} \sqrt{x'(s)^2 + y'(s)^2 - r'(s)^2}\, ds \text{ and } \int^{\bar{s}} \sqrt{\bar{x}'(\bar{s})^2 + \bar{y}'(\bar{s})^2 - \bar{r}'(\bar{s})^2}\, d\bar{s},$$

by p, *then* $f(s)$, $\bar{f}(\bar{s})$, $F(s)$, $\bar{F}(\bar{s})$ *may be looked upon as functions of* p, *so that*

$$f(s) = \phi(p), \quad \bar{f}(\bar{s}) = \Phi(p),$$
$$F(s) = \psi(p), \quad \bar{F}(\bar{s}) = \Psi(p),$$

and for the transformability of two series of oriented circles by a proper Laguerre transformation it is necessary and sufficient that

$$\phi(p) \equiv \Phi(p) \text{ and } \psi(p) \equiv \Psi(p)$$

hold. For the transformability of two series of oriented circles by an improper Laguerre transformation it is necessary and sufficient that

$$\phi(p) \equiv \Phi(p) \text{ and } \psi(p) \equiv -\Psi(p)$$

hold.

3. Let us now treat the proposed problem. When two plane curves are given, we can construct a series of osculating circles for each curve. If the direction of the measurement of the curve length be assigned, the osculating circles are all oriented. It is necessary and sufficient for the transformability of two plane

curves by a Laguerre transformation that the two series of osculating oriented circles should be transformable by a Laguerre transformation. But whether this condition is satisfied or not, cannot be determined by means of article 2, as the condition

$$x'(s)^2 + y'(s)^2 - r'(s)^2 \not\equiv 0$$

is not satisfied in this case.

In treating this problem we may assume without any loss of generality that the two plane curves have no points at which the radius of curvature becomes infinite.

If we transform our problem to the problem in space, then it takes the following form:

Having given two minimal curves in space: required to find the necessary and sufficient condition in order that the two minimal curves are congruent or symmetrical.

This problem has already been treated by Sophus Lie and E. Study*. In applying it to our problem it is convenient to use Study's fundamental theorem:

Putting the equation of the minimal curve in the form

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

where

$$x'(t)^2 + y'(t)^2 + z'(t)^2 \equiv 0,$$

determine the natural parameter p in such a manner that

$$\left(\frac{d^2x}{dp^2}\right)^2 + \left(\frac{d^2y}{dp^2}\right)^2 + \left(\frac{d^2z}{dp^2}\right)^2 = -1 \qquad \text{......(1)},$$

and

$$\begin{vmatrix} \frac{dx}{dp} & \frac{dy}{dp} & \frac{dz}{dp} \\ \frac{d^2x}{dp^2} & \frac{d^2y}{dp^2} & \frac{d^2z}{dp^2} \\ \frac{d^3x}{dp^3} & \frac{d^3y}{dp^3} & \frac{d^3z}{dp^3} \end{vmatrix} = -1 \qquad \text{......(2)}.$$

At the corresponding points of the two congruent minimal curves the values of the natural parameters must be equal. Also the quantities

$$\left(\frac{d^3x}{dp^3}\right)^2 + \left(\frac{d^3y}{dp^3}\right)^2 + \left(\frac{d^3z}{dp^3}\right)^2$$

formed for the two given congruent minimal curves are also equal. Conversely if for the two given minimal curves the quantities

$$\left(\frac{d^3x}{dp^3}\right)^2 + \left(\frac{d^3y}{dp^3}\right)^2 + \left(\frac{d^3z}{dp^3}\right)^2$$

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* Lie-Scheffers, *Vorlesungen über continuierliche Gruppen*, S. 694. E. Study, "Zur Differentialgeometrie der analytischen Curven," *Transactions of the American Mathematical Society*, 10, 1909.

formed for the two minimal curves are equal at the corresponding points at which the values of p are equal, then the two minimal curves are congruent.

In cases where the two minimal curves are symmetrical we have to determine the natural parameter p for one of the minimal curves so that

$$\left(\frac{d^2x}{dp^2}\right)^2 + \left(\frac{d^2y}{dp^2}\right)^2 + \left(\frac{d^2z}{dp^2}\right)^2 = -1,$$

and

$$\begin{vmatrix} \frac{dx}{dp} & \frac{dy}{dp} & \frac{dz}{dp} \\ \frac{d^2x}{dp^2} & \frac{d^2y}{dp^2} & \frac{d^2z}{dp^2} \\ \frac{d^3x}{dp^3} & \frac{d^3y}{dp^3} & \frac{d^3z}{dp^3} \end{vmatrix} = +1$$

are satisfied. At the corresponding points the values of p for both curves are equal and also the values of

$$\left(\frac{d^3x}{dp^3}\right)^2 + \left(\frac{d^3y}{dp^3}\right)^2 + \left(\frac{d^3z}{dp^3}\right)^2$$

for both curves are equal. This condition is also sufficient in order that the two minimal curves are symmetrical.

We will now apply this theorem to the solution of our problem.

Let $x(s)$, $y(s)$ be the rectangular coordinates of the point of the plane curve considered as analytic functions of the curve length s, and let the direction cosines of the tangent be denoted by α and β and the radius of the circle of curvature by r, so that the centre of the circle of curvature is given by

$$\xi = x - r\beta,$$
$$\eta = y + r\alpha.$$

Then it follows from the above considerations and Study's theorem that

$$\left(\frac{d^2\xi}{ds^2}\right)^2 + \left(\frac{d^2\eta}{ds^2}\right)^2 - \left(\frac{d^2r}{ds^2}\right)^2$$

is invariant by Laguerre transformations.

By differentiating we obtain

$$\frac{d\xi}{ds} = -r'\beta, \qquad \frac{d\eta}{ds} = r'\alpha,$$

$$\frac{d^2\xi}{ds^2} = -r''\beta - \frac{r'}{r}\alpha, \quad \frac{d^2\eta}{ds^2} = r''\alpha - \frac{r'}{r}\beta,$$

from which we get

$$\left(\frac{d^2\xi}{ds^2}\right)^2 + \left(\frac{d^2\eta}{ds^2}\right)^2 - \left(\frac{d^2r}{ds^2}\right)^2 = \frac{r'^2}{r^2} \qquad \text{......(1).}$$

Hence the quantity $\frac{1}{r^2}\left(\frac{dr}{ds}\right)^2$ must be invariant by Laguerre transformations.

Now we have to choose the parameter p in such a manner that

$$\left(\frac{d^2\xi}{dp^2}\right)^2+\left(\frac{d^2\eta}{dp^2}\right)^2-\left(\frac{d^2r}{dp^2}\right)^2=1.$$

For that purpose we have to determine p so that

$$\left(\frac{ds}{dp}\right)^4=\frac{r^2}{r'^2} \qquad \text{......(2)}.$$

In addition to the above condition we must choose p so that

$$\begin{vmatrix} \frac{d\xi}{dp} & \frac{d^2\xi}{dp^2} & \frac{d^3\xi}{dp^3} \\ \frac{d\eta}{dp} & \frac{d^2\eta}{dp^2} & \frac{d^3\eta}{dp^3} \\ \frac{dr}{dp} & \frac{d^2r}{dp^2} & \frac{d^3r}{dp^3} \end{vmatrix}=1 \qquad \text{......(3)}.$$

By differentiation we have

$$\frac{d^3\xi}{ds^3}=\left(-r'''+\frac{r'}{r^2}\right)\beta+\left(\frac{r'^2}{r^2}-\frac{2r''}{r}\right)\alpha,$$

$$\frac{d^3\eta}{ds^3}=-\left(-r'''+\frac{r'}{r^2}\right)\alpha+\left(\frac{r'^2}{r^2}-\frac{2r''}{r}\right)\beta.$$

From the relation (3) we get

$$\left(\frac{ds}{dp}\right)^6\begin{vmatrix} -r'\beta, & -r''\beta-\frac{r'}{r}\alpha, & \left(-r'''+\frac{r'}{r^2}\right)\beta+\left(\frac{r'^2}{r^2}-\frac{2r''}{r}\right)\alpha \\ r'\alpha\;, & r''\alpha-\frac{r'}{r}\beta\;, & -\left(-r'''+\frac{r'}{r^2}\right)\alpha+\left(\frac{r'^2}{r^2}-\frac{2r''}{r}\right)\beta \\ r'\;, & r''\;, & r''' \end{vmatrix}$$

$$=\left(\frac{r'}{r}\right)^3\left(\frac{ds}{dp}\right)^6=1 \qquad \text{......(4)}.$$

From relations (2) and (4) we obtain

$$\left(\frac{ds}{dp}\right)^2=\frac{r}{r'}.$$

Thus *the quantity*

$$\frac{dr\,ds}{r}$$

is unaltered by proper Laguerre transformations, while only its sign is changed by any improper Laguerre transformation.

Therefore $$p = \int \sqrt{\frac{1}{r}\frac{dr}{ds}}\, ds$$

is an integral invariant of Laguerre transformation and it may be called the Laguerre parameter. The invariance of

$$\frac{dr\, ds}{r}$$

by Laguerre transformations can also be proved by using H. Liebmann's theorem in the geometry of inversion. Prof. H. Liebmann* has obtained an invariant for the space curve

$$\frac{ds\, dr}{r^2}\,\frac{R}{(R^2 - r^2)^{\frac{1}{2}}},$$

where r denotes the radius of curvature and R denotes the radius of the osculating sphere. Suppose a space curve lies entirely on a sphere of radius R, then, denoting the radius of geodesic curvature by r_g, we obtain, by virtue of the relation

$$r = R \sin \frac{r_g}{R},$$

$$dr = \cos \frac{r_g}{R}\, dr_g,$$

and $$\frac{ds\, dr}{r^2}\,\frac{R}{(R^2 - r^2)^{\frac{1}{2}}} = \frac{ds \cos \frac{r_g}{R}\, dr_g}{R^2 \sin^2 \frac{r_g}{R}}\,\frac{R}{R \cos \frac{r_g}{R}} = \frac{ds\, dr_g}{R^2 \sin^2 \frac{r_g}{R}}.$$

As any inversion on a spherical surface can also be looked upon as obtained by an inversion in space, the above quantity is invariant by the inversion on the spherical surface. Thus we get the theorem:

The quantity $$\frac{ds\, dr_g}{R^2 \sin^2 \frac{r_g}{R}}$$

formed with respect to a curve on the spherical surface of radius R, is invariant by inversion on the spherical surface.

By dual observation we get the theorem:

If $d\tau$ be the angle of contingence of the curve on the spherical surface of radius R, then

$$\frac{d\tau\, d\left(\frac{1}{r_g}\right)}{R^2 \sin^2 \frac{1}{Rr_g}} = -\frac{d\tau\, dr_g}{R^2 r_g^2 \sin^2 \frac{1}{Rr_g}}$$

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* H. Liebmann, "Beiträge zur Inversionsgeometrie der Kurven," *Sitzungsberichte der Münchener Akademie*, 1923.

is unaltered by Laguerre transformation on the spherical surface, as r_g and $\frac{1}{r_g}$, ds and $d\tau$ are completely dual.

If R becomes infinitely great, then the spherical curve becomes a plane curve and

$$\frac{d\tau\, dr_g}{R^2 r_g^{\,2} \sin^2 \frac{1}{Rr_g}}$$

converges to $d\tau\, dr$.

Consequently it is shown that

$$d\tau\, dr = \frac{dr\, ds}{r}$$

is an invariant of the Laguerre transformation on the plane.

For example if we reverse the signs of all radii of circles of curvature of a plane curve, then the direction of the measurement of the curve length is also reversed and

$$\frac{dr\, ds}{r}$$

changes its sign. This transformation is a kind of improper Laguerre transformation. By a Laguerre inversion it also changes its sign. But by "Dilatation" or by symmetrical transformation with respect to a line on the plane $\frac{dr\, ds}{r}$ remains unchanged.

Thus by a precise investigation it can be shown that $\frac{dr\, ds}{r}$ remains unchanged by any proper Laguerre transformation and changes its sign only by any improper Laguerre transformation, as we have already seen.

Now let us consider the expression

$$\left(\frac{d^3\xi}{dp^3}\right)^2 + \left(\frac{d^3\eta}{dp^3}\right)^2 - \left(\frac{d^3 r}{dp^3}\right)^2,$$

and let $\omega_1, \omega_2, \omega_3$ be independent variables; then we have

$$\begin{vmatrix} \frac{d\xi}{dp} & \frac{d^3\xi}{dp^3} & \omega_1 \\ \frac{d\eta}{dp} & \frac{d^3\eta}{dp^3} & \omega_2 \\ \frac{dr}{dp} & \frac{d^3 r}{dp^3} & \omega_3 \end{vmatrix}$$

$$\equiv -\left(\omega_1 \frac{d^2\xi}{dp^2} + \omega_2 \frac{d^2\eta}{dp^2} - \omega_3 \frac{d^2 r}{dp^2}\right) \quad .$$

by virtue of relation (3). From the above relation we obtain

$$\begin{vmatrix} \dfrac{d\xi}{dp} & \dfrac{d^3\xi}{dp^3} & \dfrac{d^4\xi}{dp^4} \\ \dfrac{d\eta}{dp} & \dfrac{d^3\eta}{dp^3} & \dfrac{d^4\eta}{dp^4} \\ \dfrac{dr}{dp} & \dfrac{d^3r}{dp^3} & \dfrac{d^4r}{dp^4} \end{vmatrix}$$

$$= \left(\frac{d^3\xi}{dp^3}\right)^2 + \left(\frac{d^3\eta}{dp^3}\right)^2 - \left(\frac{d^3r}{dp^3}\right)^2$$

$$= -\left(\frac{d^2\xi}{dp^2}\frac{d^4\xi}{dp^4} + \frac{d^2\eta}{dp^2}\frac{d^4\eta}{dp^4} - \frac{d^2r}{dp^2}\frac{d^4r}{dp^4}\right).$$

Denoting
$$\frac{d^m\xi}{ds^m}\frac{d^n\xi}{ds^n} + \frac{d^m\eta}{ds^m}\frac{d^n\eta}{ds^n} - \frac{d^mr}{ds^m}\frac{d^nr}{ds^n}$$
by (m, n), and
$$\frac{d^m\xi}{dp^m}\frac{d^n\xi}{dp^n} + \frac{d^m\eta}{dp^m}\frac{d^n\eta}{dp^n} - \frac{d^mr}{dp^m}\frac{d^nr}{dp^n}$$
by $(m, n)_p$, we obtain

$$-(33)_p = \frac{4\,(22)\,(24) + 8\,(22)\,(33) - 15\,(23)^2}{4\,(22)^{\frac{5}{2}}}.$$

By easy calculation we find

$$(33) = \frac{r'^2}{r^4} + \frac{r'^4}{r^4} + \frac{4r''^2}{r^2} - \frac{2r'r'''}{r^2} - \frac{4r'^2r''}{r^3}.$$

Also by differentiation

$$\frac{d^4\xi}{ds^4} = \frac{d}{ds}\left(-\frac{2r''}{r} + \frac{r'^2}{r^2}\right)\alpha - \frac{d}{ds}\left(r''' - \frac{r'}{r^2}\right)\beta$$
$$-\left(-\frac{2r''}{r} + \frac{r'^2}{r^2}\right)\frac{\beta}{r} - \left(r''' - \frac{r'}{r^2}\right)\frac{\alpha}{r},$$

$$\frac{d^4\eta}{ds^4} = \frac{d}{ds}\left(\frac{r'^2}{r^2} - \frac{2r''}{r}\right)\beta - \frac{d}{ds}\left(-r''' + \frac{r'}{r^2}\right)\alpha$$
$$+\left(-r''' + \frac{r'}{r^2}\right)\frac{\beta}{r} + \left(\frac{r'^2}{r^2} - \frac{2r''}{r}\right)\frac{\alpha}{r},$$

from which follows

$$(24) = -\frac{r'}{r}\frac{d}{ds}\left(-\frac{2r''}{r} + \frac{r'^2}{r^2}\right) + r''\frac{d}{ds}\left(r''' - \frac{r'}{r^2}\right)$$
$$+\frac{r'}{r^2}\left(r''' - \frac{r'}{r^2}\right) + \left(-\frac{2r''}{r} + \frac{r'^2}{r^2}\right)\frac{r''}{r} - r''r^{\text{iv}}$$
$$= \frac{3r'r'''}{r^2} - \frac{r''r'^2}{r^3} - \frac{3r''^2}{r^2} + \frac{2r'^4}{r^4} - \frac{r'^2}{r^4}.$$

Also we have $$(23) = \frac{r'r''}{r^2} - \frac{r'^3}{r^3}.$$

Consequently we get

$$-(33)_p = \frac{4\,(24)\,(22) + 8\,(33)\,(22) - 15\,(23)^2}{4\,(22)^{\frac{5}{2}}}.$$

$$= \frac{5}{4}\frac{r''^2 r}{r'^3} + \frac{1}{4}\frac{r'}{r} + \frac{1}{rr'} - \frac{rr'''}{r'^2} - \frac{3}{2}\frac{r''}{r'}.$$

Thus the above expression is also an invariant of proper Laguerre transformations. By improper Laguerre transformation the above expression changes its sign, and therefore it is a semi-invariant.

Consequently we obtain:

In order that the two plane curves should be transformable into each other by a proper Laguerre transformation the relation

$$\frac{ds\,dr}{r} = \frac{d\bar{s}\,d\bar{r}}{\bar{r}}$$

must hold at the corresponding points. Thus

$$\int \sqrt{\frac{1}{r}\frac{dr}{ds}}\,ds$$

is an integral invariant of the Laguerre group. When the two plane curves are transformable into each other by an improper Laguerre transformation we must have necessarily

$$\frac{ds\,dr}{r} = -\frac{d\bar{s}\,d\bar{r}}{\bar{r}}.$$

Thus the expression $\frac{ds\,dr}{r}$ *is a semi-invariant. Also the expression*

$$I \equiv \frac{5}{4}\frac{r''^2 r}{r'^3} + \frac{1}{4}\frac{r'}{r} + \frac{1}{rr'} - \frac{rr'''}{r'^2} - \frac{3}{2}\frac{r''}{r'}$$

is another semi-invariant. It is unaltered by a proper Laguerre transformation while it changes its sign by any improper Laguerre transformation.

In order that the two plane curves should be transformable into each other by a proper Laguerre transformation it is necessary and sufficient that we can make the correspondence in such a manner that, at the corresponding points where

$$\frac{ds\,dr}{r} = \frac{d\bar{s}\,d\bar{r}}{\bar{r}},$$

the relation $$I = \bar{I}$$

is also satisfied.

In order that the two plane curves should be transformable into each other by an improper Laguerre transformation it is necessary and sufficient that we can make the correspondence in such a manner that, at the corresponding points where

$$\frac{ds\,dr}{r} = -\frac{d\bar{s}\,d\bar{r}}{\bar{r}},$$

the relation $$I = -\bar{I}$$

is also satisfied.

Take the Laguerre parameter

$$p = \int \sqrt{\pm\frac{1}{r}\frac{dr}{ds}}\,ds = \int \sqrt{\pm\frac{1}{\bar{r}}\frac{d\bar{r}}{d\bar{s}}}\,d\bar{s},$$

where the ambiguous signs are to be so chosen that p becomes real.

Then for the transformability of two given plane-curves by a proper Laguerre transformation it is necessary and sufficient that the above ambiguous signs are of the same sign and that I *and* $\bar{I}$ *are the same function of* p.

For improper Laguerre transformations the above ambiguous signs must be of opposite signs and the other conditions remain the same.

On the Penetration of Waves and their Rays into a Medium of Gradually Varying Properties. By J. A. WILCKEN, B.Sc. (Copenhagen), Ph.D. (La Plata); late of Christ's College. (Communicated by Professor Sir JOSEPH LARMOR.)

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The question has frequently been raised, whether the laws of ordinary Geometrical Optics, especially with regard to the reflexion and refraction at the confines of two media, are equally applicable when the transition is accomplished in a gradual manner. The problem has an important bearing on physical phenomena of a widely varying character, such as the propagation of sound and light in the atmosphere, with occurrence of mirage or zones of silence, the transmission of seismic disturbances through the earth, tidal waves in a canal of varying cross-section, and others.

On account of the divergence of views still prevalent, on points of vital issue in the theory, an attempt is made, in the present paper, to elucidate the following questions:

(i) Does a gradual change of refractive index cause any disintegration of an incoming wave-system, comparable to the reflexion as known in abrupt transition?

(ii) Is the ordinary refraction-law, $\mu \sin \theta =$ const., strictly applicable to a medium with a continuously varying index?

I. The first question has been treated by the late Lord Rayleigh(1) as early as 1880 and also referred to in a later paper(2). It is found, that reflexion does actually take place, though the reflected intensity is negligible when the transition is "slow," the slowness being clearly defined as such, that the medium may be considered homogeneous within several wave-lengths. The fact that the conclusions are established on the assumption of a specially defined form of variation of the index, does not impair the qualitative generality of the result. As shown below, the investigation applies to normal incidence only, but it would appear, that in this case alone would a reflected wave be expected to reveal itself in the usual form in the solution of the equation of motion. In oblique propagation namely, the direction of the incoming ray changes continuously, and so the infinitesimal reflexions would not readily integrate into a definite wave.

It seems that this paper has been overlooked by subsequent writers, since the question as to the presence of reflexion has been raised anew(3, 4), and even caused some controversies (*e.g.* Seeliger(5), *v.* Schmidt(6), Emden(7), *v.* Wallot(8)).

To sum up: Reflexion takes place in a gradually varying medium, but the attenuation of the incoming wave may be negligible in many practically important cases, where the variation is inappreciable within several wave-lengths, such as for sound- and light-waves in a normal atmosphere. For electro-magnetic disturbances of great wave-length (9), or for sound- and light-waves in layers of air with a rapid change of density, reflexion may become considerable.

If indeed reflexion were rigorously incompatible with a continuous variation of index, as has been argued, even the ordinary theory of reflexion would need revision, being based upon the assumption of a thin, continuous transition-layer between *any* two media, to account for the observed phenomena of polarisation (10).

II. In most of the investigations published hitherto, the validity of the refraction-law has been, implicitly or explicitly, assumed. It is definitely denied, by Uller (11), that Snell's law is cognate with Fermat's principle. On the other hand, Boussinesq (12) and Gans (4) have attempted to verify that the ordinary law holds, also for continuous variation of refractive index, at least approximately; but it would appear from the argument below, that the assumptions on which this result is established somehow involve the law itself. Again, when the ray becomes nearly parallel to the stratification, that is, where total reflexion becomes possible, Gans finds a deviation from the usual ("gewöhnlich angegebenen") direction, and so here rejects Snell's law.

Consider the general wave-equation, after separating the time-variable:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0 \qquad \ldots\ldots(1),$$

where k is now a real, continuous function of x only, of the usual meaning

$$k = \frac{2\pi}{\lambda} = \frac{2\pi n\mu}{c} \qquad \ldots\ldots(2).$$

The known solution for $k =$ const., viz.

$$U = e^{\pm ik(lx+my)}$$

suggests as a convenient form for the solution covering the more general case, what may be termed a generalized wave-function

$$U = \phi\,(x, y)\, e^{i\psi(x,y)} \qquad \ldots\ldots(3),$$

ϕ and ψ being real functions of x and y which, on substitution, are found subject to two real equations

$$\frac{\partial}{\partial x}\left(\phi^2 \frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi^2 \frac{\partial\psi}{\partial y}\right) = 0 \qquad \ldots\ldots(4),$$

$$\frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + k^2 - \left(\frac{\partial\psi}{\partial x}\right)^2 - \left(\frac{\partial\psi}{\partial y}\right)^2 = 0 \quad \ldots\ldots(5).$$

It has been customary to split up equation (1), postulating

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{......(6)},$$

$$\psi = X_1 + X_2 y \qquad \text{......(7)},$$

X_1 and X_2 being functions of x only, thus obtaining

$$\frac{d}{dx}\left\{\phi^2\left(\frac{dX_1}{dx} + y\frac{dX_2}{dx}\right)\right\} = 0 \qquad \text{......(4}a\text{)},$$

which implies
$$\frac{dX_2}{dx} = 0,$$

$$X_2 = \text{const.}$$

This is the mathematical expression of Snell's law, and is a necessary but not sufficient condition for the reduction of equation (1), it being also required that the amplitude shall be independent of y, or the surfaces of equal intensity be planes parallel to the stratification. There seems to be no *a priori* reason for this assumption. In the case of normal incidence, however, there is perfect symmetry around the incident ray, and it is legitimate to confine the discussion to the equation in one dimension

$$\frac{d^2U}{dx^2} + k^2U = 0 \qquad \text{......(1}a\text{)}.$$

This equation can be dealt with in various ways (*e.g.* (13, 14)). But no conclusions drawn from a consideration of this simpler case can be accepted as final with regard to oblique incidence. We return to the general equation (1) and attempt this type of solution in the more compact form

$$U = e^{\Phi},$$

Φ being now a complex function of x and y, subject to

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + k^2 = 0 \qquad \text{......(8)}.$$

A slow variation of the medium is specified in the form

$$k = k_0 + k_1 x + \ldots \qquad \text{......(9)},$$

k_1 being a small constant, higher powers of which may be neglected as compared to k_0.

Let the non-homogeneous medium be bounded to the left by $x = 0$, so that $k = k_0$ for $x < 0$, the stratification being normal to the X-axis, and let waves coming from the homogeneous medium, $x < 0$, impinge on the boundary, XY being the incidence-plane.

The solution is approached by taking

$$\Phi = \pm i(lx + my) + px + qy + \tfrac{1}{2}rx^2 + \tfrac{1}{2}sy^2 + txy \quad \text{...(10)},$$

where $l^2 + m^2 = k_0^2$, so that l and m are proportional to the direction cosines of the incident ray, or wave-normal, and p, q, r ... acquire the meaning of small coefficients, not necessarily real, to correcting terms, to be added on account of the (slow) variation of k.

Substituting in (8) and neglecting higher powers of small quantities, we establish the equations of condition

$$\left.\begin{aligned} \pm i(lr + mt) + k_0 k_1 &= 0 \\ ms + lt &= 0 \\ \pm 2i(lp + mq) + r + s &= 0 \end{aligned}\right\} \quad \text{......(11)},$$

which are satisfied by taking

$$\left.\begin{aligned} p &= -k_1 l/2k_0^2 \\ q &= -k_1 m/2k_0^2 \\ r &= \pm ik_1(k_0 l + m^2)/k_0^2 \\ s &= \mp ik_1(k_0 l - l^2)/k_0^2 \\ t &= \pm ik_1(k_0 m - lm)/k_0^2 \end{aligned}\right\} \quad \text{......(12)}.$$

Expressing k in terms of the wave-length

$$\begin{aligned} k &= k_0 + k_1 x + \dots \\ &= \frac{2\pi}{\lambda_0}\left(1 - \frac{x}{\lambda_0}\frac{d\lambda}{dx} + \dots\right) \end{aligned} \quad \text{......(13)},$$

or $$k_1/k_0 = -\lambda'/\lambda_0,$$

and writing $$l = k_0 \cos\theta_0,$$

$$m = k_0 \sin\theta_0,$$

θ_0 being the angle of incidence, the wave-system in the non-homogeneous medium can be written, with the notation $\lambda' = d\lambda/dx$,

$$\begin{aligned} \text{const.} \times e^{(x\cos\theta_0 + y\sin\theta_0)\lambda'/2\lambda_0} \cos 2\pi\{nt + \text{const.} - (x\cos\theta_0 + y\sin\theta_0)/\lambda_0 \\ + [x^2(\cos\theta_0 + \sin^2\theta_0) - y^2\cos\theta_0(1-\cos\theta_0) \\ + 2xy\sin\theta_0(1-\cos\theta_0)]\lambda'/2\lambda_0^2\}. \end{aligned}$$

Including further terms in the development

$$k = k_0 + k_1 x + \tfrac{1}{2}k_2 x^2 + \dots,$$

the approximation may be carried to terms of x^3 and y^3, but for our present purpose the discussion may be limited to the solution given above.

A comparison with the generalized wave-function, as defined by equation (3), shows the amplitude

$$\phi = e^{-k_1/2k_0^2 . (lx+my)}$$

to be such that $$\frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = k_1{}^2/4k_0{}^2 \qquad \text{......(14)},$$

and therefore, with sufficient approximation

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = k^2 \qquad \text{......(15)}.$$

If θ denote the angle between the ray and the X-axis at any point within the medium, we may thus write down the two equivalent equations

$$k\cos\theta = \frac{\partial\psi}{\partial x}, \quad k\sin\theta = \frac{\partial\psi}{\partial y},$$

and, substituting $$k = \frac{2\pi n\mu}{c}, \quad = \frac{2\pi n}{c}\left(\mu_0 + x\frac{d\mu}{dx}\right) \qquad \text{......(16)},$$

$$k_1/k_0 = \mu'/\mu_0,$$

these become, on differentiating the phase or argument of the cosine in the wave-form, two equations still practically equivalent over a long reach of the slightly varying medium, μ' representing $d\mu/dx$,

$$\mu\cos\theta = \mu_0\cos\theta_0 + \mu'\{(x\cos\theta_0 + y\sin\theta_0)(1-\cos\theta_0) + x\} \quad \text{(17)},$$

$$\mu\sin\theta = \mu_0\sin\theta_0 + \mu'\{(x\sin\theta_0 - y\cos\theta_0)(1-\cos\theta_0)\} \quad \text{......(18)}.$$

The latter form is the modification of the ordinary refraction law across the plane stratified medium, which would have involved the principle that the ray emerges from a plate of the medium in the original direction; the deviation $\delta\theta$ of the ray from its usual course specified by the ordinary law of refraction is the small amount given by

$$\mu_0\cos\theta_0\,\delta\theta = \mu'(x\sin\theta_0 - y\cos\theta_0)(1-\cos\theta_0) \quad \text{...(19)},$$

the origin being the initial point on the ray: it is dependent on the angle of incidence, vanishing as normal incidence is approached. Here $x\sin\theta_0 - y\cos\theta_0$ is the very small sideway shift of the ray measured along the final wave-front, from a straight path normal to it.

If the medium becomes more refringent with increasing x, we have $\mu' > 0$ and, since the ray is bent towards the normal,

$$x\sin\theta_0 - y\cos\theta_0 \nless 0,$$

and so the deviation is positive, or the ray is bent *less* than it would have been according to Snell's law. If, on the other hand, the medium becomes *less* refringent, $\mu' < 0$, but then

$$x \sin \theta_0 - y \cos \theta_0 \not> 0,$$

so that again the ray is refracted *less* than usually.

The path of the ray can be obtained as orthogonal trajectory to the surfaces $\psi = \text{const.}$, but the integration of the general equation leads to complicated expressions that do not readily admit of interpretation. Two special cases, however, are of interest.

The differential equation to the ray path (which the argument from Fresnel's zones still makes the orthogonal trajectory of the surfaces of constant phase, when the amplitude is variable on these fronts) is

$$\frac{dy}{dx} = \frac{\mu_0 \sin \theta_0 + \mu' (1 - \cos \theta_0)(x \sin \theta_0 - y \cos \theta_0)}{\mu_0 \cos \theta_0 + \mu' \{(1 - \cos \theta_0)(x \cos \theta_0 + y \sin \theta_0) + x\}} \quad (20).$$

(*a*) For nearly normal incidence $1 - \cos \theta_0$ is small, and neglecting the small quantity $\mu' (1 - \cos \theta_0)$, we have

$$\frac{dy}{dx} = \mu_0 \sin \theta_0 / (\mu_0 \cos \theta_0 + \mu' x) \quad \ldots\ldots(21),$$

integrating into

$$y = (\mu_0 \sin \theta_0 / \mu') \log (\mu_0 \cos \theta_0 + \mu' x) + \text{const.} \quad \ldots(22),$$

and determining the constant so that the ray is incident at the origin,

$$y = (\mu_0 \sin \theta_0 / \mu') \log (1 + \mu' x / \mu_0 \cos \theta_0) \quad \ldots\ldots(22a),$$

or approximately, taking $\cos \theta_0 = 1$,

$$\mu' y = \mu_0 \sin \theta_0 \log (\mu / \mu_0) \quad \ldots\ldots(23).$$

(*b*) For grazing incidence $\cos \theta_0$ is small, and if the layer be thin, we have, neglecting terms involving $\mu' \cos \theta_0$ and $\mu' x$,

$$\frac{dy}{dx} = \mu_0 \sin \theta_0 / (\mu_0 \cos \theta_0 + \mu' y \sin \theta_0) \quad \ldots\ldots(24),$$

which integrates into

$$x = y \cot \theta_0 + \tfrac{1}{2} \mu' y^2 / \mu_0 \quad \ldots\ldots(25),$$

determining the constant of integration as above. It is seen that the ray will meet the boundary $x = 0$ at the points $y = 0$ and $y = -2\mu_0 \cot \theta_0 / \mu'$, and so the ray bends round, becoming parallel to the stratification at the point

$$\left.\begin{aligned} y &= -\mu_0 \cot \theta_0 / \mu' \\ x &= -\tfrac{1}{2} \mu_0 \cot^2 \theta_0 / \mu' \end{aligned}\right\} \quad \ldots\ldots(26),$$

and emerging at an angle θ_1 determined by

$$\begin{aligned}\tan\theta_1 &= \mu_0 \sin\theta_0/(\mu_0\cos\theta_0 - 2\mu_0\cos\theta_0)\\ &= -\tan\theta_0 \qquad\qquad ......(27),\end{aligned}$$

that is, the angle of emergence is equal to the angle of incidence. In fact, when the equation of propagation is linear, the phase-function for a periodic train of waves does not involve t explicitly, and therefore the rays are reversible. It is seen to be necessary that $\mu' < 0$ in order that the point given by (26) be within the medium, or total reflexion can occur only when the medium becomes less refringent with increasing x. Also, the least thickness a of a layer, requisite for total reflexion, is given by

$$a = \left| \tfrac{1}{2}\mu_0 \cot^2\theta_0/\mu' \right|,$$

or, in terms of the wave-length

$$a = \tfrac{1}{2}\lambda_0 \cot^2\theta_0/\lambda' \qquad\qquad ......(28).$$

The limiting angle of incidence, for a layer of given depth a, is determined by

$$\begin{aligned}\theta_1 &= \cot^{-1}\{2a\mu'/\mu_0\}^{\frac{1}{2}}\\ &= \cot^{-1}\{2a\lambda'/\lambda_0\}^{\frac{1}{2}} \qquad\qquad ......(29).\end{aligned}$$

This result is identical with what would be immediately obtained by taking the ray as a circular arc of curvature μ'/μ_0.

Conclusion. A non-homogeneous medium of slowly varying properties causes reflexion, but of negligible intensity.

In such a varying medium the ordinary refraction law of the ray paths, determined as orthogonal to the fronts of constant phase, does not hold, the refraction being slightly less in amount than according to Snell's law. The deviation becomes less as normal incidence is approached.

Total reflexion may occur for a very oblique ray, if the layer is thick enough, the ray path then bending round in a continuous curve and emerging at an angle equal to that of incidence.

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Dissociation under a Temperature Gradient. By P. A. M. Dirac, St John's College. (Communicated by Mr R. H. Fowler, Trinity College.)

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If a gas, each of whose molecules is capable of dissociating into two similar molecules, be kept at a constant temperature, it would attain an equilibrium state in which the rate at which double molecules dissociate is equal to the rate at which they are formed by recombination of single molecules, there being a different equilibrium state for each different temperature. If, now, the gas be subjected to a temperature gradient, the concentrations necessary for equilibrium would be different at different points, so there would be diffusion between the two kinds of molecules, a steady state being attained when the rate of diffusion of double molecules into any region is equal to their excess rate of decomposition over their rate of formation within that region.

We shall consider the case of a tube of the gas of unit section, closed at each end, the ends being maintained at different temperatures. There will be a continuous flow of double molecules from the cold end to the hot end, where they dissociate and flow back to the cold end. Since two single molecules possess more energy than a double one, these flows will cause a transfer of energy across the central sections of the tube in addition to that transferred by ordinary thermal conduction. So if the sides of the tube are non-conducting, there must be a greater temperature gradient at the ends of the tube than in the middle, in order that the same amount of energy may cross each section of the tube. Also, at the hot end double molecules dissociate more rapidly than they are formed, so that there must be an excess in the concentration of the double molecules here over their concentration when in equilibrium at the same temperature with no temperature gradient. Similarly there is a deficiency of double molecules at the cold end.

Hence if observations could be made of the concentrations or temperature gradient at either end, the 'velocity' of dissociation and recombination could be calculated, in terms of the other physical constants of the gas.

§ 1. *Equations for the Steady State.*

Let z be the co-ordinate of a point along the tube, T the temperature, and ν_1 and ν_2 the numbers per unit volume of single and double molecules at that point.

The pressure must be constant all along the tube. This gives, assuming the ideal gas laws to hold,

$$(\nu_1 + \nu_2)\,T = \text{constant}.$$

Let μ_1 and μ_2 be the numbers per unit volume of single and double molecules when in equilibrium at the same temperature and pressure with no temperature gradient. Then

$$(\mu_1 + \mu_2)\,T = \text{same constant},$$

so that

$$\nu_1 = \mu_1 + \sigma, \quad \nu_2 = \mu_2 - \sigma \qquad \text{......(i)},$$

where σ is a measure of the difference in the state of the gas caused by the temperature gradient.

The total mass crossing any section of the tube is zero. This gives

$$\nu_1 U_1 + 2\nu_2 U_2 = 0 \qquad \text{......(ii)},$$

where U_1 and U_2 are the mean z components of the velocities of single and double molecules.

The total energy crossing a section of the tube is the same all the way along the tube. Thermal conduction will cause a transfer of energy across any section at the rate $-\kappa \frac{dT}{dz}$, where κ is the thermal conductivity at that section. In addition, there is an excess of $U_1\nu_1$ single molecules crossing the section and returning as double molecules. This results in a transfer of energy across the section at the rate $\frac{1}{2}kU_1\nu_1$, where k is the heat of dissociation per double molecule (kinetic energy as well as potential). We therefore get the equation

$$-\kappa \frac{dT}{dz} + \tfrac{1}{2}kU_1\nu_1 = A \qquad \text{......(iii)},$$

where A is the total rate of flow of energy along the tube.

For ordinary diffusion with D as diffusion coefficient, we have the equations

$$U_1\nu_1 = -D\frac{d\nu_1}{dz}, \quad U_2\nu_2 = -D\frac{d\nu_2}{dz} = D\frac{d\nu_1}{dz},$$

$$U_1 - U_2 = -D\left(\frac{1}{\nu_1}\frac{d\nu_1}{dz} - \frac{1}{\nu_2}\frac{d\nu_2}{dz}\right) = -D\frac{d}{dz}\log\frac{\nu_1}{\nu_2}.$$

The first two of these equations are true only when the total number of molecules crossing any section is zero. When this condition is not satisfied, as in the present case, the only equation given by the laws of diffusion is the third one, which involves only relative velocities. If there is a temperature gradient, this equation becomes

$$U_1 - U_2 = -D\frac{d}{dz}\log\frac{\nu_1}{\nu_2} + D'\frac{d}{dz}\log T \qquad \text{......(iv)},$$

where D' is related to the thermal diffusion coefficient.

The excess rate of recombination of single molecules over their rate of formation per unit volume at a given temperature, will be proportional to σ if σ is small. This means that if a small part of the gas were isolated, it would tend to its equilibrium state according to the law $\frac{d\sigma}{dt} = -H\sigma$, or $\sigma \propto \exp(-Ht)$, where H is a constant for any given temperature, and is a measure of the velocity of dissociation and recombination. This excess must be balanced by the excess number of single molecules flowing into unit volume by diffusion, so that

$$H\sigma = -\frac{d}{dz}(U_1\nu_1) \qquad \text{......(v).}$$

We thus have five equations for the five unknown variables ν_1, ν_2, U_1, U_2, T. The coefficients κ, D and D' are functions of the temperature and of σ, but as we are assuming σ is small, we may take for these coefficients their values when σ is zero, so that they are functions of T only. H and k are also functions of T only.

From (i) $\log\frac{\nu_1}{\nu_2} = \log\frac{\mu_1}{\mu_2} + \log\left(1+\frac{\sigma}{\mu_1}\right) - \log\left(1-\frac{\sigma}{\mu_2}\right)$

$$= \log\frac{\mu_1}{\mu_2} + \sigma',$$

where $$\sigma' = \left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\sigma,$$

neglecting σ^2.

Van 't Hoff's isochore gives

$$\frac{2}{\mu_1}\frac{d\mu_1}{dT} - \frac{1}{\mu_2}\frac{d\mu_2}{dT} = \frac{d}{dT}\log\frac{\mu_1^2}{\mu_2} = \frac{k}{RT^2} \qquad \text{......(vi),}$$

R being Boltzmann's gas constant.

From the equation $(\mu_1+\mu_2)T = \text{constant}$, we have

$$\frac{1}{\mu_1+\mu_2}\left(\frac{d\mu_1}{dT}+\frac{d\mu_2}{dT}\right) = -\frac{1}{T}.$$

Subtracting this equation from (vi) we find that

$$\frac{\mu_1+2\mu_2}{\mu_1+\mu_2}\left(\frac{1}{\mu_1}\frac{d\mu_1}{dT} - \frac{1}{\mu_2}\frac{d\mu_2}{dT}\right) = \frac{k}{RT^2}+\frac{1}{T},$$

or $$\frac{d}{dT}\log\frac{\mu_1}{\mu_2} = \frac{\mu_1+\mu_2}{\mu_1+2\mu_2}\left(\frac{k}{RT^2}+\frac{1}{T}\right).$$

Equation (iv) now becomes

$$U_1 - U_2 = -D\frac{d}{dz}\log\frac{\mu_1}{\mu_2} + D'\frac{d}{dz}\log T - D\frac{d\sigma'}{dz}$$

$$= -D\frac{\mu_1+\mu_2}{\mu_1+2\mu_2}\left(\frac{k}{RT^2}+\frac{1}{T}\right)\frac{dT}{dz} + \frac{D'}{T}\frac{dT}{dz} - D\frac{d\sigma'}{dz}.$$

Eliminating U_2 by means of (ii),

$$U_1\left(1+\frac{\nu_1}{2\nu_2}\right)=-\left[D\frac{\mu_1+\mu_2}{\mu_1+2\mu_2}\left(\frac{k}{RT^2}+\frac{1}{T}\right)-\frac{D'}{T}\right]\frac{dT}{dz}-D\frac{d\sigma'}{dz}\ldots\text{(vii)}.$$

Substituting this value for U_1 in (iii),

$$A=-\kappa\frac{dT}{dz}-\frac{\nu_1\nu_2 k}{\nu_1+2\nu_2}\left\{\left[D\frac{\mu_1+\mu_2}{\mu_1+2\mu_2}\left(\frac{k}{RT^2}+\frac{1}{T}\right)-\frac{D'}{T}\right]\frac{dT}{dz}+D\frac{d\sigma'}{dz}\right\}.$$

The ν's may be replaced by the corresponding μ's in this equation without affecting its order of accuracy. This gives

$$A=-\kappa'\frac{dT}{dz}-\frac{\mu_1\mu_2}{\mu_1+2\mu_2}kD\frac{d\sigma'}{dz}\qquad\ldots\ldots\text{(viii)},$$

where $$\kappa'=\kappa+\frac{\mu_1\mu_2 k}{\mu_1+2\mu_2}\left[D\frac{\mu_1+\mu_2}{\mu_1+2\mu_2}\left(\frac{k}{RT^2}+\frac{1}{T}\right)-\frac{D'}{T}\right].$$

κ' may be called the apparent thermal conductivity of the gas, since $\frac{d\sigma'}{dz}$ is negligible throughout most of the length of the tube, so that κ' determines the amount of heat flowing along the tube per unit temperature gradient.

From (iii) and (v)

$$-\frac{d}{dz}\left(\frac{2A}{k}+\frac{2\kappa}{k}\frac{dT}{dz}\right)=H\sigma=\frac{\mu_1\mu_2}{\mu_1+\mu_2}H\sigma'\qquad\ldots\ldots\text{(ix)}.$$

We now have two equations for the two unknowns T and σ'. To obtain an approximate solution for an end of the tube, we may give to the coefficients κ, κ', μ_1, μ_2, k, D and H their constant values at the temperature of the end.

Equations (viii) and (ix) now reduce to

$$\sigma'=-\frac{\mu_1+\mu_2}{\mu_1\mu_2}\frac{2\kappa}{kH}\frac{d^2T}{dz^2}$$

$$=\frac{\mu_1+\mu_2}{\mu_1+2\mu_2}\frac{2\kappa D}{\kappa' H}\frac{d^2\sigma'}{dz^2},$$

or $$\frac{d^2\sigma'}{dz^2}=\beta^2\sigma',$$

where $$\beta=\sqrt{\frac{\mu_1+2\mu_2}{\mu_1+\mu_2}\frac{\kappa' H}{2\kappa D}}.$$

This gives $$\sigma'=Be^{\pm\beta(z-z_0)},$$

where z_0 is the co-ordinate of the end of the tube considered, and the + or − sign must be chosen so as to make σ very small in the middle of the tube.

From (viii) $\dfrac{dT}{dz} = -\dfrac{A}{\kappa'} - \dfrac{\mu_1 \mu_2}{\mu_1 + 2\mu_2} \dfrac{kD}{\kappa'} \dfrac{d\sigma'}{dz}$,

whence $$T = -\frac{A}{\kappa'} z - \frac{\mu_1 \mu_2}{\mu_1 + 2\mu_2} \frac{kD}{\kappa'} B e^{\pm \beta (z - z_0)} + \text{constant}.$$

The constant of integration B can be determined from the terminal condition that $U_1 = U_2 = 0$ when $z = z_0$. Using (iii) and (vii) we get for the end of the tube

$$-\frac{dT}{dz} = \frac{A}{\kappa},$$

and $$\frac{A}{\kappa}\left[D \frac{\mu_1 + \mu_2}{\mu_1 + 2\mu_2}\left(\frac{k}{RT^2} + \frac{1}{T}\right) - \frac{D'}{T}\right] = D \frac{d\sigma'}{dz} = \pm DB\beta.$$

The value of σ at the end is therefore

$$\sigma_0 = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} B = \pm \frac{\mu_1 \mu_2}{\mu_1 + 2\mu_2} \frac{A}{\kappa\beta}\left[\frac{k}{RT^2} + \frac{1}{T} - \frac{D'}{DT} \frac{\mu_1 + 2\mu_2}{\mu_1 + \mu_2}\right]$$
$$= \pm \frac{\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \frac{\kappa' - \kappa}{\kappa} \frac{A}{Dk\beta},$$

while the exponential term in the expression for the temperature has the end value

$$-\frac{\mu_1 \mu_2}{\mu_1 + 2\mu_2} \frac{kD}{\kappa'} \sigma_0' = \mp \frac{\kappa' - \kappa}{\kappa\kappa'} \frac{A}{\beta}$$

§ 2. *Approximate Numerical Values.*

To obtain an estimate of the order of magnitude of the various quantities, we may assume the following numerical values for the coefficients in C.G.S. units and ° C. and calories:—

$$D = 0{\cdot}1;\ \kappa = 5 \times 10^{-5};\ T = 300;\ (\mu_1 + \mu_2) k = 0{\cdot}2$$

(corresponding to a heat of dissociation of 5000 cals. per grm. molecule); $\dfrac{A}{\kappa'} =$ mean temperature gradient $= 10$.

From experiments on thermal diffusion by Chapman and Dootson* it appears that D' (which is related to Chapman's thermal diffusion coefficient D_t by the formula $D' = \dfrac{(\nu_1 + \nu_2)^2}{\nu_1 \nu_2} D_t$) can be neglected without causing an error of more than about 2 °/₀. We now find roughly $\beta = 4{\cdot}2 \sqrt{H}$,

$$\sigma_0 = 0{\cdot}03 H^{-\frac{1}{2}} (\mu_1 + \mu_2),$$

terminal value of exponential part of $T = 3H^{-\frac{1}{2}}$.

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* *Phil. Mag.* XXXIII. p. 248 (1917).

§ 3. *Conclusion.*

The temperature gradient will cause a measurable change in the concentrations at the ends of the tube if H is of the order of magnitude 1 sec.$^{-1}$, while an even larger value for H would give a measurable deviation from uniformity in the temperature curve. Observations of either the concentrations (which could be made optically in the case of a gas whose colour changes when it dissociates) or the temperature at an end would enable the velocity H to be calculated, provided the other physical constants of the gas were known. The only coefficients which cannot be found easily (other than D' which can be neglected) are κ and D, but it would be possible to obtain approximate values for them from the Kinetic Theory from observations on viscosity and diffusion with other gases, and κ', which can be measured directly, would provide a check for these values.

The difficulty in making the required measurements at the end of the tube is that the quantities to be measured fall off rather rapidly according to an exponential law with space constant β^{-1}, which is about 2 mm. with the numerical values previously taken when H is 1 sec.$^{-1}$.

The analysis holds unchanged for a gas whose molecules dissociate into two different kinds of molecules, provided each of these kinds of molecules has the same diffusion coefficient with the undissociated gas, it being only necessary to let ν_1 mean the total number of single molecules of both kinds per unit volume. When this diffusion condition is not satisfied, the concentrations of the two kinds of single molecules will no longer be everywhere equal, but even in this more general case the same end effects will occur.

The Electromagnetic Equations as Basis of Einstein's Quadratic Form. By Mr R. HARGREAVES, St John's College.

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§ 1. The groundwork of the present paper is a general transformation of Maxwell's equations. As relativist speculation had its origin in a simple transformation, it was to be expected that something more would result when the general transformation was explored: the determining influence has proved stronger than I had anticipated.

The transformation is here effected, not by direct treatment of the differential equations, but by use of the Electromagnetic and Magneto-electric Integral forms introduced in an earlier paper*. They are here written in more general form as

$$\Omega_2(e)=\int X_1\,dx_2dx_3+X_2\,dx_3dx_1+X_3\,dx_1dx_2+\beta_1\,dx_4dx_1+\beta_2\,dx_4dx_2+\beta_3\,dx_4dx_3 \quad \ldots(\mathrm{I}),$$

$$\Omega_2(m)=\int B_1\,dx_2dx_3+B_2\,dx_3dx_1+B_3\,dx_1dx_2-\xi_1\,dx_4dx_1-\xi_2\,dx_4dx_2-\xi_3\,dx_4dx_3 \quad \ldots(\mathrm{II}),$$

the greater generality consisting in the use of four triads instead of two. In Maxwell's equations ξ is X and β is B for free ether (the reason for the notation adopted); while for a dielectric, specific constants connect ξ with X and β with B. The form (II) is taken to have no circuital content, and therefore to be derivable from a linear form; while to (I) is assigned invariance of circuital content as dependent solely on the equation of continuity of charge and current.

In dealing with coordinates other than rectangular and in admitting that the fourth variable ($x_4 = Vt$) shares in the transformation, it is apparent that the triads X and ξ are differently transformed (as also B and β), and so the maintenance of like forms $\xi = X$ and $\beta = B$ becomes impossible. We proceed to examine how this involves a constitutive relation, and in what way it is dependent on the transformation used.

§ 2. When a general transformation of the type (1) below is applied to (I) or (II), a linear connexion is found between symbols with and without dash, the coefficients of which are second minors of the determinant of transformation. Thus symbols X' and β' are linear functions of X and β, B' and ξ' of B and ξ. When the system with dashes is Maxwell's scheme for free ether in rectangular coordinates, conditions $\beta' = B'$ and $\xi' = X'$ hold for each component. If we equate the values found for β' and B', the first in terms of $X\beta$, the second in terms of $B\xi$, and treat ξ' and X' in the same way, we obtain six equations connecting the symbols without dash, twelve in number, i.e. four triads, which occur in (I) and (II). Solution consists in manipulating them so as to find space-time symbols (Greek) in terms of space symbols (Roman) regarded as known, *or* the symbols of (I) in terms of those of (II), or *vice versâ* in each case. The former plan conforms more closely to the conception of constitutive relations, the latter rather to a correlation of the two forms; but if either type of solution is reached there is a ready passage from one to the other. These algebraic relations reduce the number of independent constituents to six, and are sufficient with the differential equations proper to the forms to make the scheme determinate.

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* *Cambridge Phil. Trans.*, "Integral forms and their connexion with Physical Equations," vol. XXI. 1908.

If the same transformation converts $dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2$ to $g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + \ldots$, then the outcome of the solution is that the coefficients in the constitutive relations depend on the elements of the determinant of transformation *only through the forms into which these enter to form the g's.* It is on this feature that the coordination of an electromagnetic system with a quadratic form depends: it is in complete contrast to the relations between symbols with and without dash which show no such restriction. On the ground of this natural development it seems proper to base the use of Einstein's quadratic in a primary way on Maxwell's scheme, and a further reason will appear shortly.

§ 3. Although the special methods of relativity are not used in this section of the enquiry, the forms (I) and (II) clearly lend themselves to such treatment, which is made easier by quoting a third integral form

$$\Omega_1(m) = \int F\,dx_1 + G\,dx_2 + H\,dx_3 - \psi\,dx_4 \text{ or } \int \phi_1\,dx_1 + \phi_2\,dx_2 + \phi_3\,dx_3 + \phi_4\,dx_4 \ldots\text{(III)},$$

of which (II) is the derivative. Here ϕ's are covariant elements of single index, and their derivatives in (II) covariants of double index; that is if B_1 is of index type (23) and ξ_1 of type (14), each term of (II) presents upper and lower indices which are exactly balanced. The constitutive relations then make X and β contravariants, X_1 of type (14), β_1 of type (32), which present each member of (I) as of the order of four-dimensional volume, with indices referable to the order 1234 by an even number of interchanges of adjacent indices. This connexion with volume also appears in the fact of its derivative assuming the form

$$\Omega_3(e) = \int \rho\,dx_1\,dx_2\,dx_3 - \rho u_1\,dx_4\,dx_2\,dx_3 - \rho u_2\,dx_4\,dx_3\,dx_1 - \rho u_3\,dx_4\,dx_1\,dx_2 \ldots\ldots\text{(IV)},$$

in which density appears.

In the primary scheme the two triads (X) and (B) are immediately expressed in terms of potentials; but whereas one group of Maxwell's equations is then satisfied identically, in the other the singularities ρ, ρu are only attached to separate potentials by the use of

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} + \frac{\partial \psi}{V \partial t} = 0,$$

a condition which belongs to an integral form of the third order. The transformation of this third-order form is distinct from that of the linear form (III), and just as before the consequence was a division of two triads into four related triads, so here a single four-group is divided into two related groups. In this case also the connecting coefficients depend on the elements of the determinant of transformation only through the combinations into which these enter to form the *g*'s.

In respect to the new group of contravariant potentials it is natural to enquire whether the condition which it satisfies functions as a separating agent. The answer is that, subject to one proviso, it does secure the expression of the singularities by derivatives of the separate contravariant potentials, these derivatives being formed in the manner of the variant calculus. The proviso is that the analytical conditions which Einstein postulates for the existence and determination of a gravitational form must be satisfied. If we extend the type of Electromagnetic equations in coordination with a quadratic form, and apply the methods found for the reducible case, I take it to be a primary office of these limitations that they satisfy a reasonable demand of Electromagnetic theory.

§ 4. If instead of the equations for free ether we take those for a dielectric, writing $X' = K\xi'$ and $B' = M\mathfrak{B}'$, the transformation leads to the same dependence of coefficients of the constitutive relations on those of a quadratic; but the quadratic is now the transformed value of $dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2/\mu^2$. The Lorentz-Einstein transformation then yields constitutive relations proper to a state of uniform relative motion. The coefficients are constant and therefore a plane wave possible; its velocity of propagation is found to agree with that required for the identification of wave arguments in each status under the transformation. The velocity is not constant, and therefore involves ray velocity and direction distinct from those of wave propagation. The components of ray velocity obtained in this way answer to the requirements of the addition theorem proper to the transformation.

The next example is the transformation of Maxwell's equations for free ether to correspond with a uniformly accelerated standpoint. The transformation confirms the propriety of the Minkowski-Einstein choice of form for acceleration to centre, called the gravitational form. The forms are different because one is, the other is not, reducible to Maxwell's case; but the present form points unmistakably to that of Einstein.

The example may I hope prove serviceable as an illustration of Einstein's method. The Lorentz-Einstein transformation gives an unaltered constitutive relation and is on that ground of less use for such a purpose; the present gives new constitutive relations, involving variable coefficients though in a simple manner. Thus each correlation can be checked, and the need for it shown, by a direct use of transformation formulæ carried through without reference to preconceived ideas as to the result which should be reached.

Then follows a dynamical theorem touching ignoration, which admits of application to the theory of relativity. It is significant in respect to the determination of time, and to the origin of the potential energy of gravitation. Finally the stress-equations, and the forms of expression for energy and kinetic potential are briefly considered. The object in the first case is to deal with the special terms introduced by the acceleration (using the material of the second example above) in respect to the inertia of energy content. In the second matter a connecting link is found between the variations of coefficients in two forms, analogous to that for Hamiltonian and Lagrangian forms in Dynamics.

§ 5. The notation used for the general transformation is

$$dx_1' = k_{11}\,dx_1 + k_{12}\,dx_2 + k_{13}\,dx_3 + k_{14}\,dx_4,\quad dx_2' = k_{21}\,dx_1 + k_{22}\,dx_2 + \dots \quad (1);$$

and the transformation of (I) leads to relations

$$\left.\begin{aligned} X_1 &= a_{11}X_1' + a_{21}X_2' + a_{31}X_3' + c_{11}B_1' + c_{21}B_2' + c_{31}B_3' \\ \beta_1 &= b_{11}B_1' + b_{21}B_2' + b_{31}B_3' + d_{11}X_1' + d_{21}X_2' + d_{31}X_3' \end{aligned}\right\} \dots(2),$$

in which the coefficients are second minors of the determinant Δ of (1), representing Jacobian forms. They are classified as *a*'s with no index 4, *b*'s with two indices 4, *c*'s with one index 4 in the first place, and *d*'s with one index 4 in the second place. With like indices *b*'s are adjugates of *a*'s, and *d*'s of *c*'s; e.g. b_{12} of a_{12}. It is not to be assumed that $a_{12} = a_{21}$, and in table (3) three values are defined, and cyclic order is used for 1, 2, 3:

$$\left.\begin{aligned} a_{11} &= k_{22}k_{33} - k_{23}k_{32},\ & a_{12} &= k_{23}k_{31} - k_{33}k_{21},\ & a_{21} &= k_{13}k_{32} - k_{33}k_{12} \\ b_{11} &= k_{44}k_{11} - k_{41}k_{14},\ & b_{12} &= k_{44}k_{12} - k_{42}k_{14},\ & b_{21} &= k_{44}k_{21} - k_{41}k_{24} \\ c_{11} &= k_{13}k_{42} - k_{12}k_{43},\ & c_{12} &= k_{11}k_{43} - k_{13}k_{41},\ & c_{21} &= k_{23}k_{42} - k_{22}k_{43} \\ d_{11} &= k_{31}k_{24} - k_{21}k_{34},\ & d_{12} &= k_{32}k_{24} - k_{22}k_{34},\ & d_{21} &= k_{11}k_{34} - k_{31}k_{14} \end{aligned}\right\} \dots(3).$$

For our purpose the transformation of form (II) must be taken in reverse order; it is

$$\left.\begin{aligned} \Delta\beta_1' &= b_{11}B_1 + b_{12}B_2 + b_{13}B_3 - c_{11}\xi_1 - c_{12}\xi_2 - c_{13}\xi_3 \\ \Delta X_1' &= a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 - d_{11}B_1 - d_{12}B_2 - d_{13}B_3 \end{aligned}\right\} \dots\dots(4).$$

The reversal of order leads to the use of adjugates, and as $\Delta dx_1 = K_{11} dx_1' + K_{21} dx_2' + \dots$ there is also a reversal of the order of indices, while the negative sign is a result of that sign appearing in (II). Combining (2) with (4) we have

$$\left.\begin{array}{rl}
\Delta X_1 = a_{11}\,(a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 - d_{11}B_1 - d_{12}B_2 - d_{13}B_3) & \\
+ a_{21}\,(a_{21}\xi_1 + \qquad\qquad\qquad - d_{21}B_1 \qquad\qquad\qquad) & \\
+ a_{31}\,(a_{31}\xi_1 + \qquad\qquad\qquad - d_{31}B_1 \qquad\qquad\qquad) & \\
+ c_{11}\,(b_{11}B_1 + b_{12}B_2 + b_{13}B_3 - c_{11}\xi_1 - c_{12}\xi_2 - c_{13}\xi_3) & \\
+ c_{21}\,(b_{21}B_1 + \qquad\qquad\qquad - c_{21}\xi_1 \qquad\qquad\qquad) & \\
+ c_{31}\,(b_{31}B_1 + \qquad\qquad\qquad - c_{31}\xi_1 \qquad\qquad\qquad) & \\
\Delta \beta_1 = b_{11}\,(b_{11}B_1 + b_{12}B_2 + b_{13}B_3 - c_{11}\xi_1 - c_{12}\xi_2 - c_{13}\xi_3) & \\
+ b_{21}\,(b_{21}B_1 + \qquad\qquad\qquad - c_{21}\xi_1 \qquad\qquad\qquad) & \\
+ b_{31}\,(b_{31}B_1 + \qquad\qquad\qquad - c_{31}\xi_1 \qquad\qquad\qquad) & \\
+ d_{11}\,(a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 - d_{11}B_1 - d_{12}B_2 - d_{13}B_3) & \\
+ d_{21}\,(a_{21}\xi_1 + \qquad\qquad\qquad - d_{21}B_1 \qquad\qquad\qquad) & \\
+ d_{31}\,(a_{31}\xi_1 + \qquad\qquad\qquad - d_{31}B_1 \qquad\qquad\qquad) &
\end{array}\right\} \dots(5)$$

as expressions for the elements of (I) in terms of those of (II).

§ 6. Suppose the transformation to make

$$dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2 = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + \dots \quad \dots(6),$$

then
$$g_{12} = k_{11}k_{12} + k_{21}k_{22} + k_{31}k_{32} - k_{41}k_{42},$$

and
$$-G_{12} = K_{11}K_{12} + K_{21}K_{22} + K_{31}K_{32} - K_{41}K_{42} \quad \dots\dots(7),$$

where G_{12} is a first minor of Δ_g, and the second formula is to be considered shortly. When determinants Δ and $-\Delta$ in the form where first rows are $k_{11}k_{21}k_{31}k_{41}$ and $k_{11}k_{21}k_{31}-k_{41}$, are multiplied together the constituents of the product determinant are the g's, i.e. $\Delta_g = -\Delta^2$. Also in the product form of the square of Δ a second minor is the sum of products of second minors of the original, and the change of sign in one column as above only affects this result through the sign of half the products. To express the second minors of Δ_g use $a'b' \dots$ defined with g_{pq} for k_{pq} in (3): the symmetry of g's entails $a_{12}' = a_{21}'$, $b_{12}' = b_{21}'$, and $d_{12}' = c_{21}'$, thus rendering d' unnecessary symbols, but $c_{12}' \neq c_{21}'$. The product theorem for second minors enables us to write specimen terms of ΔX_1 in the forms

$$\left.\begin{array}{l}
\xi_2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} - c_{11}c_{12} - c_{21}c_{22} - c_{31}c_{32}) = \xi_2(g_{23}g_{31} - g_{33}g_{12}) = \xi_2 a_{21}' \\
B_2(-a_{11}d_{12} - a_{21}d_{22} - a_{31}d_{32} + c_{11}b_{12} + c_{21}b_{22} + c_{31}c_{32}) = B_2(g_{22}g_{43} - g_{23}g_{42}) = -B_2c_{21}'
\end{array}\right\} (8),$$

and so $\Delta X_1 = a_{11}'\xi_1 + a_{21}'\xi_2 + a_{31}'\xi_3 - c_{11}'B_1 - c_{21}'B_2 - c_{31}'B_3 \dots(9)$.

The algebra required to deal with G_{12} may be shortened by using two connected equations

$$\left.\begin{aligned} &x_1' = k_{11}x_1 + \ldots + k_{14}x_4, \ \ x_1'' = k_{11}x_1' + k_{21}x_2' + k_{31}x_3' - k_{41}x_4', \\ &\text{and therefore } \ \Delta x_1' = K_{11}x_1'' + K_{12}x_2'' + K_{13}x_3'' + K_{14}x_4'', \\ &\text{but} \qquad\qquad -\Delta x_4' = K_{41}x_1'' + K_{42}x_2'' + K_{43}x_3'' + K_{44}x_4'' \end{aligned}\right\} \quad (10).$$

We then have a correlation of x and x'' with reference to the pseudosphere, viz.

$$\left.\begin{aligned} &x_1'^2 + x_2'^2 + x_3'^2 - x_4'^2 \\ &\quad = x_1x_1'' + x_2x_2'' + x_3x_3'' + x_4x_4'' \\ &\quad = x_1^2(k_{11}^2 + k_{21}^2 + k_{31}^2 - k_{41}^2) \\ &\qquad + 2x_1x_2(k_{11}k_{12} + k_{21}k_{22} + k_{31}k_{32} - k_{41}k_{42}) + \ldots \\ &\quad = x_1''^2\Delta^{-2}(K_{11}^2 + K_{21}^2 + K_{31}^2 - K_{41}^2) \\ &\qquad + 2x_1''x_2''\Delta^{-2}(K_{11}K_{12} + K_{21}K_{22} + K_{31}K_{32} - K_{41}K_{42}) + \ldots \end{aligned}\right\} \quad (11).$$

Comparing these correlations with

$$x_1'' = g_{11}x_1 + g_{12}x_2 + g_{13}x_3 + g_{14}x_4$$

and

$$\Delta_g x_1 = G_{11}x_1'' + G_{12}x_2'' + G_{13}x_3'' + G_{14}x_4'',$$

we reach (7). It should be noted that the new variable is only introduced to ease the passage to G_{12}: a differential form is not used because there would be no guarantee that dx'' would be a differential if related to dx and dx' in the linear form above. It may be remarked that though the determinant work does not specially involve differentials, they are implied at the outset, and I have verified that when the integral forms are translated to differential equations, conditions like $\dfrac{\partial k_{13}}{\partial x_2} = \dfrac{\partial k_{12}}{\partial x_3}$ are essential.

§ 7. We proceed to write the whole group of relations constitutive or correlative. To find space-time vectors (Greek symbols) in terms of fluxes we have the constitutive relations

$$\left.\begin{aligned} &G_{44}\xi_1 + G_{24}B_3 - G_{34}B_2 = \sqrt{-\Delta_g}\,(g_{11}X_1 + g_{12}X_2 + g_{13}X_3) \\ &G_{44}\beta_1 - G_{24}X_3 + G_{34}X_2 = \sqrt{-\Delta_g}\,(g_{11}B_1 + g_{12}B_2 + g_{13}B_3) \\ &\text{to which} \\ &G_{41}\xi_1 + G_{42}\xi_2 + G_{43}\xi_3 = -\sqrt{-\Delta_g}\,(g_{41}X_1 + g_{42}X_2 + g_{43}X_3) \end{aligned}\right\} \quad (12)$$

may be added, and a similar equation in β, both derivable from the prime relations.

To express members of (I) in terms of members of (II) or *vice versâ* (correlation of forms) we have

$$\left.\begin{aligned} X_1\sqrt{-\Delta_g} &= a_{11}'\xi_1 + a_{21}'\xi_2 + a_{31}'\xi_3 - (c_{11}'B_1 + c_{21}'B_2 + c_{31}'B_3) \\ \beta_1\sqrt{-\Delta_g} &= -(b_{11}'B_1 + b_{12}'B_2 + b_{13}'B_3) + c_{11}'\xi_1 + c_{12}'\xi_2 + c_{13}'\xi_3 \\ -\xi_1\sqrt{-\Delta_g} &= b_{11}'X_1 + b_{12}'X_2 + b_{13}'X_3 + c_{11}'\beta_1 + c_{12}'\beta_2 + c_{13}'\beta_3 \\ B_1\sqrt{-\Delta_g} &= a_{11}'\beta_1 + a_{21}'\beta_2 + a_{31}'\beta_3 + c_{11}'X_1 + c_{21}'X_2 + c_{31}'X_3 \end{aligned}\right\} \quad (13).$$

Lastly, expressions for X and β may be given in terms of second minors of g's with upper index, e.g. $a_{12}'' = g^{23}g^{31} - g^{33}g^{21}$, where $g^{12} = G_{12}/\Delta_g$, viz.

$$\left.\begin{aligned} -X_1/\sqrt{-\Delta_g} &= b_{11}''\xi_1 + b_{12}''\xi_2 + b_{13}''\xi_3 - c_{11}''B_1 - c_{12}''B_2 - c_{13}''B_3 \\ \beta_1/\sqrt{-\Delta_g} &= a_{11}''B_1 + a_{21}''B_2 + a_{31}''B_3 - c_{11}''\xi_1 - c_{21}''\xi_2 - c_{31}''\xi_3 \end{aligned}\right\} \quad (14).$$

The first of (13) is the equation (9) already found; with multipliers g_{11}, g_{12}, g_{13} the first of (12) is obtained. The second of (12) follows by analogy, and these two give the whole group. My original work was directed to solving six linear equations with Greek symbols treated as the unknowns, and a solution was found in the form (12). The intricacy of the determinant work made it an undesirable prelude to a paper, but although in the present mode of proving the equations it is not immediately obtained, it is given first place in the group. It has the conspicuous advantage of being easy to remember and simple in expression, and is specially serviceable in the important cases where only principal terms g_{pp} exist. Formula (14) lends itself to the passage from covariant to contravariant forms.

All these inner relations of the transformed group, whether expressing space-time symbols in terms of fluxes, or the correlation of the two integral forms, are dependent on the k's of the transformation by which they are derived, *only* through the forms into which they enter to give the values of g's.

§ 8. Next in importance to (12) is the connexion between two types of potential. The reference of charge and components of current to separate potentials in Maxwell's theory depends on a differential equation which expresses the vanishing of the derivative of the integral form

$$\int \phi_1' dx_2' dx_3' dx_4' + \ldots + \phi_4' dx_1' dx_2' dx_3' \quad \ldots\ldots(15),$$

ϕ_4' replacing $-\psi$. The transformation of (III) gives

$$\Delta\phi_1' = K_{11}\phi_1 + K_{12}\phi_2 + K_{13}\phi_3 + K_{14}\phi_4 \quad \ldots\ldots(16),$$

while that of (15) is

$$\begin{aligned} &\int (K_{11}\phi_1' + K_{21}\phi_2' + K_{31}\phi_3' - K_{41}\phi_4')\, dx_2 dx_3 dx_4 + \ldots \\ &\qquad - (K_{14}\phi_1' + K_{24}\phi_2' + K_{34}\phi_3' - K_{44}\phi_4')\, dx_1 dx_2 dx_3 \\ &= \int \Delta^{-1} [\{(K_{11}^2 + K_{21}^2 + K_{31}^2 - K_{41}^2)\phi_1 \\ &\qquad + (K_{11}K_{12} + K_{21}K_{22} + K_{31}K_{32} - K_{41}K_{42})\phi_2 + \ldots\}\, dx_2 dx_3 dx_4 + \ldots] \\ &= -\int \Delta^{-1} [(G_{11}\phi_1 + G_{12}\phi_2 + G_{13}\phi_3 + G_{14}\phi_4)\, dx_2 dx_3 dx_4 + \ldots \\ &\qquad - (G_{41}\phi_1 + G_{42}\phi_2 + G_{43}\phi_3 + G_{44}\phi_4)\, dx_1 dx_2 dx_3] \\ &= \int \sqrt{-\Delta_g}\, [g^{1p}\phi_p dx_2 dx_3 dx_4 + \ldots - g^{4p}\phi_p dx_1 dx_2 dx_3]. \end{aligned}$$

Thus with the definition $\phi^1 = g^{1p}\phi_p$ the divergence stands

$$\frac{\partial}{\partial x_1}(\sqrt{-\Delta_g}\phi^1) + \frac{\partial}{\partial x_2}(\sqrt{-\Delta_g}\phi^2) + \ldots = 0 \text{ or } \phi_q{}^q = 0 \quad \ldots(17),$$

its form and the connexion between the types of potential dependent only on the elements of Δ_g.

The harmonic operator of the new system is obtained by writing $\frac{\partial\psi}{\partial x_1}, \frac{\partial\psi}{\partial x_2}, \ldots$ for $\phi_1', \phi_2', \ldots$ in the above, when the derivatives of the two members yield

$$\int \Box\psi\, dx_1'\, dx_2'\, dx_3'\, dx_4' = \int \frac{\partial}{\partial x_p}\left(g^{pq}\sqrt{-\Delta_g}\frac{\partial\psi}{\partial x_q}\right) dx_1 dx_2 dx_3 dx_4;$$

or as $$dx_1' dx_2' dx_3' dx_4' = \sqrt{-\Delta_g}\, dx_1 dx_2 dx_3 dx_4,$$

this is $$\Box\psi = \frac{1}{\sqrt{-\Delta_g}}\frac{\partial}{\partial x_p}\left(\sqrt{-\Delta_g}\, g^{pq}\frac{\partial\psi}{\partial x_q}\right) \quad \ldots\ldots(18a).$$

An alternative form with

$$g^{pq}\left[\frac{\partial^2\psi}{\partial x_p \partial x_q} - \{pq, r\}\frac{\partial\psi}{\partial x_r}\right] \quad \ldots\ldots(18b)$$

is readily deducible.

§ 9. The potential ϕ^m has been defined as $g^{mq}\phi_q$; if its covariant derivative is formed normally then $\phi_p{}^m = g^{mq}\phi_{qp}$ with a definite order of cancelling. The variants of (I) and (II) are then expressed in terms of the potentials as follows:

$$\left.\begin{array}{ll} \xi_1 = \phi_{41} - \phi_{14}, \ldots & B_1 = \phi_{32} - \phi_{23}, \ldots \\ X_1\sqrt{-g} = g^{4p}\phi_p{}^1 - g^{1p}\phi_p{}^4, \ldots & \beta_1/\sqrt{-g} = g^{2p}\phi_p{}^3 - g^{3p}\phi_p{}^2, \ldots \end{array}\right\} (19).$$

The expressions for X and β are given most readily by (14), but may also be obtained from (12) or (13). From these we can deduce values of ρ defined as $\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3}$, $-\rho u_1$ defined as

$$\frac{\partial X_1}{\partial x_4} + \frac{\partial \beta_2}{\partial x_3} - \frac{\partial \beta_3}{\partial x_2},$$

and other components, with the special aim of taking into account the simplification given by (17). Here we set out from

$$-\rho = \frac{\partial}{\partial x_r}(\sqrt{-g}\, g^{rp}\phi_p{}^4) - \frac{\partial}{\partial x_r}(\sqrt{-g}\, g^{4q}\phi_q{}^r),$$

or $$-\rho/\sqrt{-g} = (g^{pq}\phi_p{}^4)_q - (g^{4p}\phi_p{}^q)_q$$

$$= g^{pq}\phi_{pq}{}^4 + \phi_p{}^4 g_q{}^{pq} - g^{4p}\phi_{pq}{}^q - \phi_p{}^q g_q{}^{4p} = g^{pq}\phi_{pq}{}^4 - g^{4p}\phi_{pq}{}^q \quad \ldots(20),$$

since the three-index symbols in g vanish, when understood to represent normal covariant derivatives. We then write

$$\frac{\partial}{\partial x_p}\phi_q^q = \frac{\partial}{\partial x_p}\left[\frac{\partial\phi^q}{\partial x_q} + \{rq, q\}\,\phi^r\right]$$

$$= \frac{\partial}{\partial x_q}[\phi_p^q - \{pr, q\}\,\phi^r] + \phi^r\frac{\partial}{\partial x_p}\{rq, q\} + \{rq, q\}\,[\phi_p^r - \{ps, r\}\,\phi^s]$$

$$= \phi_{pq}^q + \{pq, r\}\,\phi_r^q - \{rq, q\}\,\phi_p^r - \phi^r\frac{\partial}{\partial x_q}\{pr, q\} - \{pr, q\}$$

$$\times\,[\phi_q^r - \{qs, r\}\,\phi^s] + \phi^r\frac{\partial}{\partial x_p}\{rq, q\} + \{rq, q\}\,[\phi_p^r - \{ps, r\}\,\phi^s].$$

Here recognising the summation of q as well as of r,

$$\{pq, r\}\,\phi_r^q = \{pr, q\}\,\phi_q^r,$$

and interchanging s and r in the ϕ^s section, this is reduced to

$$\phi_{pq}^q + \phi^r\left[\frac{\partial}{\partial x_p}\{rq, q\} - \frac{\partial}{\partial x_q}\{rp, q\} + \{sp, q\}\,\{qr, s\} - \{sq, q\}\,\{rp, s\}\right].$$

Then with Einstein's notation for the bracket, we have

$$g^{4p}\frac{\partial}{\partial x_p}\phi_q^q = g^{4p}\phi_{pq}^q + g^{4p}B_{rpq}^q\,\phi^r \qquad \text{......(21)},$$

which gives a new form to the vanishing quantity on the left-hand and makes (20) replaceable by

$$-\rho/\sqrt{-g} = g^{pq}\phi_{pq}^4 + g^{4p}B_{rpq}^q\,\phi^r \qquad \text{......(22)}.$$

If then B_{rpq}^q or B_{rp} is supposed to vanish we have the last of the equations

$$-(\rho u_1, \rho u_2, \rho u_3, \rho)/\sqrt{-g} = g^{pq}(\phi_{pq}^1, \phi_{pq}^2, \phi_{pq}^3, \phi_{pq}^4) \quad \text{...(23)}.$$

Under the conditions stated the four singularities are then referred to the four contravariant potentials in the sense of operation by the variant calculus, which gives a systematic account of the interdependence of derivatives, as will appear clearly in an example to be given later. Further, it should be noted that if B_{rp} in (22) is taken to be proportional to g_{rp}, then the final term is $\lambda g^{4p}g_{rp}\phi^r$ or $\lambda\phi^4$; with the consequence that the exclusive reference to ϕ^r as operand in the potential expression for ρ is still secured.

§ 10. To obtain the transformation of the expression

$$\rho\,(Fu + Gv + Hw - V\psi)$$

of Maxwell's theory we first write that of (IV), which is

$$\int \rho'\,dx_1'\,dx_2'\,dx_3' - \rho'\,u_1'\,dx_2'\,dx_3'\,dx_4' - \dots$$

$$= \int (K_{14}\rho'\,u_1' + K_{24}\rho'\,u_2' + K_{34}\rho'\,u_3' + K_{44})\,dx_1\,dx_2\,dx_3$$

$$-(K_{11}\rho'\,u_1' + K_{21}\rho'\,u_2' + K_{31}\rho'\,u_3' + K_{41}\rho')\,dx_2\,dx_3\,dx_4 - \dots,$$

or
$$\rho = \rho' (K_{14} u_1' + K_{24} u_2' + K_{34} u_3' + K_{44}),$$
$$\rho u_1 = \rho' (K_{11} u_1' + K_{21} u_2' + K_{31} u_3' + K_{41}), \dots \quad \dots\dots(24).$$

Combining this with (16),

$$\begin{aligned} &\rho (u_1 \phi_1 + u_2 \phi_2 + u_3 \phi_3 + \phi_4) \\ &\quad = \rho' \phi_1 (K_{11} u_1' + K_{21} u_2' + K_{31} u_3' + K_{41}) + \rho' \phi_2 (K_{12} u_1' + K_{22} u_2' + \dots) \\ &\quad = \rho' u_1' (K_{11} \phi_1 + K_{12} \phi_2 + K_{13} \phi_3 + K_{14} \phi_4) + \rho' u_2' (K_{21} \phi_1 + \dots) \\ &\quad = \rho' (u_1' \phi_1' + u_2' \phi_2' + u_3' \phi_3' + \phi_4') \Delta, \end{aligned}$$

and therefore

$$\begin{aligned} &\int \rho' (u_1' \phi_1' + u_2' \phi_2' + u_3' \phi_3' + \phi_4') \, dx_1' dx_2' dx_3' dx_4' \\ &\qquad = \int \rho (u_1 \phi_1 + u_2 \phi_2 + u_3 \phi_3 + \phi_4) \, dx_1 dx_2 dx_3 dx_4 \quad \dots\dots(25). \end{aligned}$$

A reference to (22) then gives as invariant forms $g^{pq} \phi_r \phi_{pq}{}^r$, or this with $+ \lambda \phi_r \phi^r$ added according as we write $B_{rp} = 0$ or $= \lambda g_{rp}$.

The elimination of $\rho : \rho'$ from (24) gives a scheme of equations connecting velocities. But if these are in each system measured in the usual way the formula of transformation (1) gives

$$u_1 dx_4 = \Delta^{-1} (K_{11} u_1' + K_{21} u_2' + K_{31} u_3' + K_{41}) \, dx_4',$$
$$dx_4 = \Delta^{-1} (K_{14} u_1' + \dots) \, dx_4' \quad \dots\dots(26),$$

where the elimination of $dx_4 : dx_4'$ leads to the same scheme; and indeed the equations correspond exactly if $\rho' \Delta dx_4 = \rho dx_4'$, which implies $\rho \, dx_1 dx_2 dx_3 = \rho' dx_1' dx_2' dx_3'$ in accordance with our definition of ρ. Cf. *infra* (38 *a*).

We now sketch the argument for the invariance of kinetic potential in the form in which integration over the whole field is required. It turns on the use of equations

$$\left. \begin{aligned} k_{11} X_1 + k_{12} X_2 + k_{13} X_3 &= K_{44} X_1' + K_{34} B_2' - K_{24} B_3' \\ \Delta (k_{11} B_1' + k_{21} B_2' + k_{31} B_3') &= K_{44} \beta_1 + K_{43} X_2 - K_{42} X_3 \end{aligned} \right\} \quad (27),$$

and a correlative pair with X, B interchanged, β replaced by ξ, and the sign of skew terms reversed. The first is derived from (2) by multipliers k_{11}, k_{12}, k_{13}, the second by multipliers k_{11}, k_{21}, k_{31} from the reciprocal of (2). From (27),

$$\begin{aligned} &B_1 (K_{44} \beta_1 + K_{43} X_2 - K_{42} X_3) + B_2 (\quad) + \dots \\ &\qquad = \Delta \, [B_1 (k_{11} B_1' + k_{21} B_2' + \dots) + B_2 (k_{12} B_1' + \dots) + \dots], \end{aligned}$$

or
$$\begin{aligned} &K_{44} \Sigma B \beta + \Sigma K_{43} (B_1 X_2 - B_2 X_1) \\ &\qquad = \Delta \, [B_1' (k_{11} B_1 + k_{12} B_2 + \dots) + B_2' (k_{21} B_1' + \dots) + \dots] \\ &\qquad = \Delta \, [B_1' (K_{44} B_1' - K_{34} X_2' + K_{24} X_3') + B_2' (K_{44} B_2' + \dots)] \\ &\qquad = \Delta \, [K_{44} \Sigma B'^2 - \Sigma K_{34} (B_1' X_2' - B_2' X_1')], \end{aligned}$$

and also
$$\begin{aligned} &K_{44} \Sigma X \xi + \Sigma K_{43} (B_1 X_2 - B_2 X_1) \\ &\qquad = \Delta \, [K_{44} \Sigma X'^2 - \Sigma K_{34} (B_1' X_2' - B_2' X_1')]. \end{aligned}$$

By subtraction

$$\left.\begin{aligned} &\Sigma B\beta - \Sigma X\xi = \Delta\,(\Sigma B'^2 - \Sigma X'^2),\\ \text{or}\qquad &(\Sigma B\beta - \Sigma X\xi)/\sqrt{-g} = \Sigma B'^2 - \Sigma X'^2\\ \text{and}\qquad &\\ &\int(\Sigma B\beta - \Sigma X\xi)\, dx_1 dx_2 dx_3 dx_4 = \int(\Sigma B'^2 - \Sigma X'^2)\, dx_1' dx_2' dx_3' dx_4' \end{aligned}\right\} \quad (28).$$

If we add instead of subtracting, and regard $\Sigma B\beta + \Sigma X\xi$ as an energy form, then clearly the expressions involve different energy-contents so long as space-time coefficients $k_{14} k_{41} \ldots$ exist.

§ 11. At the close of this section some comment may be made on the general position. As a result of transformation we reach a new system in which the constituents of the fundamental forms (I) and (II) are connected by relations (12) dependent only on the coefficients of a quadratic. The quadratic is expressed in general terms, but its derivation from a special form presumes conditions. What is the position in respect to a more general electromagnetic scheme in which (12) is used without the limitations implied in reducibility to Maxwell's form? The scheme is fully defined because the constituents of (I) and (II) involve only six independent quantities, and there are six differential equations. The auxiliary forms in which the potentials occur are likewise connected in a way dependent only on the quadratic. Lastly the special character of (II), interpreted as a form free from circuital content, and so derivable from a linear form, has the valuable feature of complete independence of metrical relations. An extended scheme is then feasible and the real question is, has it any support in physical facts? The course of the work has revealed one set of conditions, less exacting than those of reducibility to Maxwell's scheme, viz. $B_{rpq}{}^q = 0$ or $= \lambda g_{rp}$. That conditions should arise in the way suggested, that is through formulæ connecting current with potential, is made reasonable by a comparison of (24) and (26), in which it appears that the velocity implied in current is subject to transformation on the same footing as other velocities.

In the first of two interesting papers* by F. Kottler, which are specially concerned with the metrical aspect, the method of integral forms is applied to the Newtonian law of gravitation. Viewed merely as a method of transforming Poisson's equation, the case may be stated in a way (rather different from Kottler's method) which is precisely analogous to the use of these forms for the electromagnetic equations.

In respect to a linear form $\int X_1' dx_1' + X_2' dx_2' + X_3' dx_3'$ absence of circuital content is assumed in analogy with the use of (II). This implies $X_1' = \frac{\partial U}{\partial x_1'}$, ... and a transformation

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* *Newton'sches Gesetz und Metrik, Maxwell'sche Gleichungen und Metrik*, Vienna, 1922.

$$\Delta X_1' = K_{11} X_1 + K_{12} X_2 + K_{13} X_3.$$

We then transform a flux form

$$\begin{aligned}
&\int X_1' dx_2' dx_3' + X_2' dx_3' dx_1' + X_3' dx_1' dx_2' \\
&\quad = \int (K_{11} X_1' + K_{21} X_2' + K_{31} X_3')\, dx_2 dx_3 + \dots \\
&\quad = \int \Delta^{-1} [(K_{11}^2 + K_{21}^2 + K_{31}^2)\, X_1 \\
&\qquad\qquad + (K_{11} K_{12} + K_{21} K_{22} + K_{31} K_{32})\, X_2 + \dots]\, dx_2 dx_3 + \dots \\
&\quad = \int \Delta^{-1} (G_{11} X_1 + G_{12} X_2 + G_{13} X_3)\, dx_2 dx_3 + \dots \\
&\quad = \int \sqrt{\Delta_g}\, (g^{1p} X_p dx_2 dx_3 + g^{2p} X_p dx_3 dx_1 + g^{3p} X_p dx_1 dx_2).
\end{aligned}$$

Now assign to this form the property that its derivative

$$\int \left(\frac{\partial X_1'}{\partial x_1'} + \frac{\partial X_2'}{\partial x_2'} + \frac{\partial X_3'}{\partial x_3'} \right) dx_1' dx_2' dx_3'$$

is equal to $\int 4\pi\rho' dx_1' dx_2' dx_3'$; and we have after transformation

$$\int \left[\frac{\partial}{\partial x_1} \left(\sqrt{\Delta_g}\, g^{1p} \frac{\partial U}{\partial x_p} \right) + \frac{\partial}{\partial x_2} \left(\sqrt{\Delta_g}\, g^{2p} \frac{\partial U}{\partial x_p} \right) + \frac{\partial}{\partial x_3} \left(\sqrt{\Delta_g}\, g^{3p} \frac{\partial U}{\partial x_p} \right) \right] dx_1 dx_2 dx_3$$
$$= \int 4\pi \rho' \sqrt{\Delta_g}\, dx_1 dx_2 dx_3 \;\dots\dots (29).$$

When only g_{pp} exist, $X_1' \sqrt{\Delta_g} = \sqrt{\frac{g_{22} g_{33}}{g_{11}}} \frac{\partial U}{\partial x_1}$, the most generally useful formula.

The principles applied to these forms are exact analogues of those applied to (II) and (I), but the details of transformation correspond rather with the connexion of potentials because first minor Jacobians occur in each case. Whereas for four dimensions correlations for orders 1 and 3 and for 2 and 2 were possible, in three dimensions the sole correlation is that of orders 1 and 2.

§ 12. In Maxwell's system for a dielectric $X' = K\xi'$, $B' = M\beta'$ with $KM = \mu^2$ and usually $M = 1$. The problem is then modified, and equations (12) are replaced by

$$\left.\begin{aligned}
G_{44} \xi_1 + G_{24} B_3 - G_{34} B_2 &= \frac{\Delta}{K} (g_{11} X_1 + g_{12} X_2 + g_{13} X_3) \\
G_{44} \beta_1 - G_{24} X_3 + G_{34} X_2 &= \frac{\Delta}{M} (g_{11} B_1 + g_{12} B_2 + g_{13} B_3)
\end{aligned}\right\} \dots (30),$$

in which Δ is the determinant of the transformation, but the g quadratic now represents the transformed value of

$$dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2/\mu^2, \text{ and } \Delta_g = -\mu^{-2} \times \Delta^2.$$

These equations will be applied to furnish constitutive relations when the transformation is that of Lorentz-Einstein, viz.

$$\left.\begin{aligned} x_1' &= x_1 - u_1 t + (\gamma - 1)\, u_1 \Sigma u_1 (x_1 - u_1 t)/\Sigma u^2, \\ t' &= \gamma\,(t - \Sigma ux/V^2) \text{ or } t' = t/\gamma - \gamma \Sigma u\,(x - ut)/V^2 \end{aligned}\right\} \ldots(31),$$

where $\gamma^2 (1 - \Sigma u^2/V^2) = 1$. Then taking the velocity of light as unit, and writing $\epsilon = 1 - \dfrac{1}{\mu^2}$,

$$\left.\begin{aligned} &x_1'^2 + x_2'^2 + x_3'^2 - x_4'^2/\mu^2 \\ &\quad = x_1'^2 + x_2'^2 + x_3'^2 - x_4'^2 + \epsilon x_4'^2 \\ &\quad = x_1^2 + x_2^2 + x_3^2 - x_4^2 + \epsilon\gamma^2 (x_4 - u_1 x_1 - u_2 x_2 - u_3 x_3)^2 \end{aligned}\right\} \ldots(32)$$

is the g quadratic; and equations (30) are replaced by

$$\left.\begin{aligned} \xi_1 (1 - \Sigma u^2/\mu^2) + \epsilon\,(u_2 B_3 - u_3 B_2) &= \{(1 - \Sigma u^2)\, X_1 + \epsilon u_1 \Sigma u X\}/\mu^2 \\ \beta_1 (1 - \Sigma u^2/\mu^2) - \epsilon\,(u_2 X_3 - u_3 X_2) &= (1 - \Sigma u^2)\, B_1 + \epsilon u_1 \Sigma u B \end{aligned}\right\} (33).$$

Apart from these special values, *whenever the coefficients are constant* a plane-wave solution can be found, and if

$$lx + my + nz - \Omega t$$

is its argument the differential equations (63) give

$$\Omega X_1 = V(n\beta_2 - m\beta_3), \quad \Omega B_1 = V(m\xi_3 - n\xi_2) \ldots\ldots(34),$$

to be taken in conjunction with the constitutive relations. Thus with (30) we find

$$\begin{aligned} &(G_{44}\Omega - VlG_{41} - VmG_{42} - VnG_{43})\, X_1 \\ &\qquad = \Delta V\,\{n\,(g_{21} B_1 + g_{22} B_2 + g_{23} B_3) - m\,(g_{31} B_1 + g_{32} B_2 + g_{33} B_3)\}/M, \end{aligned}$$

which combined with a similar equation for B (but with K for M) leads to

$$\begin{aligned} &(G_{44}\Omega - VlG_{41} - VmG_{42} - VnG_{43})^2 \\ &\qquad = (a_{11}' l^2 + 2a_{12}' lm + \ldots)\, \Delta^2 V^2/\mu^2 \ldots\ldots(35a). \end{aligned}$$

In the special case we are considering, the value obtained is

$$\Omega\,(1 - \Sigma u^2/\mu^2 V^2) - \epsilon \Sigma lu = (V^2 - \Sigma u^2/\mu^2 - \epsilon\, \overline{\Sigma lu}|^2)^{\frac{1}{2}}/\mu\gamma \ldots(35b).$$

§ 13. This value should agree with that implied in the transformation of a plane-wave argument from

$$p'(l'x' + m'y' + n'z' - Vt'/\mu) \quad \text{to} \quad p\,(lx + my + nz - \Omega t),$$

where we now write the more usual (xyz), (uvw). The comparison of the two members gives

$$\left.\begin{array}{l} p'l' = p\,\{l + (\gamma-1)\,u\Sigma lu/\Sigma u^2 - \gamma u\Omega\}, \ldots\ p'/\mu = p\gamma(\Omega - \Sigma lu); \\ \text{and then }\ \mu\gamma l'(\Omega - \Sigma lu) = l + (\gamma - 1)\,u\Sigma lu/\Sigma u^2 - \gamma u\Omega \end{array}\right\} \quad (36),$$

is an equation clear of the ratio $p:p'$. Squaring and adding we find after some reduction $V^2 - \Omega^2 = \gamma^2(\mu^2 - 1)(\Omega - \Sigma lu)^2$, which is in agreement with the solution (35 *b*).

We now consider the connexion with ray velocity and direction. The value of Ω may be made a homogeneous function of degree 1 in (lmn) by writing under the root sign $(V^2 - \Sigma u^2/\mu^2)(l^2 + m^2 + n^2)$, and then Drude's method gives $(\lambda, \mu, \nu)\,\omega = \frac{\partial\Omega}{\partial l}, \frac{\partial\Omega}{\partial m}, \frac{\partial\Omega}{\partial n}$. The elimination of (lmn) leads to a quadratic

$$\left.\begin{array}{l} \omega^2(1 - \Sigma u^2 + \epsilon\overline{\Sigma\lambda u}\,|^2) - 2\epsilon\omega\Sigma\lambda u = \dfrac{1}{\mu^2} - \Sigma u^2 \\ \text{of which the solution is} \\ \omega(1 - \Sigma u^2 + \epsilon\overline{\Sigma\lambda u}\,|^2) = \epsilon\Sigma\lambda u + \dfrac{1}{\gamma}\left(\dfrac{1}{\mu^2} - \Sigma u^2 + \epsilon\overline{\Sigma\lambda u}\,|^2\right)^{\frac{1}{2}} \end{array}\right\} \quad (37).$$

To connect this with the addition theorem we use (26) in the reciprocal form and write

$$\dot{x}_1' = (k_{14} + k_{11}\dot{x}_1 + k_{12}\dot{x}_2 + k_{13}\dot{x}_3)/(k_{44} + k_{41}\dot{x}_1 + k_{42}\dot{x}_2 + k_{43}\dot{x}_3) \quad (38\,a),$$

where *k*'s are the coefficients in (31), and so find

$$\frac{Vl'}{\mu} = \frac{\omega\,\{\lambda + (\gamma - 1)\,u\Sigma\lambda u/\Sigma u^2\} - \gamma u}{\gamma(V - \omega\sigma/V^2)} \quad \ldots\ldots(38\,b).$$

Squaring and adding we obtain the quadratic just given. The ray velocity apart from direction may be obtained more readily by writing

$$dt\sqrt{V^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} = dt'\sqrt{V^2 - \dot{x}'^2 - \dot{y}'^2 - \dot{z}'^2},$$
$$dt' = \gamma dt(1 - \omega\Sigma\lambda u/V^2),$$

and then taking $\dot{x} = \lambda\omega$, $\dot{x}' = Vl'/\mu$ when the above quadratic is presented in the form $V^2 - \omega^2 = \epsilon\gamma^2(V - \omega\Sigma\lambda u/V^2)$.

The transformation as applied to a plane wave gives material to determine velocity as well as direction. For free ether that velocity is of course V, but for a dielectric a variable value is found; and I take it we must use the same transformation in both cases. A variable velocity in wave transmission implies distinct values for ray velocity and direction, and we have found that these agree with the correspondence of velocities required by the transformation, or in other words the components of ray velocity may be treated as other velocities in what is called the addition theorem: it is properly a correspondence theorem.

The value $\Omega = V/\mu + \epsilon\Sigma lu$, which one has been accustomed to regard as the ideal solution of Fresnel's problem, is here replaced by a less simple value, but by way of compensation a satisfactory account of the use of the addition theorem is given.

To deal with refraction at a surface moving with normal velocity w, we write $u = 0 = v$ in the value of Ω. If then we denote similar quantities in free ether by letters with a zero index, the problem is to translate conditions $p'l' = p_0'l_0'$, $p'm' = p_0'm_0'$, $p' = \mu p_0'$ into terms of plm. The result is

$$\left.\begin{array}{l} pl = p_0l_0,\ pm = p_0m_0, \text{ involving } p^2(1-n^2) = p_0^2(1-n_0^2);\\ \text{and}\quad p\{1 - wn/\mu V - \epsilon w^2(1+n^2)/2V^2\} = \mu p_0(1 - wn_0/V), \end{array}\right\} \quad (39)$$

the last representing to the second order the exact equation $p(\Omega - nw) = p_0(V - n_0w)$. With regard to the use of equations (33) in this problem, there are two cases of refraction differing with the polarisation; and if m is taken $= 0$ in the wave argument one has quantities $X_2B_1B_3$ existing, the other $B_2X_1X_3$. It may be sufficient to state that in the first problem the conditions to be expressed are that $\xi_2 + wB_1/V$ and $\beta_1 + wX_2/V$ are continuous at a surface of separation for which z is constant.

§ 14. The quadratic form used by Einstein to correspond with a central acceleration is of course not referable to Maxwell's type: it satisfies the conditions referred to above which play so important a part in the connexion between the various potentials and the physical quantities to which they are attached. For an acceleration constant in magnitude and direction, a transformation exists which in its simplest form stands,

$$\left.\begin{array}{c} x_1' = x_1,\ \ x_2' = x_2,\ \ 1 + fx_3' = \sqrt{1+2fx_3}\cosh(fx_4)\\ fx_4' = \sqrt{1+2fx_3}\sinh(fx_4) \end{array}\right\} \ldots(40),$$

and will be found to give

$$dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2 = dx_1^2 + dx_2^2 + \frac{dx_3^2}{1+2fx_3} - (1+2fx_3)dx_4^2 \ldots(41).$$

The adjustment of correspondence is made to secure variables of the two types vanishing together. A more general form is obtained by using

$$\left.\begin{array}{c} \alpha + f(x_3' - b_3) = \sqrt{\alpha^2 + 2f(x_3 - a_3)}\cosh f(x_4 - a_4)\\ f(x_4' - b_4) = \sqrt{\alpha^2 + 2f(x_3 - a_3)}\sinh f(x_4 - a_4) \end{array}\right\} \ldots(42),$$

with correspondence at arbitrary values, and moreover a constant α at our disposal. A third form which gives isotropy, but with a determinant differing from -1, is obtained by using

$$\alpha + fx_3' = (\alpha + fx_3)\cosh fx_4,\ \ fx_4' = (\alpha + fx_3)\sinh fx_4 \ldots(43),$$

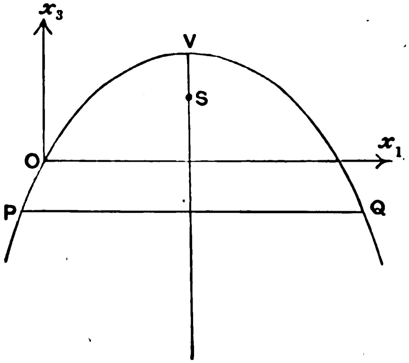
which gives $dx_1^2 + dx_2^2 + dx_3^2 - (a + fx_3)^2\, dx_4^2$ in lieu of (41). In general we shall be concerned with the simple type (40) and (41).

The resemblance of (41) to Einstein's form with $1 - 2m/r$ is very obvious: the acceleration is here $-m/r^2$, and if the direction of x_3 is that of the radius, then $2fx_3$ agrees with $-2m/r$. Thus an adjustment of correspondence is possible at any one point, but there is divergence in passing to neighbouring points.

§ 15. In respect to the orbits which attach to (41), whether the path of light or the orbit of a particle, it is clear that they are deducible directly from the transformation. If we take the case of light and for convenience suppose there is no motion in the direction x_2', then the path is given by $x_1' = lx_4'$, $x_3' = nx_4'$, where $l^2 + n^2 = 1$. Since $x_1 = x_1' = lx_4'$, and $nx_1 = nx_1' = lx_3'$, equations (40) may be written

$$\left.\begin{array}{l} l + nfx_1 = l\sqrt{1 + 2fx_3}\cosh(fx_4),\ fx_1 = l\sqrt{1 + 2fx_3}\sinh(fx_4) \\ \text{and so } l^2(1 + 2fx_3) = (l + nfx_1)^2 - f^2 x_1^2, \text{ or } 2lx_3 = 2nx_1 - lfx_1^2 \end{array}\right\} \quad (44),$$

and the orbit is parabolic*. The extreme position $1 + 2fx_3 = 0$ at which the transformation ceases to be valid gives $x_1 = (n \pm 1)/lf$. With the positive sign $lfx_1 = 1 + n$ or $fx_1(1 - n) = l$ makes $\sinh(fx_4) = \cosh(fx_4)$ and corresponds to $x_4 = +\infty$; the negative sign makes $\sinh(fx_4) = -\cosh(fx_4)$, and corresponds to $x_4 = -\infty$. Thus the spatial extent of the orbit is limited while the time-coordinate has a complete range of values; and when the transformation makes the orbit in x' (a straight line of course) agree at O with the orbit in x, the total course of the latter is from P through OV to Q, where PQ is parallel to Ox_1 at distance $1/2f$ below it.



In this problem with only two space dimensions we may for the purpose of graphical illustration represent time by a coordinate perpendicular to the plane of the diagram. The $(x_1'x_3')$ orbit is an infinite straight line, tangent to the parabola at O, or if x_4' is represented the line through O is tilted upwards. In the x scheme the space orbit is in the plane, but with x_4 represented it is a curved line on a parabolic cylinder, axis perpendicular to the diagram; which agrees at O with the direction of the x' path, comes from a point at infinite distance below P and passes to a point at infinite distance above Q.

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* Parabolic motion is always reducible to uniform linear motion. Thus if $x = ut$, $z = wt$ the transformation $a + fz/w = \sqrt{a^2 + 2fy}$ makes $1 = \dfrac{dy}{dt}\dfrac{f}{\sqrt{a^2 + 2fy}}$, or $\dfrac{d^2y}{dt^2} = \dfrac{f}{\sqrt{a^2 + 2fy}}\dfrac{dy}{dt} = f$; and also $a + fx/u = \sqrt{a^2 + 2fy}$.

With the transformation as in (40), when $l=1$, $n=0$ correspondence is made at the vertex V, the parabola is $2x_3=-fx_1^2$, and the critical position $1+2fx_3=0$ makes $fx_1=\pm 1$ occurring therefore at the ends of the latus rectum and giving a total deviation of a right angle. With l small and n nearly 1 the point of adjustment is at a great distance from the vertex $x_1=n/lf$, $x_3=n^2/2l^2f$. Here $x_1=(n\pm 1)/lf$ gives one very large value $x_1=2/lf$ and a small value $-l/2f$, the total deviation approaching two right angles. The reason for values so considerable in comparison with the central case is that the influence persists undiminished instead of showing sensible value only near its maximum and then falling off rapidly. The distance concerned in the critical value $1+2fx_3=0$ or $V^2+2fx_3=0$ is in general very great; for example, with the acceleration of gravity at the surface of the earth it is $4{\cdot}6\times 10^{12}$ km. On the other hand the whole effect given by Einstein's formula for deviation due to the sun's action would be produced by the constant acceleration at the surface acting for a space equal to the sun's radius on each side of the grazing point (i.e. from $x_1=-a$ to $x_1=+a$), a small distance relatively to the critical value for the constant acceleration.

§ 16. The quadratic (41) involves the values

$$\left.\begin{aligned} &g_{11}=1=g_{22},\quad g_{33}=1/(1+2fx_3),\quad g_{44}=-(1+2fx_3);\\ &\text{with } \{33,3\}=-f/(1+2fx_3),\ \{34,4\}=\{43,4\}=f/(1+2fx_3),\\ &\text{and}\qquad .\ \{44,3\}=f(1+2fx_3)\end{aligned}\right\}\quad(45)$$

for the Christoffel symbols. The constitutive relations are therefore, by (12),

$$\left.\begin{aligned}(\xi_1,\xi_2)&=(1+2fx_3)(X_1,X_2),\quad \xi_3=X_3\\ (\beta_1,\beta_2)&=(1+2fx_3)(B_1,B_2),\quad \beta_3=B_3\end{aligned}\right\}\quad\ldots\ldots(46).$$

The connexion of potentials is

$$\phi^1=\phi_1,\ \phi^2=\phi_2,\ \phi^3=(1+2fx_3)\,\phi_3,\ \phi^4=-\,\phi_4/(1+2fx_3);\ \ldots(47),$$

while the contravariant potentials satisfy

$$\frac{\partial\phi^p}{\partial x_p}=0\qquad\ldots\ldots(48).$$

The form of $\Box\psi$ is given by

$$\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}+\frac{\partial}{\partial x_3}(1+2fx_3)\frac{\partial\psi}{\partial x_3}-\frac{1}{1+2fx_3}\frac{\partial^2\psi}{\partial x_4^2}\ \ldots\ldots(49).$$

The density is given by

$$-\rho = \phi_{11}{}^4 + \phi_{22}{}^4 + (1+2fx_3)\,\phi_{33}{}^4 - \frac{1}{1+2fx_3}\,\phi_{44}{}^4 \quad \ldots(50),$$

and this form is applicable with alteration of the upper index to $-\rho u_1, \ldots$. These are internal relations of the system.

We now give a brief *résumé* of the transformations. We have here

$$\left.\begin{aligned} k_{11}=1,\ k_{22}=1,\ k_{33}=\cosh(fx_4)/\sqrt{1+2fx_3},\ k_{34}=\sqrt{1+2fx_3}\sinh(fx_4)\\ k_{43}=\sinh(fx_4)/\sqrt{1+2fx_3},\ k_{44}=\sqrt{1+2fx_3}\cosh(fx_4)\end{aligned}\right\}(51),$$

other k's are zero, and the determinant is 1. The transformation of (II) gives

$$\left.\begin{aligned} B_1 = k_{33}B_1' + k_{43}X_2',\ \ B_2 = k_{33}B_2' - k_{43}X_1',\ \ B_3 = B_3'\\ \xi_1 = k_{44}X_1' - k_{34}B_2',\ \ \xi_2 = k_{44}X_2' + k_{34}B_1',\ \ \xi_3 = X_3'\end{aligned}\right\} \ (52),$$

with formulæ for (I) which can be written down at sight, and the conjunction of the two groups will be found in agreement with (46). The transformation of (III) gives

$$\phi_1 = \phi_1',\ \phi_2 = \phi_2',\ \phi_3 = k_{33}\phi_3' + k_{43}\phi_4',\ \phi_4 = k_{44}\phi_4' + k_{34}\phi_3' \ \ldots(53),$$

and that of (IV)

$$\rho u_1 = \rho' u_1',\ \rho u_2 = \rho' u_2',\ \rho u_3 = \rho'(k_{44}u_3' - k_{34}),\ \rho = \rho'(k_{33} - k_{43}u_3') \ \ (54),$$

which together involve the invariance of $\rho(u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + \phi_4)$.

A property of the coefficients k may be noted, viz., that each satisfies $\Box k_{mn} = 0$. This is obvious in respect to $k_{44} = 1 + fx_3'$ and $k_{34} = fx_4'$; it is also true for the others, since

$$2(k_{33},\ k_{43}) = \frac{1}{1+fx_3' - fx_4'} \pm \frac{1}{1+fx_3' + fx_4'},$$

as follows from $\quad 1 + 2fx_3 = (1+fx_3')^2 - f^2x_4'^2$.

In respect to all these transformations there is dependence on the k's otherwise than as they affect the g's, in this case on the variable x_4. At the same time the whole group of relations given above as particular cases of general formulæ may be obtained by direct transformation. To any one like myself passing through a novitiate in this technique it may be serviceable to seek conviction by this more lengthy route. It is also useful to verify (50) by setting out from the value of ξ as curl of a covariant potential, transfer to the contravariant by (47), and then connect X with ξ by (46). The expression obtained has two terms which do not occur in the form $\Box\phi^4$, and this is true also for ρu_3, while $\Box$ suffices for ρu_1 and ρu_2. In each case the reference to a separate potential requires the use of (48). The orbital problem may also be treated as a dynamical problem with $dx_1/ds, \ldots$ for velocities. There are three ignorable coordinates giving rise to first integrals, and a fourth is given by the equation of energy. The results when transformed to coordinates x' must give uniform motion in a straight line.

§ 17. It is proposed to give here a dynamical theorem touching ignoration, which will first be stated in general form, but is meant for application to the theory of relativity.

The system has a potential energy U, a kinetic energy T_0, where

$$2T_0 = \gamma\dot{\phi}^2 + a_{11}\dot{x}_1^2 + 2a_{12}\dot{x}_1\dot{x}_2 + \dots, \text{ say } \gamma\dot{\phi}^2 + 2T \quad \dots(55),$$

and ϕ is an ignorable coordinate. It is desired to eliminate t in favour of ϕ, so as to direct attention to orbital form, and also to recognize that ultimately time measures are largely dependent on observations with respect to an ignorable coordinate.

Let T' in a new scheme be defined by

$$2T' = \frac{h}{\gamma}\left[a_{11}\left(\frac{dx_1}{d\phi}\right)^2 + 2a_{12}\frac{dx_1}{d\phi}\frac{dx_2}{d\phi} + \dots\right] \quad \dots\dots(56).$$

$$\left.\begin{aligned} &\text{say } 2T' = \frac{h}{\gamma}(a_{11}x_1'^2 + 2a_{12}x_1'x_2' + \dots) = \frac{\gamma}{h}(a_{11}\dot{x}_1^2 + 2a_{12}\dot{x}_1\dot{x}_2 + \dots), \\ &\text{that is} \qquad T' = \frac{\gamma}{h}T = \frac{\gamma}{h}\left(E_0 - U - \frac{h^2}{2\gamma}\right); \end{aligned}\right\} \quad (57)$$

where $h = \gamma\dot{\phi}$, and E_0 is constant value of total energy. A momentum p_1 expressed by $\frac{\partial T}{\partial \dot{x}_1}$ in one scheme is given by $\frac{\partial T'}{\partial x_1'}$ in the other. Thus an equation of motion is

$$\frac{d}{d\phi}\frac{\partial T'}{\partial x_1'} = \frac{\gamma}{h}\frac{dp_1}{dt} = \frac{\gamma}{h}\left[\dot{x}_1^2\frac{\partial}{\partial x_1}(\tfrac{1}{2}a_{11}) + \dot{x}_1\dot{x}_2\frac{\partial}{\partial x_1}(a_{12}) + \dots - \frac{\partial}{\partial x_1}\left(U + \frac{h^2}{2\gamma}\right)\right]$$

$$= \frac{h}{\gamma}\left[x_1'^2\frac{\partial}{\partial x_1}(\tfrac{1}{2}a_{11}) + x_1'x_2'\frac{\partial}{\partial x_1}(a_{12}) + \dots\right] - \frac{\gamma}{h}\frac{\partial}{\partial x_1}\left(U + \frac{h^2}{2\gamma}\right).$$

But $$\frac{h}{\gamma}x_1'x_2'\frac{\partial}{\partial x_1}a_{12} = x_1'x_2'\left(\frac{\partial}{\partial x_1}\frac{ha_{12}}{\gamma} + \frac{h}{\gamma^2}\frac{\partial\gamma}{\partial x_1}a_{12}\right)$$

$$= x_1'x_2'\frac{\partial}{\partial x_1}\frac{ha_{12}}{\gamma} + \frac{1}{h}\frac{\partial\gamma}{\partial x_1}a_{12}\dot{x}_1\dot{x}_2;$$

and therefore

$$\frac{d}{d\phi}\frac{\partial T'}{\partial x_1'} = \frac{\partial T'}{\partial x_1} + \frac{T}{h}\frac{\partial\gamma}{\partial x_1} - \frac{\gamma}{h}\frac{\partial}{\partial x_1}\left(U + \frac{h^2}{2\gamma}\right)$$

$$= \frac{\partial T'}{\partial x_1} + \frac{1}{h}\frac{\partial\gamma}{\partial x_1}\left(E_0 - U - \frac{h^2}{2\gamma}\right) - \frac{\gamma}{h}\frac{\partial U}{\partial x_1} + \frac{h}{2\gamma}\frac{\partial\gamma}{\partial x_1}$$

$$= \frac{\partial T'}{\partial x_1} - \frac{\partial}{\partial x_1}\left(\frac{\gamma U}{h} - \frac{\gamma E_0}{h}\right) \quad \dots\dots(58).$$

The new system has therefore a potential energy $U' = \frac{\gamma}{h}(U - E_0)$, a kinetic energy $\frac{\gamma}{h}(E_0 - U) - \frac{h}{2}$, and a total energy $-\frac{h}{2}$. Thus it behaves as a true dynamical system in which the ignorable coordinate takes the place of time. This is applied to the scheme

$$-1 = 2T_0 = -\gamma\left(\frac{dt}{ds}\right)^2 + a_{11}\left(\frac{dx_1}{ds}\right)^2 + 2a_{12}\frac{dx_1}{ds}\frac{dx_2}{ds} + \dots \quad \dots(59),$$

in which t is supposed ignorable and $\gamma\frac{dt}{ds} = h$. The new system then has

$$2T' = \frac{h}{\gamma}\left[a_{11}\left(\frac{dx_1}{dt}\right)^2 + 2a_{12}\frac{dx_1}{dt}\frac{dx_2}{dt} + \dots\right] \quad \dots\dots(60).$$

Also $U' = \frac{\gamma}{2h}$, and since

$$2T' = \frac{\gamma}{h}\left[a_{11}\left(\frac{dx_1}{ds}\right)^2 + \dots\right] = \frac{\gamma}{h}\left(-1 + \frac{h^2}{\gamma}\right) = h - 2U',$$

the new system has a total energy $h/2$; while the originating system had $E_0 = -1/2$, and $U = 0$.

§ 18. With Einstein's value $\gamma = 1 - 2m/r$ the new potential energy is $\frac{1}{2h}\left(1 - \frac{2m}{r}\right)$, and when we drop the convention that the velocity of light V is taken for unit, the new kinetic energy is found to be $\frac{hV^2}{2} - \frac{1}{2h}\left(V^2 - \frac{2m}{r}\right)$, the potential energy $\frac{1}{2h}\left(V^2 - \frac{2m}{r}\right)$, and the total energy $\frac{V^2}{2h}$. Under these circumstances h is a pure number differing little from unity. The potential energy vanishes at the critical radius.

The method applied to the case of uniform acceleration gives for the new potential energy $\frac{1}{2h}(1 + 2fx_3)$. The application to the isotropic form gives a potential energy $\frac{1}{2h}(\alpha + fx_3)^2$ suggesting an oscillation type; but the occurrence of the critical value in what for an oscillation problem must be the precise place of equilibrium, presents a difficulty. Here arises the question as to the meaning of the alternative form $B_{pq} = \lambda g_{pq}$, cf. p. 146.

With a notation

$$\left.\begin{aligned} &\gamma^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 - \gamma\, dt^2 \equiv g_{11}dx_1^2 + \dots + g_{44}dx_4^2 \\ &\text{we find}\quad B_{11} = \frac{1}{2r^2\gamma}\frac{d}{dr}\left(r^2\frac{d\gamma}{dr}\right),\ B_{44} = -\frac{\gamma}{2r^2}\frac{d}{dr}\left(r^2\frac{d\gamma}{dr}\right), \\ &\qquad B_{22} = \frac{d}{dr}(r\gamma) - 1,\ B_{33} = B_{22}\sin^2\theta \end{aligned}\right\} \quad (61).$$

The condition in the cases of B_{11} and B_{44} gives

$$\frac{d}{dr}\left(r^2\frac{d\gamma}{dr}\right) = 2\lambda r^2, \text{ or } \gamma = \frac{\lambda r^2}{3} + \frac{A}{r} + C;$$

while that for B_{22} and B_{33} gives $\gamma = 1 + \frac{\lambda r^2}{3} + \frac{A}{r}$. They can both be satisfied by taking $C = 1$. But I find it difficult to accept the use of forms r^2 and r^{-1} in conjunction, or to resist the conclusion that the former applies to the status of the quadratic form within the attracting body, the latter to an external position. Thus if we take $\lambda = 4\pi\rho_0$, $\gamma = 1 + 4\pi\rho_0 r^2/3$, where ρ_0 is density of matter, there is continuity of acceleration at the surface. The contrast between (B_{11}, B_{44}) on the one hand and (B_{22}, B_{33}) on the other will be noticed, the first showing an equation of the second order, of potential type, capable of adjustment to the full continuity of an attraction potential, the second only capable of meeting continuity of the acceleration term or attractive force.

For the case of an infinite cylinder, if we write

$$\left.\begin{aligned} &\gamma^{-1}d\varpi^2 + \varpi^2 d\theta^2 + dz^2 - \gamma dt^2 \equiv g_{11}dx_1^2 + \ldots + g_{44}dx_4^2 \\ &\text{we find* } B_{11} = \frac{1}{2\varpi\gamma}\frac{d}{d\varpi}\left(\varpi\frac{d\gamma}{d\varpi}\right),\quad B_{44} = -\frac{\gamma}{2\varpi}\frac{d}{d\varpi}\left(\varpi\frac{d\gamma}{d\varpi}\right), \\ &\qquad\qquad B_{22} = \varpi\frac{d\gamma}{d\varpi},\quad B_{33} = 0 \end{aligned}\right\} \quad (62).$$

For an external position $\gamma = 1 + 4\pi\rho_0 a^2 \log_e \varpi$ makes B_{11} and B_{44} vanish, B_{33} vanishes, but $B_{22} = 4\pi\rho_0 a^2$. For an internal position $\gamma = 1 + 2\pi\rho_0\varpi^2$ makes $(B_{11}, B_{22}, B_{44}) = 4\pi\rho_0(g_{11}, g_{22}, g_{44})$, but B_{33} is still zero while $g_{33} = 1$. The possibility of adjustment for B_{11} and B_{44} to the full continuity of an attraction potential is as clear as for the sphere. There is no adjustment for the tangential terms to the continuity of acceleration.

The general argument is then as follows. With reference to a quadratic form the orbit of a point is taken to be a geodesic. In general dynamics when there is no potential energy the equations of motion are precisely geodesic equations in which t is a geodesic length. The most general integrating principle is that of ignoration, the outstanding feature of which is the introduction of a term which functions as a potential energy. In the case of Einstein's quadratic formula for central acceleration the variable coefficient attached to the time, which is an ignorable coordinate, provides the potential energy corresponding to gravitation.

As applied to a central acceleration varying as r^{-2} outside a

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* It may be of use to give the Christoffel symbols for this case. They are

$$\{11, 1\} = -\frac{1}{2\gamma}\frac{d\gamma}{d\varpi},\ \{12, 2\} = \frac{1}{\varpi},\ \{14, 4\} = \frac{1}{2\gamma}\frac{d\gamma}{d\varpi},\ \{22, 1\} = -\varpi\gamma,\ \{44, 1\} = \tfrac{1}{2}\gamma\frac{d\gamma}{d\varpi}.$$

given radius and there changing continuously to an acceleration varying as r, the agreement is complete. The passage to the cylindrical case with acceleration varying as ϖ^{-1} outside and as ϖ inside a given radius is in perfect conformity with the first case in respect to a radial component, but is at fault in respect to components transverse to the acceleration. If we had any assurance that this position could be maintained for the acceleration of gravity in less simple cases, instead of being an index of greater difficulty in securing even this degree of conformity, as it may well be, then the breach of conditions in respect to transverse forms might be accepted as pointing to a gravitational influence, minute but direct, on the circulation of transverse electric currents.

§ 19. The differential equations which correspond to the integral forms (I) and (II) are

$$\frac{\partial X_1}{\partial x_4} + \rho u_1 = \frac{\partial \beta_3}{\partial x_2} - \frac{\partial \beta_2}{\partial x_3}, \quad \frac{\partial B_1}{\partial x_4} = \frac{\partial \xi_2}{\partial x_3} - \frac{\partial \xi_3}{\partial x_2} \quad \ldots\ldots(63).$$

Energy and stress equations then appear first in a form

$$\left.\begin{aligned}
&-\rho\Sigma u\xi = \frac{\partial}{\partial x_1}(\xi_2\beta_3 - \xi_3\beta_2) + \ldots \\
&+\left(\xi_1\frac{\partial X_1}{\partial x_4} + \xi_2\frac{\partial X_2}{\partial x_4} + \xi_3\frac{\partial X_3}{\partial x_4} + \beta_1\frac{\partial B_1}{\partial x_4} + \beta_2\frac{\partial B_2}{\partial x_4} + \beta_3\frac{\partial B_3}{\partial x_4}\right), \\
&-\rho(\xi_3 + u_1B_2 - u_2B_1) = \frac{\partial}{\partial x_4}(X_1B_2 - X_2B_1) - \frac{\partial}{\partial x_1}(B_1\beta_3 + X_1\xi_3) - \ldots \\
&+B_1\frac{\partial \beta_1}{\partial x_4} + B_2\frac{\partial \beta_2}{\partial x_4} + B_3\frac{\partial \beta_3}{\partial x_4} + X_1\frac{\partial \xi_1}{\partial x_4} + X_2\frac{\partial \xi_2}{\partial x_4} + X_3\frac{\partial \xi_3}{\partial x_4}
\end{aligned}\right\}(64),$$

where values of ρ, $\rho u_1, \ldots$ from (63) are given to the left-hand members, which then are somewhat altered by using the second group of (63). If now we write $2E = \Sigma X\xi + \Sigma B\beta$ as a general value of energy, then suppose it expressed as a quadratic in X and B, and write as $\left(\frac{\partial E}{\partial x}\right)_C$ the result of differentiating coefficients *only* in this quadratic expression, we obtain the more convenient form

$$\left.\begin{aligned}
-\rho\Sigma u\xi &= \frac{\partial}{\partial x_1}(\xi_2\beta_3 - \xi_3\beta_2) + \frac{\partial}{\partial x_2}(\xi_3\beta_1 - \xi_1\beta_3) \\
&\quad + \frac{\partial}{\partial x_3}(\xi_1\beta_2 - \xi_2\beta_1) + \frac{\partial E}{\partial x_4} - \left(\frac{\partial E}{\partial x_4}\right)_C, \\
-\rho(\xi_3 + u_1B_2 - u_2B_1) &= \frac{\partial}{\partial x_4}(X_1B_2 - X_2B_1) - \frac{\partial}{\partial x_1}(B_1\beta_3 + X_1\xi_3) \\
&\quad - \frac{\partial}{\partial x_2}(B_2\beta_3 + X_2\xi_3) - \frac{\partial}{\partial x_3}(B_3\beta_3 + X_3\xi_3 - E) + \left(\frac{\partial E}{\partial x_3}\right)_C
\end{aligned}\right\}(65).$$

When the coefficients g are constant, and therefore the coefficients of the constitutive relations constant, the final term of these equations vanishes. In the form which represents a uniform acceleration, some coefficients depend on x_3 and this final term exists in the last of equations (65). It represents a body-force which it is important to examine.

§ 20. In this problem a reference to (46) shows that we have

$$2E(B,X) = (1+2fx_3)(B_1^2+B_2^2+X_1^2+X_2^2)+B_3^2+X_3^2 \quad \ldots(66),$$

and therefore $$\left(\frac{\partial E}{\partial x_3}\right)_C = f(B_1^2+B_2^2+X_1^2+X_2^2) \quad \ldots\ldots(67).$$

We now suppose that in the x' system from which this has been transformed we are dealing with plane waves, and that we are only concerned with mean values of energy-content or streams of energy. Also it is sufficient to consider places near the origin from which the transformation is made so that we can write $x_3 = 0$, $x_4 = 0$ after any differentiations; and in (67) we may replace BX by $B'X'$. Next examine the term $\frac{\partial}{\partial x_4}(X_1B_2 - X_2B_1)$, using

$$X_1B_2 - X_2B_1 = (k_{33}^2+k_{43}^2)(X_1'B_2' - X_2'B_1') \\ - k_{33}k_{43}(B_1'^2+B_2'^2+X_1'^2+X_2'^2).$$

For mean values at the origin the sole contribution to $\frac{\partial}{\partial t}(X_1B_2 - X_2B_1)$ is through $\frac{\partial}{\partial x_4}k_{43} = f$, and is

$$-f(B_1'^2+B_2'^2+X_1'^2+X_2'^2) \quad \ldots\ldots(68).$$

The stress variations with $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ contribute nothing. The third stress or

$$\tfrac{1}{2}(B_1\beta_1 + B_2\beta_2 - B_3\beta_3) + \tfrac{1}{2}(X_1\xi_1 + X_2\xi_2 - X_3\xi_3)$$

is $\tfrac{1}{2}(k_{33}k_{44}+k_{34}k_{43})(X_1'^2+X_2'^2+B_1'^2+B_2'^2)$
$$-(k_{34}k_{33}+k_{44}k_{43})(X_1'B_2' - X_2'B_1') - \tfrac{1}{2}(B_3'^2+X_3'^2).$$

As $k_{33}k_{44} - k_{34}k_{43} = 1$, and the mean value of $(X_1'^2 + \ldots)$ or of $X_1'B_2' - X_2'B_1'$ is constant, the differential coefficient with regard to x_3 of this stress will be found to vanish for $x_3 = 0$, and again there is no contribution. Thus the last equation of (65) assumes for this case the form $\frac{\partial p_3}{\partial t} + \left(\frac{\partial E}{\partial x_3}\right)_C = 0$, or with the notation of the next section $\frac{\partial p_3}{\partial t} = \left(\frac{\partial L}{\partial x_3}\right)_C$, where p_3 is a momentum.

It is clear then that the contribution of energy to inertia is given by a form $(B_1'^2+B_2'^2+X_1'^2+X_2'^2)/V^2$, which contains twice the sum of energy components transverse to the direction of acceleration. Using a wave argument $lx + nz - Vt$ it is readily seen that this expression is $E(1 + \cos^2\theta)/V^2$, where $\cos\theta$ is the angle between the normal to the wave and the direction of acceleration. The maximum is for a wave normal in the line of acceleration, because all its components are transverse; the mean value for all directions is $4E/3V^2$. The contribution is usually taken to be E/V^2, and it might be supposed that the other term represents pressure. I do not think that view is tenable, for a stress equation contains only rates of change of stress. With such a view in mind the values of separate terms have been given in detail, and the examination points to the whole formula having one interpretation.

The first equation of (64) treated in the same way yields a result

$$\frac{\partial E}{\partial t} + \frac{f}{V}(X_1'B_2' - X_2'B_1') = 0, \text{ or } \frac{\partial E}{\partial t} + fp_3 = 0, \text{ or } \frac{\partial E}{\partial t} + \frac{nfE}{V} = 0 \ldots (69),$$

if $lx' + my' + nz' - Vt$ is wave argument in the x' system, and we are still dealing with a mean energy-content. Motion with constant velocity in a uniform field gives no rate of change of energy-content, motion with constant acceleration gives a rate of change defined in this way.

§ 21. The expression of E as a quadratic in BX is not essential, but we are restricted in our choice of constituents for a quadratic form in lieu of the bilinear form, to one in which either E or $L = \frac{1}{2}(\Sigma B\beta - \Sigma X\xi)$ will furnish an equivalent of the term $\left(\frac{\partial E}{\partial x_3}\right)_C$ above. It is proposed to examine the expression $L(\beta, X)$ and to show that this gives the required value with sign reversed. With this expression $2\delta L = \Sigma(\beta\delta B - X\delta\xi)$, where δ is applied to the coefficients of B and ξ expressed as linear functions of β and X; while for $E(B, X)$, $2\delta E = \Sigma(B\delta\beta + X\delta'\xi)$ B and ξ being linear functions of B and X, but the variation of ξ different from the above. Thus $2(\delta L + \delta E) = \Sigma\{\beta\delta B + B\delta\beta + X(\delta'\xi - \delta\xi)\}$ is the quantity we are to examine. Apply to the first relation of (12), viz. $G_{44}\xi_1 + G_{24}B_3 - G_{34}B_2 = \sqrt{-\Delta_g}\,(g_{11}X_1 + g_{12}X_2 + g_{13}X_3)$, the following procedure:—perform the variations for the L scheme with $\delta\xi$ and δB both occurring, for the E scheme with only $\delta'\xi$ occurring, and subtract the one varied form from the other. The variations of coefficients shown explicitly in (12) are cancelled in the subtraction, with the result $G_{44}(\delta\xi_1 - \delta'\xi_1) + G_{24}\delta B_3 - G_{34}\delta B_2 = 0$, from which we get

$$G_{44}\Sigma X(\delta\xi - \delta'\xi) = \Sigma X_1(G_{34}\delta B_2 - G_{24}\delta B_3) = \Sigma\delta B_1(G_{24}X_3 - G_{34}X_2).$$

Hence $2(\delta L + \delta E)$

$$= \Sigma\,[\beta\delta B + B\delta\beta - \delta B_1\,(G_{24}X_3 - G_{34}X_2)/G_{44}]$$

$$= \Sigma\,[B\delta\beta + \delta B_1\,(G_{44}\beta_1 - G_{24}X_3 + G_{34}X_2)/G_{44}]$$

$$= \Sigma\,[B\delta\beta + \delta B_1\,(g_{11}B_1 + g_{12}B_2 + g_{13}B_3)\sqrt{-\Delta_g}/G_{44}], \text{ by the second (12)},$$

$$= \Sigma\,[B\delta\beta + \Sigma B_1\,(g_{11}\delta B_1 + g_{12}\delta B_2 + g_{13}\delta B_3)\sqrt{-\Delta_g}/G_{44}].$$

Now take variations of the second of (12), for the L case with B only varied, for E with β only varied. On subtraction we have $-G_{44}\delta\beta_1 = \sqrt{-\Delta_g}\,(g_{11}\delta B_1 + g_{12}\delta B_2 + g_{13}\delta B_3)$ in virtue of which

$$\delta L + \delta E = 0, \text{ i.e., } \delta L(\beta, X) + \delta E(B, X) = 0 \quad \text{......(70)}.$$

In each formula the variation touches only the coefficients of the quadratic expression, and accordingly we may write $\left(\frac{\partial L}{\partial x_3}\right)_C$ in (65) in lieu of $\left(\frac{\partial E}{\partial x_3}\right)_C$. The case of constant acceleration with

$$2L(\beta, X) = (\beta_1^2 + \beta_2^2)/(1 + 2fx_3) - (1 + 2fx_3)(X_1^2 + X_2^2) + \beta_3^2 - X_3^2$$

provides an example.

The various forms of quadratic expression are readily written down by use of (12), (13), (14); it may be sufficient to make a brief comment on the forms. The form $E(B, X)$ contains products of B and X, and gives $\beta\xi$ as $\frac{\partial E}{\partial B}$ and $\frac{\partial E}{\partial X}$; $L(B, X)$ is the difference of quadratics in B and X, ξ and β not derivable from it. The form $E(\beta, X)$ is the sum of quadratics in β and X; while $L(\beta, X)$ contains also products of β and X, and gives B as $\frac{\partial L}{\partial \beta}$ and ξ as $-\frac{\partial L}{\partial X}$. The dynamical analogue of (B, X) forms is expression in terms of momenta, that of (β, X) forms is expression in terms of velocity for one type momenta for the other. Expression in terms of (ξ, B) would make L' a reciprocal of L, and presumably would give $\partial L(\beta, X) + \partial L'(B, \xi) = 0$.

On the Polygons Inscribed in one Conic and Circumscribed to Another. By Dr W. BURNSIDE, Pembroke College, Cambridge.

[*Read* 3 March 1924.]

This problem is in general treated in connection with the division of the periods of the elliptic functions. It is the object of the present note to shew, from a purely algebraical point of view, that the condition of closing of the polygon depends solely on a difference equation of the form

$$(p_n - p_1)^2 p_{n-1} p_{n+1} = (p_n p_1 - 1)^2.$$

Every two real conics, one at least of which is not reducible, can be projected into

$$S_1 \text{ or } x^2 + y^2 - z^2 = 0,$$
$$S_2 \text{ or } ax^2 + by^2 - cz^2 = 0,$$

a, b, c being real numbers.

Any point on S_1 is $(1 - \alpha^2, 2\alpha, 1 + \alpha^2)$; and the chord joining the points whose parameters are α, β is

$$x(1 - \alpha\beta) + y(\alpha + \beta) - z(1 + \alpha\beta) = 0.$$

This touches S_2 if

$$b(a - c)\alpha^2\beta^2 - ca(\alpha^2 + \beta^2) + 2(bc - ca + ab)\alpha\beta + b(a - c) = 0.$$

Hence if s_i, s_{i+1} are the parameters of successive vertices of a polygon inscribed in S_1 and circumscribed to S_2, then

$$b(a-c)s_i^2 s_{i+1}^2 - ca(s_i^2 + s_{i+1}^2) + 2(bc - ca + ab)s_i s_{i+1} + b(a - c) = 0.$$

In general the values of s_{i+1} given by this quadratic are irrational in s_i. If however S_1 and S_2 have double contact, i.e. if either $a = b$, $b = c$, or $c = a$, s_{i+1} is a rational linear function of s_i. In these cases there are constant numbers, α, β, κ, such that for every positive or negative integral value of i,

$$\frac{s_i - \alpha}{s_i - \beta} = \kappa^i \frac{s_0 - \alpha}{s_0 - \beta}.$$

If S_2 consists of two straight lines, i.e. if either a, b, or c is zero, a chord of S_1 which meets S_2 in two coincident points must pass through the intersection of the two lines. In this case the polygon has just two vertices and two (coincident) sides.

In what follows it is assumed that

$$abc(b - c)(c - a)(a - b) \neq 0.$$

Then the relation between s_i and s_{i+1} may be written

$$s_i^2 s_{i+1}^2 + p_1(s_i^2 + s_{i+1}^2) + 2q_1 s_i s_{i+1} + 1 = 0,$$

where $$a(1 - q_1 + p_1) = 2bp_1 = c(1 + q_1 - p_1),$$
and the polygon has at least three sides.

If s_{i+1} is eliminated between the preceding equation and
$$s_{i+1}^2 s_{i+2}^2 + p_1(s_{i+1}^2 + s_{i+2}^2) + 2q_1 s_{i+1} s_{i+2} + 1 = 0,$$
the result is
$$(s_i - s_{i+2})^2 [4p_1 q_1^2 s_i^2 s_{i+2}^2 + (p_1^2 - 1)^2 (s_i^2 + s_{i+2}^2)$$
$$+ 2\{(p_1^2 - 1)^2 - 2q_1^2 (p_1^2 + 1)\} s_i s_{i+2} + 4p_1 q_1^2] = 0.$$

Since the polygon has more than two sides
$$s_i \neq s_{i+2}.$$

If p_1 were zero, either a or c would be zero, which has been assumed not to be the case.

If $q_1 = 0$, the equation is
$$(s_i + s_{i+2})^2 = 0,$$
so that $$s_i = s_{i+4},$$
and the polygon is a quadrilateral.

If $q_1 \neq 0$, the equation may be written
$$s_i^2 s_{i+2}^2 + p_2(s_i^2 + s_{i+2}^2) + 2q_2 s_i s_{i+2} + 1 = 0,$$
where $$p_2 = \frac{(p_1^2 - 1)^2}{4p_1 q_1^2}, \quad q_2 = \frac{(p_1^2 - 1)^2 - 2q_1^2 (p_1^2 + 1)}{4p_1 q_1^2}.$$

Suppose now that it has been shewn that for all values of the suffix r not exceeding n, the relation between s_i and s_{i+r} is
$$s_i^2 s_{i+r}^2 + p_r(s_i^2 + s_{i+r}^2) + 2q_r s_i s_{i+r} + 1 = 0,$$
where p_r, q_r are finite numbers.

Between the two equations
$$\alpha^2\gamma^2 + p_n(\alpha^2 + \gamma^2) + 2q_n \alpha\gamma + 1 = 0,$$
$$\beta^2\gamma^2 + p_1(\beta^2 + \gamma^2) + 2q_1 \beta\gamma + 1 = 0,$$
let γ be eliminated. The result is
$$E \equiv [(p_n - p_1)(\alpha^2\beta^2 - 1) + (p_n p_1 - 1)(\alpha^2 - \beta^2)]^2$$
$$- 4[\alpha\beta(p_1 q_n \beta - p_n q_1 \alpha) + q_n \alpha - q_1 \beta]$$
$$[\alpha\beta(q_1\alpha - q_n\beta) + p_n q_1 \beta - p_1 q_n \alpha] = 0,$$
an equation of the fourth degree in both α and β. Now if α is s_i, then γ is s_{i+n} or s_{i-n} and β is s_{i+n+1}, s_{i+n-1}, s_{i-n+1} or s_{i-n-1}; while if β is s_i then α has one of these four values. Hence the previous equation must be symmetric in α and β. The necessary and sufficient condition for this is found to be
$$p_n(p_1^2 - q_1^2 + 1) = p_1(p_n^2 - q_n^2 + 1).$$

Now, on the assumption that has been made,

$$s_i^2 s_{i+n-1}^2 + p_{n-1}(s_i^2 + s_{i+n-1}^2) + 2q_{n-1}s_i s_{i+n-1} + 1 = 0.$$

Hence if $F \equiv \alpha^2\beta^2 + p_{n-1}(\alpha^2+\beta^2) + 2q_{n-1}\alpha\beta + 1,$

then F is a factor of E.

First suppose that $p_n - p_1 \neq 0.$

Then $$E \equiv (p_n - p_1)^2 FG,$$

where G is a symmetric quadrato-quadratic function of α and β, with $\alpha^2\beta^2$ for the leading term and unity for the constant term. Moreover a comparison of the two sides of the identity shews that G has no terms in $\alpha\beta(\alpha+\beta)$ or in $\alpha+\beta$, so that

$$G \equiv \alpha^2\beta^2 + p_{n+1}(\alpha^2+\beta^2) + 2q_{n+1}\alpha\beta + 1,$$

where, on comparison,

$$\begin{aligned}
(p_n-p_1)^2(p_{n-1}+p_{n+1}) &= 2(p_n q_1^2 + p_1 q_n^2),\\
(p_n-p_1)^2(q_{n-1}+q_{n+1}) &= -2(p_n+p_1)q_1q_n,\\
(p_n-p_1)^2(p_{n-1}q_{n+1}+p_{n+1}q_{n-1}) &= -2(p_np_1+1)q_1q_n,\\
(p_n-p_1)^2 p_{n-1}p_{n+1} &= (p_np_1-1)^2,\\
(p_n-p_1)^2 q_{n-1}q_{n+1} &= -(p_n-p_1)^2-(p_np_1-1)^2\\
&\quad + (p_n^2+1)q_1^2 + (p_1^2+1)q_n^2.
\end{aligned}$$

These equations give finite values for p_{n+1} and q_{n+1}, and three equations of condition which are necessarily satisfied by the previously determined p's and q's. It follows that s_i and s_{i+m} are connected by a relation of the form

$$s_i^2 s_{i+m}^2 + p_m(s_i^2 + s_{i+m}^2) + 2q_m s_i s_{i+m} + 1 = 0,$$

where p_m, q_m can be calculated successively until a point is reached at which $p_m = p_1$.

Suppose now that, in E, $p_n - p_1 = 0$. Then

$$\begin{aligned}
E \equiv (p_1^2-1)^2(\alpha^2-\beta^2)^2 &+ 4p_1\alpha^2\beta^2(q_n\beta - q_1\alpha)^2\\
&+ 4(p_1^2+1)\alpha\beta(q_n\beta-q_1\alpha)(q_n\alpha-q_1\beta) + 4p_1(q_n\alpha-q_1\beta)^2.
\end{aligned}$$

It is still true that $\alpha^2\beta^2 + p_{n-1}(\alpha^2+\beta^2) + 2q_{n-1}\alpha\beta + 1$ is a factor of E, and the other factor must be

$$4p_1(q_n\alpha - q_1\beta)^2.$$

Now it follows from a preceding equation that if $p_n = p_1$, then $q_n^2 = q_1^2$.

If $q_n = q_1$,

$$E \equiv 4p_1q_1^2(\alpha-\beta)^2\left(\alpha^2\beta^2 + \frac{(p_1^2-1)^2}{4p_1q_1^2}(\alpha^2+\beta^2) + 2\cdot\frac{(p_1^2-1)^2 - 2q_1^2(p_1^2+1)}{4p_1q_1^2}\alpha\beta + 1\right),$$

so that
$$s_i = s_{i+n+1},$$
$$p_{n-1} = p_2,\ q_{n-1} = q_2.$$

If $q_n = -q_1$,
$$E \equiv 4p_1q_1^2(\alpha+\beta)^2\left(\alpha^2\beta^2 + \frac{(p_1^2-1)^2}{4p_1q_1^2}(\alpha^2+\beta^2) - 2.\frac{(p_1^2-1)^2 - 2q_1^2(p_1^2+1)}{4p_1q_1^2}\alpha\beta + 1\right),$$
so that
$$s_i = -s_{i+n+1} = s_{i+2n+2},$$
$$p_{n-1} = p_2,\ q_{n-1} = -q_2.$$

The combined result of the two cases may be stated as follows:

If $p_1, p_2, \ldots, p_{n-1}$ are all distinct, the polygon, if closed, must have more than n sides. If in addition $p_n = p_1$, so that $q_n^2 = q_1^2$, then the polygon is closed. It has $n+1$ sides when $q_n = q_1$; and $2n+2$ sides when $q_n = -q_1$.

Since p_i $(i = 3, 4, \ldots)$ are determined rationally in terms of p_1 and p_2, by repeated use of the formula
$$(p_m - p_1)^2 p_{m-1}p_{m+1} = (p_mp_1 - 1)^2,$$
there is no difficulty in forming the equation $p_n = p_1$ as a rational algebraical equation in p_1 and p_2.

This rational equation in p_1 and p_2, or in p_1 and q_1^2, when p_2 is replaced by its value in terms of p_1 and q_1, will always be reducible. In fact, for a polygon of ν sides the relation
$$p_{\mu\nu\pm1} = p_1$$
must hold for all integral values of μ. It follows that the equation
$$p_n = p_1,$$
apart from the assumption that $p_1, p_2, \ldots, p_{n-1}$ are all distinct, must be satisfied by polygons of ν sides, where ν is equal to or a factor of either $2(n-1)$ or $2(n+1)$.

As an illustration the difference equation leads to
$$p_4 = \frac{p_1^2(p_2^2-1)^2}{p_2(p_1^2+1-2p_1p_2)^2}.$$

The relation $\quad p_4 - p_1 = 0$

is at once found to be reducible in the form
$$(p_2 - p_1)(p_1p_2^3 - 3p_1^2p_2^2 + (p_1^3 + 2p_1)p_2 - 1) = 0,$$
where the first factor corresponds to polygons of 3 and 6 sides, and the second to polygons of 5 and 10 sides. When p_2 is replaced by its value in terms of p_1 and q_1^2, each factor is found again to be reducible; one irreducible factor corresponding to each of the four cases.

The Problems of Random Flight and Conduction of Heat. By Dr W. BURNSIDE, Pembroke College, Cambridge.

[*Received* 5 April 1924.]

In a paper on random flight* Lord Rayleigh proved the following result: A number is formed by adding together n numbers each of which is equally likely to have any value from $-a$ to $+a$. Then, if $f(n, s)\,ds$ is the probability that the number so formed lies between s and $s + ds$, and if n is sufficiently great,

$$f(n, s) = \sqrt{\frac{3}{2\pi n a^2}}\, e^{-\frac{3s^2}{2na^2}} \text{ very nearly.}$$

This result may be stated as follows: A point moves discontinuously in a straight line. For a time τ it has a constant velocity. During the next time-interval τ it again has a constant velocity, and so on. Then if each of these velocities is equally likely to have any value from $-v$ to $+v$, the probability that in the time $n\tau$, the point moves a distance lying between s and $s + ds$ is $f(n, s)\,ds$, with $v\tau$ written for a.

Suppose now that N points, moving in this way, are, when $t = 0$, at the origin from which x is measured. Then if $f(n, x)\,dx$ is p, the probability that, when $t = n\tau$, just ν of them are on the segment of the line lying between x and $x + dx$ is

$$\frac{N!}{\nu!\,\overline{N-\nu}!}\,p^{\nu}.$$

This probability has its greatest value when ν is the greatest integer in $(Np + 1)/(1 + p)$. Since when dx is small p is small, the most probable number of the N points, which, when $t = n\tau$, lie in the segment dx, is very nearly equal to

$$Nf(n, x)\,dx.$$

Now suppose that points moving in the way described are distributed all along the line and that, when $t = 0$, the number of such points per unit length at x is $N(x)$. Then it follows from the preceding result that, when $t = n\tau$, the most probable number of the points that lie in the segment from x to $x + dx$ is

$$dx\int_{-\infty}^{\infty} N(x_1) f(n, x - x_1)\,dx_1.$$

In other words, if $N(x)$ represents the actual distribution of the points when $t = 0$, and $N(x, n\tau)$ the most probable distribution when $t = n\tau$, then

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* *Phil. Mag.* (1919).

$$N(x, n\tau) = \int_{-\infty}^{\infty} N(x_1) f(n, x - x_1)\, dx_1$$

$$= \sqrt{\frac{3}{2\pi n v^2 \tau^2}} \int_{-\infty}^{\infty} N(x_1)\, e^{-\frac{3(x-x_1)^2}{2nv^2\tau^2}}\, dx_1$$

or

$$N(x, t) = \sqrt{\frac{3}{2\pi v^2 \tau t}} \int_{-\infty}^{\infty} N(x_1)\, e^{-\frac{3(x-x_1)^2}{2v^2\tau t}}\, dx_1.$$

Now if $u(x)$ is the initial temperature of a straight bar of material whose conductivity is k and heat-capacity is c, and if no heat is radiated from the sides, the temperature at any point of the bar at time t is given by the known formula

$$u(x, t) = \sqrt{\frac{c}{4\pi k t}} \int_{-\infty}^{\infty} u(x_1)\, e^{-\frac{c(x-x_1)^2}{4kt}}\, dx_1.$$

If

$$v^2\tau = \frac{6k}{c}$$

these formulae are identical.

The condition that n shall be a very large number involves that, when t is a finite time-interval, τ shall be a very small time-interval. Further, $\frac{1}{3}v^2$ is the mean-square velocity of the moving points, say V^2; and for moderate values of k/c, V^2 must be great if τ is small.

Hence if the moving points be regarded as carriers of heat-energy, each point being associated with the same amount of energy, the actual distribution of heat-energy at any time, resulting from a given initial distribution, is the same as the most probable distribution at the same time given by the ideal carriers, provided that

$$V^2\tau = \frac{2k}{c}.$$

The same result as regards the distribution of heat-energy will clearly hold for very various laws of motion of the carriers, provided that the intervals during which the carriers move uniformly are sufficiently small, and the mean-square velocity is sufficiently large; while between $\bar{\tau}$ the mean interval and V^2 the mean-square velocity the relation

$$V^2\bar{\tau} = \frac{ik}{c}$$

holds, where i is a numerical constant depending on the particular law.

The Integral Expansions of Arbitrary Functions connected with Integral Equations. By J. Hyslop, St John's College. (Communicated by Professor E. W. Hobson.)

[*Received* 2 April 1924.]

INTRODUCTION.

The following paper aims at a more general treatment than has hitherto been given, of the integral expansions of arbitrary functions, from the point of view of integral equation theory.

The working consists largely of the bringing together of results from the following works, which are quoted in the text by the names of the Authors:

HELLINGER: *Göttingen Dissertation*, 1907.
HAHN: *Monatshefte für Math. u. Phys.* XXIII. (1912), pp. 161–224.
WEYL: *Math. Ann.* LXVI. pp. 273–324.
PLANCHEREL: *Rend. Palermo*, XXX. (1910), pp. 289–335.
CARLEMAN: *Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique* (Uppsala, 1923).

Part 1 contains a summary of the principal results of Hahn, Weyl, and Carleman.

Part 2 applies Hahn's results to those of Weyl and Carleman, use being made of the idea, introduced by Plancherel, of convergence on the mean of integrals. It has thus been possible to prove, under very wide conditions, an integral expansion of Fourier form, having as kernel the derivative of the characteristic differential of a complete singular symmetrical kernel.

In Part 3 an example of the theory is given, by proving the expansions connected with the kernel e^{-st}. This kernel is exceedingly interesting in itself, and also because of the efforts made to find its characteristics by Weyl[*] and Carleman[†]. Weyl, however, has only succeeded in giving its bounds, while Carleman has found the resolving kernel for $\frac{1}{s+t}$.

In conclusion I desire to express my indebtedness and grateful thanks to Professor Hobson, Cambridge, who introduced me to the subject of singular integral equations, and indicated to me the literature connected with it; to Professor Courant, Göttingen, who drew my attention to Carleman's work, and to Mr Steen, Cambridge, who drew my attention to Weyl's note on the kernel e^{-st}, and in answer to whose questioning the solution of that kernel was undertaken.

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[*] *Göttingen Dissertation, ad fin.*

[†] Pp. 158–169.

PART I. SUMMARY OF PREVIOUS WORK.

Notation. Throughout this paper Δ denotes an interval of the real axis of the variable λ or g. $\Delta f(\lambda)$ denotes the difference between the values of the function $f(\lambda)$ at the ends of the interval, i.e. if Δ denote the interval $(\lambda_1 \lambda_2)$, where $\lambda_2 > \lambda_1$, then

$$\Delta f(\lambda) = f(\lambda_2) - f(\lambda_1).$$

§ 1. *The Hellinger Integral.*

The Hellinger integral was introduced by Hellinger in connexion with the theory of infinite quadratic forms. It is defined as follows:

(1) $\rho(\lambda)$, $f(\lambda)$ denote continuous functions of λ in an interval (ab), $\rho(\lambda)$ being also non-decreasing in that interval. We divide (ab) into a finite number of sub-intervals $\Delta_1, \Delta_2, \ldots, \Delta_n$ and form the sum*

$$\sum_{r=1}^{n} \frac{[\Delta_r f(\lambda)]^2}{\Delta_r \rho(\lambda)}.$$

If, by sufficiently decreasing the maximum size of a sub-interval, this sum can be made to differ by less than an arbitrarily small quantity from a fixed number L, then L is defined to be the Hellinger integral $\int_a^b \frac{[df(\lambda)]^2}{d\rho(\lambda)}$. Hellinger has shown that this integral exists, and is equal to the upper bound, for all possible modes of subdivision, of $\sum_{r=1}^{n} \frac{[\Delta_r f(\lambda)]^2}{\Delta_r \rho(\lambda)}$, provided that upper bound exists.

(2) If $\int_a^b \frac{[df_1(\lambda)]^2}{d\rho(\lambda)}$, $\int_a^b \frac{[df_2(\lambda)]^2}{d\rho(\lambda)}$ exist, we can readily deduce the existence of the limit of $\sum_{r=1}^{n} \frac{\Delta_r f_1(\lambda) . \Delta_r f_2(\lambda)}{\Delta_r \rho(\lambda)}$ as n increases indefinitely so that the maximum subdivision tends to zero. This limit is the Hellinger integral $\int_a^b \frac{df_1(\lambda) . df_2(\lambda)}{d\rho(\lambda)}$.

(3) If $F(\lambda)$ be any continuous function of λ in (ab) we can define the integral $\int_a^b F(\lambda) \frac{df_1(\lambda) . df_2(\lambda)}{d\rho(\lambda)}$ as the limit of

$$\sum_{r=1}^{n} F(\lambda_r) \frac{\Delta_r f_1(\lambda) . \Delta_r f_2(\lambda)}{\Delta_r \rho(\lambda)},$$

where λ_r is any point of the interval Δ_r, the limit being existent if $\int_a^b \frac{df_1(\lambda) . df_2(\lambda)}{d\rho(\lambda)}$ exists.

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[*] In this summation, if $\Delta f(\lambda)$, $\Delta\rho(\lambda)$ both vanish, the fraction $\frac{[\Delta f(\lambda)]^2}{\Delta\rho(\lambda)}$ is to be replaced by zero.

(4) We may define improper Hellinger integrals, where the range is infinite, or includes singularities of $F(\lambda)$, by continuity in the usual way.

§ 2. *Hahn's Transformation.*

Hahn has shown* how to transform the Hellinger integral into a Lebesgue integral by a change of variable from λ to $\rho(\lambda)$. If $\int_a^b \frac{[df(\lambda)]^2}{d\rho(\lambda)}$ exists, then $f(\lambda)$ is expressible as a single-valued function $\phi(\rho)$ of $\rho(\lambda)$. Further, $\phi(\rho)$ is almost everywhere derivable with respect to ρ, and

$$\int_a^b \frac{[df(\lambda)]^2}{d\rho(\lambda)} = \int_{\rho(a)}^{\rho(b)} \left[\frac{d\phi(\rho)}{d\rho}\right]^2 d\rho.$$

Similarly $$\int_a^b \frac{df_1(\lambda) \,.\, df_2(\lambda)}{d\rho(\lambda)} = \int_{\rho(a)}^{\rho(b)} \frac{d\phi_1(\rho)}{d\rho}\,\frac{d\phi_2(\rho)}{d\rho}\,d\rho,$$

and if

$$F(\lambda) = \Phi\{\rho(\lambda)\},\quad \int_a^b F(\lambda)\,\frac{df_1(\lambda)\,.\,df_2(\lambda)}{d\rho(\lambda)} = \int_{\rho(a)}^{\rho(b)} \Phi(\rho)\,\frac{d\phi_1(\rho)}{d\rho}\,\frac{d\phi_2(\rho)}{d\rho}\,d\rho.$$

These results hold, by continuity, when the integrals are improper.

§ 3. *Weyl's Results.*

The singular integral equations whose "characteristic differentials" give rise to integral representations of arbitrary functions, were first studied with generality by Weyl.

The type of equations treated was that with real symmetric kernel $K(s, t)$, subject to the following conditions†:

(1) $K(s,t)$ is defined for $s \geqslant 0$, $t \geqslant 0$, and is continuous in every finite region, except possibly along a finite number of ordinary curves and at a finite number of points.

(2) $\int_0^\infty [K(s, t)]^2\,ds = [K(t)]^2$ defines a non-negative function which is continuous, except possibly for a finite or enumerable number of isolated points.

(3) For every pair of functions $u(s)$, $v(s)$ such that

$$\text{(i)}\quad \int_0^\infty [u(s)]^2\,ds \leqslant 1,\qquad \int_0^\infty [v(s)]^2\,ds \leqslant 1;$$

$$\text{(ii)}\quad \int_0^\infty K(s)\,|\,v(s)\,|\,ds \text{ exists,}$$

$$\int_0^\infty \int_0^\infty K(s, t)\,u(s)\,v(t)\,ds\,dt \leqslant M, \text{ a fixed constant.}$$

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* Hahn, §§ 8–13.

† Weyl, p. 275.

Weyl shows* how to define a set of characteristic functions $\phi_p(s)$, and of characteristic differentials $A_q(s, \lambda)$, such that

$$\int_0^\infty K(s,t)\,\phi_p(t)\,dt = \frac{1}{\lambda_p}\phi_p(s), \qquad \lambda_p \text{ a constant,}$$

$$\int_0^\infty K(s,t)\,\Delta A_q(t,\lambda)\,dt = \int_{\Delta}\frac{1}{\lambda}\,dA_q(s,\lambda).$$

The integral on the right is a Stieltjes integral with λ as variable. Also

$$\int_0^\infty \phi_p(s)\,\phi_q(s)\,ds = 0 = \int_0^\infty \Delta A_q(s,\lambda)\,.\,\Delta A_p(s,\lambda)\,ds \text{ if } p \neq q.$$

$$\int_0^\infty \Delta A_p(s,\lambda)\,\phi_q(s)\,ds = 0, \text{ for all } p, q.$$

$$\int_0^\infty [\phi_p(s)]^2\,ds = 1 \text{ and } \int_0^\infty [\Delta A_q(s,\lambda)]^2\,ds = \Delta\rho_q(\lambda).$$

And the following expansion theorems hold†, $g(s)$ being a function of integrable square in $(0, \infty)$:

$$\text{I.}\quad \int_0^\infty K(s,y)\,K(y,t)\,dy = \sum_p \frac{\phi_p(s)\,\phi_p(t)}{\lambda_p^2} + \sum_q \int_{-\infty}^{\infty} \frac{dA_q(s,\lambda)\,.\,dA_q(t,\lambda)}{\lambda^2\,.\,d\rho_q(\lambda)};$$

where the summation w.r.t. p and that w.r.t. q are extended over the set of characteristic functions and differentials respectively, a set which may be null, finite, or enumerably infinite in either case. The integrals are improper, absolutely convergent Hellinger integrals, and the series, if infinite, converge absolutely.

$$\text{II.}\quad \int_0^\infty K(s,t)\,g(t)\,dt = \sum_p \frac{\phi_p(s)}{\lambda_p}\int_0^\infty \phi_p(t)\,g(t)\,dt + \sum_q \int_{-\infty}^{\infty} \frac{dA_q(s,\lambda)\,.\,d\int_0^\infty A_q(t,\lambda)\,g(t)\,dt}{\lambda\,.\,d\rho_q(\lambda)}.$$

or, if $f(s)$ be written for the function on the left of this equality,

$$\text{II}(a).\quad f(s) = \sum_p \phi_p(s)\int_0^\infty \phi_p(t)\,f(t)\,dt + \sum_q \int_{-\infty}^{\infty} \frac{dA_q(s,\lambda)\,.\,d\int_0^\infty A_q(t,\lambda)\,f(t)\,dt}{d\rho_q(\lambda)}.$$

Here again the convergence of the integrals and of the series is absolute.

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* P. 294, Eqn. (20), and p. 301.

† For I cf. Weyl, p. 293; for II, II (*a*) cf. pp. 294–5.

Weyl goes on to state* that if:

(i) $K(s, t)$ has no characteristic functions and only one characteristic differential; $A(s, \lambda)$;

(ii) $\Delta\rho(\lambda) \equiv \int_0^\infty [\Delta A(s, \lambda)]^2 ds = \Delta(\lambda)$ for every interval in which $A(s, \lambda)$ does not vanish;

(iii) $\frac{\partial}{\partial\lambda} A(s, \lambda) = P(s, \lambda)$ exists, and is continuous;

(iv) $\int_0^\infty P(t, \lambda) f(t)\, dt$ converges uniformly w.r.t. λ in every finite part of the λ-axis;

(v) $\int_0^\infty K(s, t)\, g(t)\, dt = f(s)$, where $g(t)$ and hence also $f(s)$ are of integrable square in $(0, \infty)$,

then II (*a*) may be written in the form of a "Fourier" Integral Theorem:

$$\text{II }(b). \qquad f(s) = \int_{-\infty}^\infty P(s, \lambda)\, d\lambda \int_0^\infty P(t, \lambda) f(t)\, dt.$$

The object of the present paper is to show that an integral formula similar to II (*b*) holds under very much wider conditions, and that for all functions $f(s)$ of integrable square, not necessarily of the form $\int_0^\infty K(s, t)\, g(t)\, dt$. The convergence of the integral is in the general case convergence "on the mean".

§ 4. *Extension of Results by Carleman.*

Before passing to this proof, we may note that the restrictions laid on the kernel have been considerably reduced by Carleman, who proves Weyl's results under the following conditions†:

$K(s, t)$ is real and symmetric in a finite interval (ab).

$$(1) \quad \int_a^b [K(s, t)]^2 dt = [K(s)]^2 \text{ exists,}$$

$$(2) \quad \underset{s' \to s}{\mathrm{L}} \int_a^b |K(s, t) - K(s', t)|^2 dt = 0,$$

except possibly at an enumerable set ξ_n of values of s which have at most a finite number of limit points.

(3) Amongst these values ξ_n is a finite number $\eta_1, \eta_2, \dots, \eta_m$ such that, if $\delta(\epsilon)$ denote the sum of the intervals $|s - \eta_r| < \epsilon$, $\int_{(ab)-\delta(\epsilon)} [K(s)]^2 ds$ converges if $\epsilon > 0$.

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* Pp. 300–1.

† Carleman, p. 25 and p. 75.

(4) If E be the class of functions $f(s)$ of integrable square, for which $F(s)$, defined by the equation

$$F(s) = \int_a^b K(s, t) f(t)\, dt,$$

is also of integrable square, then, for every pair of functions $f_1(s)$, $f_2(s)$ of E,

$$\int_a^b ds \int_a^b K(s, t) f_1(s) f_2(t)\, dt = \int_a^b dt \int_a^b K(s, t) f_1(s) f_2(t)\, ds.$$

Carleman's work is applicable if (ab) be an infinite interval, if we include in $\delta(\epsilon)$ of condition (3) the part of (ab) outside the interval

$$\left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right).$$

It can be shown that conditions (1), (2), (3) above are satisfied if Weyl's first two conditions are satisfied, but not conversely. Condition (4) is much more general than Weyl's third condition, since in Weyl's case the set E includes all functions of integrable square[*]. The present writer hopes to show in a future paper that even Carleman's conditions are unnecessarily narrow, and that (2), (3) can be replaced by the simple condition, already implied in (1), that $K(s)$ be almost everywhere finite.

In addition to the formulae proved by Weyl, the following are proved in, or readily deducible from, Carleman's work[†].

III. $\sum_q \int_\Delta \frac{dA_q(s, \lambda) . dA_q(t, \lambda)}{d\rho_q(\lambda)} = \Delta\sigma(s, t \mid \lambda)$ a continuous function of s, t except for $s, t = \xi_n$.

IV. If $\psi_n(s)$ be a complete set of prepared mutually orthogonal functions which are also orthogonal to $K(s, t)$, then

$$\underset{N, P, Q, L \to \infty}{\mathrm{L}} \left[\sum_{n=1}^{N} \psi_n(s) \int_a^b \psi_n(t) g(t)\, dt + \sum_{p=1}^{P} \phi_p(s) \int_a^b \phi_p(t) g(t)\, dt \right.$$
$$\left. + \sum_{q=1}^{Q} \int_{-L}^{L} \frac{dA_q(s, \lambda) . d\int_a^b A_q(t, \lambda) g(t)\, dt}{d\rho_q(\lambda)} \right] \to g(s) \text{ on the mean.}$$

Hence in particular

IV (*a*).
$$\sum_n \left[\int_a^b \psi_n(t) g(t)\, dt\right]^2 + \sum_p \left[\int_a^b \phi_p(t) g(t)\, dt\right]^2$$
$$+ \sum_q \int_{-\infty}^{\infty} \frac{\left[d \int_a^b A_q(t, \lambda) g(t)\, dt\right]^2}{d\rho_q(\lambda)} = \int_a^b [g(t)]^2\, dt,$$

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[*] See Weyl, p. 276 and p. 286.

[†] For III see Carleman, p. 102; for IV cf. pp. 48–9 and p. 104.

and

$$\text{V.}\quad K(s,t) \sim \sum_p \frac{\phi_p(s).\phi_p(t)}{\lambda_p} + \sum_q \int_{-\infty}^{\infty} \frac{dA_q(s,\lambda).dA_q(t,\lambda)}{\lambda.d\rho_q(\lambda)}.$$

In V convergence is not merely on the mean but absolute if $K(s,t)$ is a continuous kernel of positive type*.

PART 2. THE "FOURIER" INTEGRAL THEOREM.

§ 5. *Application of Hahn's Transformation.*

Let us assume for the sake of simplicity in writing that $K(s,t)$ is complete, has no characteristic functions, and only one characteristic differential $A(s,\lambda)$, then formulae III, IV, IV (a) may be written:

$$\text{IV.}\qquad g(s) \sim \int_{-\infty}^{\infty} \frac{dA(s,\lambda).d\int_a^b A(t,\lambda)\,g(t)\,dt}{d\rho(\lambda)},$$

$$\text{IV }(a).\quad \int_a^b [g(s)]^2\,ds = \int_{-\infty}^{\infty} \frac{\left[d\int_a^b A(s,\lambda)\,g(s)\,ds\right]^2}{d\rho(\lambda)},$$

$$\text{III.}\qquad \Delta\sigma(s,t\,|\,\lambda) = \int_\Delta \frac{dA(s,\lambda).dA(t,\lambda)}{d\rho(\lambda)},$$

where in IV the sign of equivalence may be changed to one of equality if $g(s)$ is of the form $\int_a^b K(s,t)f(t)\,dt$, the integrals in this case converging absolutely.

From III, we see that $\int_\Delta \frac{[dA(s,\lambda)]^2}{d\rho(\lambda)}$ exists, and is finite and continuous for every finite interval Δ of the λ-axis and almost all values of s.

Hence, if $A(s,\lambda) = B(s,\rho(\lambda))$, $\frac{\partial B(s,\rho)}{\partial\rho}$ exists almost everywhere, and $\int_{\rho_1}^{\rho_2}\left[\frac{\partial B(s,\rho)}{\partial\rho}\right]^2 d\rho$ converges† in every finite interval $(\rho_1\rho_2)$, and for almost all values of s.

Write $\frac{\partial B(s,\rho)}{\partial\rho} = P(s,\rho)$. Since $B(s,\rho)$ may be defined to be zero at $\rho = 0$, $B(s,\rho) = \int_0^\rho P(s,\rho)\,d\rho$.

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* Cf. Carleman, p. 47.

† It is assumed that every finite interval of the λ-axis is transformed into a finite interval of the ρ-axis, and $(-\infty,\infty)$ into $(-\infty,\infty)$. The changes to be made, if this is not so, are slight, and do not affect the argument.

From IV, $g(s) \sim \int_{-\infty}^{\infty} P(s,\rho) \frac{\partial}{\partial\rho}\left[\int_a^b B(t,\rho)\, g(t)\, dt\right] d\rho.$

§ 6. *Proof of the Integral Theorem.*

In order to have the "Fourier" Integral Theorem suggested by Weyl, we have now only to examine conditions under which

$$\frac{\partial}{\partial\rho}\int_a^b B(t,\rho)\, g(t)\, dt = \int_a^b P(t,\rho)\, g(t)\, dt.$$

Since $\int_{\rho_1}^{\rho_2} [P(s,\rho)]^2\, d\rho$ converges for almost all values of s, the set of points on the s-axis, for which $\int_{-G}^{G} [P(s,\rho)]^2\, d\rho > M^2$, tends to a set of zero measure as $M \to \infty$.

This set is a function of G, M, and may be denoted by $D(G, M)$.

If $G_1 > G_0$, $D(G_1, M) \supset D(G_0, M)$.

If (ab) be an infinite interval, we include also in $D(G, M)$, by definition, all that part of (ab) outside the interval $(-G, G)$, so that the measure of $\{(ab) - D(G, M)\}$ is always finite.

Let
$$\begin{aligned} g^*(s) &= g(s), \text{ if } s \text{ be in } \{(ab) - D(G, M)\} \\ &= 0, \text{ if } s \text{ be in } D(G, M). \end{aligned}$$

Then

$$\int_a^b [g^*(t)]^2 \left\{\int_0^\rho [P(t,\rho)]^2\, d\rho\right\} dt \leqslant M^2 \int_a^b [g^*(t)]^2\, dt, \text{ if } |\rho| \leqslant G.$$

Hence, if $P(t, \rho)$ be measurable with respect to the pair of variables (t, ρ),

$$\int_a^b \int_0^\rho |P(t,\rho)\, g^*(t)|\, d\rho\, dt$$

exists, and its value does not exceed

$$M . G^{\frac{1}{2}} . [m\{(ab) - D(G, M)\}]^{\frac{1}{2}} \left[\int_a^b [g^*(t)]^2\, dt\right]^{\frac{1}{2}}, \text{ if } |\rho| \leqslant G.$$

$$\begin{aligned} \therefore \frac{\partial}{\partial\rho}\int_a^b B(t,\rho)\, g^*(t)\, dt &= \frac{\partial}{\partial\rho}\int_0^\rho d\rho \int_a^b P(t,\rho)\, g^*(t)\, dt \\ &= \int_a^b P(t,\rho)\, g^*(t)\, dt, \text{ almost everywhere, if } |\rho| \leqslant G. \end{aligned}$$

From IV (a),

$$\begin{aligned} \int_a^b [g^*(t)]^2\, dt &\geqslant \int_{-G}^{G} \left\{\frac{\partial}{\partial\rho}\int_a^b B(t,\rho)\, g^*(t)\, dt\right\}^2 d\rho \\ &\geqslant \int_{-G}^{G} \left\{\int_a^b P(t,\rho)\, g^*(t)\, dt\right\}^2 d\rho. \end{aligned}$$

So far G, M have remained undefined. Let now G go through a set of values $G_1, G_2, \ldots$ increasing to an infinite limit. Let M_n be so chosen, that the measure of d_n, the part of $D(G_n, M_n)$ contained within $(-G_n, G_n)$, does not exceed $\frac{1}{2^n}$.

Then if $\sum_{r=n}^{\infty} d_r = \Delta_n, \quad m(\Delta_n) \leqslant \frac{1}{2^{n-1}}$.

$$\therefore\ \mathrm{L}\, m(\Delta_n) \to 0.$$

Let $g_n(s) = g(s)$, if s be outside Δ_n but within $(-G_n, G_n)$
$= 0$, if s be in Δ_n or outside $(-G_n, G_n)$.

Then, as above,

$$\frac{\partial}{\partial\rho}\int_a^b B(t,\rho)\, g_n(t)\, dt = \int_a^b P(t,\rho)\, g_n(t)\, dt \qquad \ldots\ldots(1)$$

almost everywhere if $|\rho| \leqslant G_n$.

And $\int_{-\rho}^{\rho}\left[\int_a^b P(t,\rho)\, g_n(t)\, dt\right]^2 d\rho \leqslant \int_a^b [g_n(t)]^2\, dt$, if $|\rho| \leqslant G_n$.

These equations have a meaning and are satisfied if $|\rho| \leqslant G_m$, where m is any integer which exceeds n, since

$$|g_n(t)| \leqslant |g_m(t)| \text{ if } n \leqslant m, \text{ for all values of } t.$$

Hence $\displaystyle \mathop{\mathrm{L}}_{\rho\to\infty}\int_{-\rho}^{\rho}\left[\int_a^b P(t,\rho)\, g_n(t)\, dt\right]^2 d\rho$

exists, and does not exceed

$$\int_a^b [g_n(t)]^2\, dt.$$

Similarly

$$\mathop{\mathrm{L}}_{\rho\to\infty}\int_{-\rho}^{\rho}\left[\int_a^b \{g_m(t) - g_n(t)\}\, P(t,\rho)\, dt\right]^2 d\rho \leqslant \int_a^b [g_m(t) - g_n(t)]^2\, dt.$$

But $g_n(s)$ converges on the mean to $g(s)$.

$$\therefore\ \int_a^b P(t,\rho)\, g_n(t)\, dt$$

converges on the mean in $(-\infty, \infty)$ to a function which we *define* to be

$$\int_a^b P(t,\rho)\, g(t)\, dt \qquad \ldots\ldots(2).$$

From equation IV (a), by substitution of $[g(t) - g_n(t)]$ for $g(t)$, we see that $\frac{\partial}{\partial\rho}\int_a^b B(t,\rho)\, g_n(t)\, dt$ converges on the mean to

$$\frac{\partial}{\partial\rho}\int_a^b B(t,\rho)\, g(t)\, dt \qquad \ldots\ldots(3).$$

Hence, combining (1), (2), (3), we have, almost everywhere,

$$\text{VI.}\quad \frac{\partial}{\partial\rho}\int_a^b g(t)\,B(t,\rho)\,dt=\int_a^b P(t,\rho)\,g(t)\,dt,\ \text{if } |\rho|\leqslant G_n.$$

But this holds for every value of n, and $G_n \to \infty$.

Hence the equation holds for all finite values of ρ.

§ 7. *Integrals that Converge on the Mean.*

Our definition of $\int_a^b P(t,\rho)\,g(t)\,dt$ was

$$\underset{\Delta_n\to 0,\ G_n\to\infty}{\mathrm{L}}\int_{(ab).(-G_n,\,G_n)-\Delta_n} P(t,\rho)\,g(t)\,dt \to \int_a^b P(t,\rho)\,g(t)\,dt$$

on the mean.

It is clear that, while we have given Δ_n a particular form for the sake of precision, the convergence on the mean does not depend on the form of Δ_n, nor on the way in which G_n increases, provided G_n is finite and

$$\int_{-G_n}^{G_n}[P(t,\rho)]^2\,d\rho \text{ is bounded in } \{(-G_n,G_n)-\Delta_n\}.$$

Thus our definition of $\int_a^b P(t,\rho)\,g(t)\,dt$ is a natural one, and coincides, if the integral exists as a proper or improper Lebesgue integral, with the value of the integral as ordinarily defined. In examples which arise in practice, integrals known to be convergent on the mean will almost certainly converge also in the ordinary sense of the term, *but not conversely*. It is, however, scarcely possible in the general theory which we here consider, to specify simple restrictions which, imposed on $g(s)$, make convergence in the usual sense certain. That is rather a matter for examination for each particular kernel.

§ 8. *Summary of Results.*

It will be seen that the only restriction made in the course of the proof is that $P(s,\rho)$ be measurable.

We have therefore proved the "Fourier" integral theorem suggested by Weyl under the conditions:

(i) $K(s,t)$ is complete, has no characteristic functions and only one characteristic differential, $A(s,\lambda)\equiv B(s,\rho)$.

(ii) $\frac{\partial}{\partial\rho}B(s,\rho)=P(s,\rho)$, which certainly exists, is measurable, w.r.t. the pair of variables (s,ρ).

(iii) $f(s)$ is of integrable square in (ab).

The theorem also holds if $f(s) = \int_a^b K(s,t)\,g(t)\,dt$, where $g(s)$ is of integrable square, whether $K(s,t)$ is complete or not, and the second integration in this case converges absolutely.

The most general form of the theorem, in which condition (i) above is not satisfied, is, in notation already used,

$$\text{IV.}\quad g(s) \sim \sum_m \psi_m(s) \int_a^b \psi_m(t)\,g(t)\,dt + \sum_p \phi_p(s) \int_a^b \phi_p(t)\,g(t)\,dt + \sum_q \int_{-\infty}^{\infty} P_q(s,\rho)\,d\rho \int_a^b P_q(t,\rho)\,g(t)\,dt,$$

and we have also

$$\text{V.}\quad K(s,t) \sim \sum_p \frac{\phi_p(s)\,\phi_p(t)}{\lambda_p} + \sum_q \int_{-\infty}^{\infty} \frac{1}{\lambda(\rho)} P_q(s,\rho)\,P_q(t,\rho)\,d\rho,$$

$$\text{II.}\quad \int_a^b K(s,t)\,g(t)\,dt = \sum_p \frac{\phi_p(s)}{\lambda_p} \int_a^b \phi_p(t)\,g(t)\,dt + \sum_q \int_{-\infty}^{\infty} \frac{1}{\lambda(\rho)} P_q(s,\rho) \int_a^b P_q(t,\rho)\,g(t)\,dt\,d\rho.$$

§ 9. *Modified form of Hahn's Transformation.*

There is a point of practical value which may be mentioned here; a fuller discussion being reserved for a future paper.

As a particular case of formula VI,

$$\frac{\partial}{\partial\rho} \int_a^b K(s,t)\,B_q(t,\rho)\,dt \sim \int_a^b K(s,t)\,P_q(t,\rho)\,dt \sim \frac{\partial}{\partial\rho} \int_0^\rho \frac{1}{\lambda(\rho)}\,dB_q(s,\rho) \sim \frac{1}{\lambda(\rho)} P_q(s,\rho),$$

i.e. $P_q(s,\rho)$ is a "characteristic function" with a parameter. It is distinguished from the characteristic functions properly so called by the fact that $\int_a^b [P_q(s,\rho)]^2\,ds$ does not, in general, converge.

In practice, the characteristic differentials are generally found as such parametric "characteristic functions," in an equation of form

$$\int_a^b K(s,t)\,\Pi(t,g)\,dt = f(g)\,\Pi(s,g),$$

where as a rule g has not, with regard to $\Pi(s,g)$, the property of ρ with regard to $P(s,\rho)$, viz.

$$(\rho_2 - \rho_1) = \int_a^b \left[\int_{\rho_1}^{\rho_2} P(s,\rho)\,d\rho\right]^2 ds.$$

We assume however that

$$\int_a^b K(s,t)\int_{g_1}^{g_2}\Pi(t,g)\,dg\,dt=\int_{g_1}^{g_2} f(g)\,\Pi(s,g)\,dg$$

$$=\int_{g_1}^{g_2} f(g)\,d\int_0^g \Pi(s,g)\,dg \qquad \text{......(4)},$$

and $\int_a^b\left[\int_0^g \Pi(s,g)\,dg\right]^2 ds$ converges to a function $\rho(g)$ ......(5).

Equations (4), (5) show* that $\int_0^g \Pi(s,g)\,dg$ is a characteristic differential. We assume $\frac{d\rho}{dg}$ exists. This derivative is then clearly always positive.

Let $$\int_0^g \Pi(s,g)\,dg=B(s,\rho).$$

Then $$P(s,\rho)\equiv\frac{\partial}{\partial\rho}B(s,\rho)=\Pi(s,g)\left[\frac{d\rho}{dg}\right]^{-1}.$$

Now in the expansion theorems above, ρ always appears in some such combination as

$$P(s,\rho)\,P(t,\rho)\,d\rho.$$

Formally this is equal to

$$\Pi(s,g)\,\Pi(t,g)\left[\frac{d\rho}{dg}\right]^{-1}dg,$$

or $$\left\{\Pi(s,g)\left[\frac{d\rho}{dg}\right]^{-\frac{1}{2}}\right\}\left\{\Pi(t,g)\left[\frac{d\rho}{dg}\right]^{-\frac{1}{2}}\right\}dg.$$

Thus our expansion theorems hold, on the assumptions made above, if $\Pi(s,g)\left[\frac{d\rho}{dg}\right]^{-\frac{1}{2}}$ is put in place of $P(s,\rho)$, and integration is with respect to g. Here ρ may now be regarded merely as a function of g defined by (5), and the troublesome change of variable involved in Hahn's transformation is rendered unnecessary.

PART 3. EXAMPLE: THE KERNELS e^{-st}, $\frac{1}{s+t}$ FOR $(0,\infty)$.

§ 10. *The Characteristic Differentials of the Kernel* e^{-st}.

Weyl found† that the kernel e^{-st} had, for values of a between 0 and 1, the "characteristic functions":

$$\psi_1(t,a)=\sqrt{\Gamma(1-a)}\,t^{a-1}+\sqrt{\Gamma(a)}\,t^{-a},$$

$$\psi_2(t,a)=\sqrt{\Gamma(1-a)}\,t^{a-1}-\sqrt{\Gamma(a)}\,t^{-a},$$

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* Cf. Carleman, pp. 82 sqq.

† Weyl, *Göttingen Dissertation*, 1908.

and gave the formulae, which are readily verified,

$$\int_0^\infty e^{-st}\psi_1(t,a)\,dt = \sqrt{\frac{\pi}{\sin a\pi}}\,\psi_1(s,a),$$

$$\int_0^\infty e^{-st}\psi_2(t,a)\,dt = -\sqrt{\frac{\pi}{\sin a\pi}}\,\psi_2(s,a).$$

He also remarked, that these functions are inadmissible either as characteristic functions properly so called, or as parametric functions derived from characteristic differentials. The kernel e^{-st} is known to have the bound $\sqrt{\pi}$, while $\sqrt{\frac{\pi}{\sin a\pi}}$ is not bounded, and in fact takes values only outside $(-\sqrt{\pi}, \sqrt{\pi})$.

Weyl's formulae are however still valid if a be a complex number whose real part is between 0 and 1. By putting $a = \frac{1}{2} + ig$, we obtain the characteristic differentials of the kernel, as we proceed to show.

Let $\omega_1(s,g) = \sqrt{\Gamma(\frac{1}{2} - ig)}\, s^{-\frac{1}{2}+ig} + \sqrt{\Gamma(\frac{1}{2} + ig)}\, s^{-\frac{1}{2}-ig}$,

$i\omega_2(s,g) = \sqrt{\Gamma(\frac{1}{2} - ig)}\, s^{-\frac{1}{2}+ig} - \sqrt{\Gamma(\frac{1}{2} + ig)}\, s^{-\frac{1}{2}-ig}$,

the branch of $\sqrt{\Gamma(\frac{1}{2} \pm ig)}$ being chosen which has zero amplitude at $g = 0$.

Then $\omega_1(s,g)$, $\omega_2(s,g)$ are real functions of s, g.

If $\int_0^g \omega_1(s,g)\,dg = \Omega_1(s,g)$, $\int_0^g \omega_2(s,g)\,dg = \Omega_2(s,g)$,

then it is readily proved that

$$\int_0^\infty e^{-st}\Delta\Omega_1(t,g)\,dt = \int_\Delta \sqrt{\frac{\pi}{\cosh g\pi}}\,\omega_1(s,g)\,dg = \int_\Delta \sqrt{\frac{\pi}{\cosh g\pi}}\,d\Omega_1(s,g),$$

$$\int_0^\infty e^{-st}\Delta\Omega_2(t,g)\,dt = -\int_\Delta \sqrt{\frac{\pi}{\cosh g\pi}}\,d\Omega_2(s,g).$$

We now require to evaluate

$$I_1 = \int_0^\infty [\Delta\Omega_1(s,g)]^2\,ds.$$

We write for shortness $\sqrt{\Gamma(\frac{1}{2} - ig)} = u(g) + iv(g)$, $u(g)$, $v(g)$ being real.

Then

$$I_1 = \int_0^\infty \frac{ds}{s}\left[\int_\Delta \{u(g) + iv(g)\}\, s^{ig} + \{u(g) - iv(g)\}\, s^{-ig}\,dg\right]^2.$$

Or writing $$e^x = s,$$

$$I_1 = 4\int_{-\infty}^{\infty}\left[\int_\Delta u(g)\cos gx - v(g)\sin gx\,dg\right]^2 dx$$

$$= 4\int_{-\infty}^{\infty}\left[\int_\Delta u(g)\cos gx\,dg\right]^2 + \left[\int_\Delta v(g)\sin gx\,dg\right]^2 dx.$$

If $u(g)$, $v(g)$ are functions of integrable square in Δ*

$$I_1 = 4\pi\int_\Delta [u(g)]^2 + [v(g)]^2\,dg$$

$$= 4\pi\int_\Delta \sqrt{\Gamma(\tfrac{1}{2}+ig)\,\Gamma(\tfrac{1}{2}-ig)}\,dg$$

$$= 4\pi\int_\Delta \sqrt{\frac{\pi}{\cosh \pi g}}\,dg.$$

The latter integral certainly exists for every finite Δ.

$$\therefore \quad \int_\Delta [u(g)]^2\,dg, \quad \int_\Delta [v(g)]^2\,dg$$

also converge, and our assumption is justified.

If as above we write $I_1 = \rho(g_2) - \rho(g_1)$, where $\Delta = (g_1 g_2)$,

$$\frac{d\rho}{dg} = 4\pi\sqrt{\Gamma(\tfrac{1}{2}+ig)\,\Gamma(\tfrac{1}{2}-ig)}.$$

We find similarly

$$\int_0^\infty [\Delta\Omega_2(s,g)]^2\,ds = 4\pi\int_\Delta \sqrt{\Gamma(\tfrac{1}{2}+ig)\,\Gamma(\tfrac{1}{2}-ig)}\,dg.$$

Hence $\Omega_1(s,g)$, $\Omega_2(s,g)$ are characteristic differentials, and the corresponding functions† appearing in the expansion are

$$\Pi_1(s,g) = \frac{s^{-\frac{1}{2}}}{2\sqrt{\pi}}\{\sigma_1(g)\,s^{ig} + \sigma_2(g)\,s^{-ig}\},$$

$$\Pi_2(s,g) = \frac{s^{-\frac{1}{2}}}{2i\sqrt{\pi}}\{\sigma_1(g)\,s^{ig} - \sigma_2(g)\,s^{-ig}\},$$

where $$\sigma_1(g) = \sqrt[4]{\frac{\Gamma(\frac{1}{2}-ig)}{\Gamma(\frac{1}{2}+ig)}}; \quad \sigma_2(g) = \sqrt[4]{\frac{\Gamma(\frac{1}{2}+ig)}{\Gamma(\frac{1}{2}-ig)}}.$$

The simplest way of verifying our theory, and at the same time of proving that the characteristic differentials found are the only admissible ones, is to prove directly the accuracy of the expansion theorems. The range of integration with respect to g will be $(0, \infty)$.

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* An application of the ordinary Fourier Theorem, cf. Weyl, p. 315, and Plancherel.

† It is readily proved that g has, w.r.t. these functions, the characteristic property of ρ w.r.t. $P(s, \rho)$.

since, as g goes through this range, $\sqrt{\dfrac{\pi}{\cosh \pi g}}$ and $-\sqrt{\dfrac{\pi}{\cosh \pi g}}$ together cover once the complete interval $(-\sqrt{\pi}, \sqrt{\pi})$.

§ 11. *Expansion of e^{-st}.*

Our expansion theorem gives

$$e^{-st} \sim \int_0^\infty \sqrt{\frac{\pi}{\cosh g\pi}} \{\Pi_1(s,g)\,\Pi_1(t,g) - \Pi_2(s,g)\,\Pi_2(t,g)\}\, dg \equiv I_2.$$

Now

$$\begin{aligned} I_2 &= \frac{(st)^{-\frac{1}{2}}}{4\pi} \int_0^\infty \sqrt{\frac{\pi}{\cosh g\pi}} [\{\sigma_1(g)\, s^{ig} + \sigma_2(g)\, s^{-ig}\}\{\sigma_1(g)\, t^{ig} + \sigma_2(g)\, t^{-ig}\} \\ &\qquad + \{\sigma_1(g)\, s^{ig} - \sigma_2(g)\, s^{-ig}\}\{\sigma_1(g)\, t^{ig} - \sigma_2(g)\, t^{-ig}\}]\, dg \\ &= \frac{(st)^{-\frac{1}{2}}}{2\pi} \int_0^\infty \sqrt{\frac{\pi}{\cosh \pi g}} \{[\sigma_1(g)]^2 (st)^{ig} + [\sigma_2(g)]^2 (st)^{-ig}\}\, dg \\ &= \frac{1}{2\pi} \int_0^\infty [\Gamma(\tfrac{1}{2} - ig)\, (st)^{-\frac{1}{2}+ig} + \Gamma(\tfrac{1}{2} + ig)\, (st)^{-\frac{1}{2}-ig}]\, dg \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(z)\, (st)^{-z}\, dz \\ &= e^{-st}, \text{ [Mellin's Formula*].} \end{aligned}$$

Thus our expansion theorem is verified.

§ 12. *Expansion of $\dfrac{1}{s+t}$.*

$\dfrac{1}{s+t}$ is the first iteration of e^{-st}. We should therefore have

$$\frac{1}{s+t} = \int_0^\infty \frac{\pi}{\cosh \pi g} \{\Pi_1(s,g)\,\Pi_1(t,g) + \Pi_2(s,g)\,\Pi_2(t,g)\}\, dg \equiv I_3.$$

Now

$$\begin{aligned} I_3 &= \frac{(st)^{-\frac{1}{2}}}{4\pi} \int_0^\infty \frac{\pi}{\cosh \pi g} [\{\sigma_1(g)\, s^{ig} + \sigma_2(g)\, s^{-ig}\}\{\sigma_1(g)\, t^{ig} + \sigma_2(g)\, t^{-ig}\} \\ &\qquad - \{\sigma_1(g)\, s^{ig} - \sigma_2(g)\, s^{-ig}\}\{\sigma_1(g)\, t^{ig} - \sigma_2(g)\, t^{-ig}\}]\, dg \\ &= \frac{(st)^{-\frac{1}{2}}}{2\pi} \int_0^\infty \frac{1}{\cosh \pi g} \sigma_1(g)\, \sigma_2(g)\, [s^{ig} t^{-ig} + t^{ig} s^{-ig}]\, dg \\ &= (st)^{-\frac{1}{2}} \int_0^\infty \frac{\cos\left\{g \log \dfrac{s}{t}\right\}}{\cosh g\pi}\, dg. \end{aligned}$$

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* *Acta Math.* xxv (1902), p. 161.

Now we readily prove, by contour integration round the rectangular contour in the plane $z = x + iy$; $x = (-\infty, \infty)$, $y = (0, 1)$, that

$$\int_0^\infty \frac{\cos az}{\cosh \pi z}\, dz = \frac{1}{2}\operatorname{sech}\frac{a}{2}.$$

$$\therefore\ I_3 = \frac{(st)^{-\frac{1}{2}}}{\sqrt{\frac{s}{t}} + \sqrt{\frac{t}{s}}} = \frac{1}{s+t}.$$

§ 13. *The "Fourier" Integral Theorem.*

If we assume that the kernel e^{-st} is complete, then for all functions $f(s)$ of integrable square in $(0, \infty)$, we should have

$$f(s) \sim \int_0^\infty dg \int_0^\infty f(t)\, \{\Pi_1(s, g)\, \Pi_1(t, g) + \Pi_2(s, g)\, \Pi_2(t, g)\}\, dt \equiv I_4.$$

This formula we can also prove, thus at once verifying the accuracy of the formula, and proving the completeness of the kernel.

$$I_4 = \frac{1}{2\pi}\int_0^\infty dg \int_0^\infty f(t)\, \{t^{ig} s^{-ig} + t^{-ig} s^{ig}\}\, (st)^{-\frac{1}{2}}\, dt.$$

We introduce the transformation

$$s = e^x, \quad t = e^y, \quad e^{\frac{x}{2}} f(e^x) = F(x).$$

Then
$$I_4 = \frac{e^{-\frac{x}{2}}}{\pi}\int_0^\infty dg \int_{-\infty}^\infty F(y) \cos g(x - y)\, dy$$

$$\sim e^{-\frac{x}{2}} F(x) = f(s).$$

The accuracy of the last step is dependent on the fact that $\int_{-\infty}^\infty [F(x)]^2\, dx$ converges if $\int_0^\infty [f(s)]^2\, ds$ does so; in fact these integrals are equal. Hence the ordinary Fourier integral theorem is applicable to $F(x)$, the integrals concerned converging on the mean. While the above working has been used to verify the expansion theorems obtained from our integral equation, it may be noted that it also constitutes a proof of the Fourier integral theorem for functions of integrable square, and of Mellin's Formula, completeness of the characteristic differentials found being readily established otherwise.

§ 14. *The Kernel* sech $\frac{1}{2}(s-t)$.

The above theory of the kernel $\frac{1}{s+t}$ may be given a more symmetrical form by means of the transformation already used several times, $s = e^x$, $t = e^y$, thus giving the theory of the kernel: $\frac{e^{\frac{1}{2}(x+y)}}{e^x + e^y}$ or $\frac{1}{2}$ sech $\frac{1}{2}(x-y)$ in the interval $(-\infty, \infty)$. We can express the characteristic differentials in the form

$$\int_0^g \sin xg\,dg, \quad \int_0^g \cos xg\,dg,$$

and so get rid of the formidable looking Γ-functions. All the formulae become simple and symmetric, and the integral expansion theorem has the ordinary Fourier form.

On the Angle of Incidence in Soaring Flight. By E. H. Hankin, M.A., Sc.D., and J. D. North, Chief Aeronautical Engineer to Messrs Boulton and Paul, Norwich.

[*Received* 14 December 1923; *Read* 21 January 1924.]

Perhaps the most peculiar fact of observation yet recorded about the wing disposition of soaring birds is the apparent large negative angle of incidence shown by the wings of vultures when in fast horizontal flight.

A simple explanation of the phenomenon is now available. If a bird could glide horizontally by the application of an imaginary force upwards in still air, the plane of its wings would appear to lie in the direction of flight. As a matter of fact they would have a small positive angle of incidence to the bird's relative wind. Suppose the bird now begins to glide downhill, the wings, as before, will have a small positive angle of incidence to the relative wind. They will be inclined at a large angle with the horizon. Now suppose the air in which the bird is gliding changes to an ascending current of sufficient strength, then the bird will travel horizontally relatively to the earth while still gliding downhill relatively to the air. An observer, not knowing of the existence of the ascending current, will see an apparent negative angle of incidence, which in Hankin's observations sometimes seemed to amount to 30 degrees, though, in reality, as before, the plane of the wings only made a small positive angle with the bird's relative wind.

The proof given by Dr Gilbert Walker* of the probable presence of widespread ascending currents over Agra, where such observations were made, obviously yields a simple explanation of the appearance. The wings made a small positive angle with the relative wind but a large negative angle with the flight path (considered relatively to the ground), thus giving rise to an apparent large negative angle of incidence.

Cheels gliding obliquely from leeward into the ascending current reflected upwards from the Agra Fort battlements have been seen to show a rotation of the wing that was first to enter the rising air. On other occasions rotations of both wings have been observed at the moment of entering this ascending current. In this rotation the leading edge of the wing went down and the trailing edge went up. No doubt such movements were an adaptation to flight in an ascending current. Presumably they resulted in a negative angle to the ground reference flight path which, owing to the feebleness of the ascending current, was too small to be recognised. On some occasions, when in the ascending current the bird showed relaxation of the secondaries, which again appears to indicate a negative angle with the flight path too small to be recognised as such. In certain stormy winds cheels show occasional rotations or depressions, or both, of one wing. In some cases such movements may be due to the bird entering an ascending current. But in other cases they may be due to the wind pressing momentarily on the upper surface of the wing.

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* "Meteorology and the non-flapping flight of tropical birds," *Proc. Camb. Phil. Soc.* 21, Pt 4, 1922, p. 363.

Some years ago gulls were observed by Hankin gliding from the "soarable area" at the stern of a steamer into the ascending current on the windward side of the ship. While in the soarable area the wings showed an apparent positive angle of incidence. At the moment of entering the ascending current the wing-tips were rotated so that they acquired an apparent negative angle of incidence*.

If a vulture suddenly enters an ascending current, or if it enters an ascending current ascending at a greater rate than that in which it was previously flying, then, in order to continue its course as before, it would require to make a wing adjustment calculated to produce a downward dive. Such an adjustment, described by Hankin as the "double dip," occurs from time to time in gliding flight but, in Agra, no dive was observed sufficient to account for the increase of speed that followed it. Measurements with the aid of a Souchier telemeter† of apparent speeds of vultures before and just after a double dip movement showed increases of speeds (relatively to the earth) of 5, 5, 4½, 4, 4, 3, 3, and 3 metres per second. These measurements were made on different occasions on vultures flex-gliding at heights varying from 450 to 1500 metres. To produce such increases of speed a dive of several metres would be required. But in Agra, no dive at all, or at most a very small dive, was observed. Hence it did not appear feasible to ascribe the rise of speed to a dive caused by the double dip. But now, in the light of Dr Walker's ascending currents, a possible explanation may be suggested. A sufficiently ample dive perhaps did occur but, as it happened in an ascending current, it appeared shortened to a negligible amount from the point of view of the observer. The speed of the vultures may also have contributed to concealing the dive. Eagles when playing in the air sometimes make a double dip movement at the end of an upward swoop, when their speed ahead is small. The resulting dive is easily observed‡.

This explanation fits the facts so well that it is difficult to believe that it is not at least an approximation to the truth. But it must be borne in mind that it would not be safe to assume that the double dip movement only occurs at the time of entering an ascending current. When a group of circling vultures breaks up, each bird, as a rule, makes this movement at the moment of leaving the circle. In such cases the movement appears to signalise the change from gliding with gain of height to gliding in a horizontal direction.

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* "On the flight of sea gulls," *Aeronautical Journal*, 19, July–September 1915, p. 90. The two wing-tip dispositions are shown in Figs. 3 A and 3 B of that paper.

† *Animal Flight*, p. 40.

‡ *Ibid.*, p. 88.

A method of observing the angle of incidence in circling flight was suggested by the late Mr Howard Short. It is as follows. Let the track of the bird be indicated by the frustum of a cone (Fig. 1).

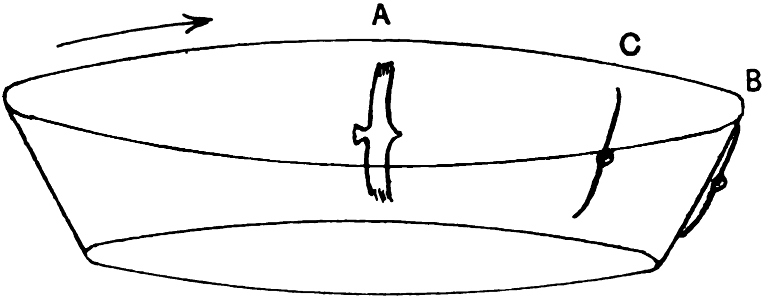

Fig. 1

At *A* the bird is shown on the far side of the track. The dorsal surface of the wings is seen. If the angle of incidence of the wings were zero, then the wings would only be seen edge on when the bird reached the limit of its travel as at *B*. But if the wings have a positive angle of incidence they will be seen edge on at some such position as *C*. This is in fact what is seen when the bird is circling in soarable air and gaining height. The wings are seen edge on some time before the bird reaches the limit of its sideway travel, much as shown in the figure, thus proving that the wings have a large apparent positive angle of incidence. This statement is not put forward as a means of measuring the apparent angle of incidence but merely as a description of what the observer actually sees. As the bird is gliding with gain of height it is obviously possible and probable that, as with gliding in an ascending current, the large angle observed is only apparent and that, in reality, the wings make a small angle with the relative wind.

Conclusion.

There is no evidence from observation that the angle of incidence in soaring flight is otherwise than what one would expect on aerodynamical grounds.

On Five Lines, in Space of Four Dimensions, which lie upon a Quadric Threefold and the Normal Quartic Curves of which they are Chords. By F. Bath, King's College. (Communicated by Mr H. W. Richmond, King's College.)

[*Received* 9 April 1924.]

The connexion between the conditions for five lines of S_4

(i) to lie upon a quadric threefold,

and (ii) to be chords of a normal quartic curve,

leads to an apparent contradiction. This difficulty is explained in the first paragraph below and, subsequently, two investigations are given of which the first uses, mainly, properties of space of three dimensions.

It is shown that five lines upon a quadric threefold are chords of a doubly infinite number of quartic curves and that any one of these is determined by the assignment of one of its intersections with each of two of the five lines. Further, (i) and (ii) are mutually inclusive.

The condition for six lines in S_4 to be chords of a normal quartic is clearly of importance, and I am indebted to Mr P. Fraser of Bristol for a suggestion which leads to the discovery of this condition in the form given in § 8 (Theorem III). This form immediately suggests theorems, already known, concerning conics triangularly circumscribed to the conics which touch five of six lines in a plane, and an interesting connexion between the two is established in § 9 by projection. In the special case when the six lines lie upon a quadric threefold, there is a unique quartic of which they are chords.

My thanks are due to Mr Richmond and to Prof. H. F. Baker for criticism during the writing of this paper.

§ 1. Theorem I. *If five lines in S_4 are chords of a normal quartic curve they lie upon a quadric threefold, Q_3.*

For, a Q_3 is represented by a quadratic equation in five homogeneous variables and is therefore determined, in general uniquely, by the prescription of fourteen points which are to lie upon it. Now any Q_3 containing nine points of the quartic must contain the whole curve and, in particular, two points of each line. Taking one further point on each line, there will be a unique quadric threefold containing these five and the quartic; this quadric contains three points of each of the five lines and therefore contains all five lines entirely.

Clearly it is not possible in general to find a Q_3 containing five given lines of S_4 and we may enquire whether the existence of a normal quartic curve having the lines as chords is a necessary (as well as sufficient) condition for the existence of such a Q_3. But, at first sight, it would appear that *any* five lines of S_4 are chords of a normal quartic—indeed of an infinite number of normal quartics—and so that any five lines of S_4 lie on a Q_3! For, a unique quartic (in general) passes through seven points P_1, P_2, ... P_7. viz. the one defined by the three quadric 'cones' of the second kind,

(1) vertex P_1P_2 passing through P_3, P_4, P_5, P_6, and P_7;
(2) ,, P_3P_4 ,, ,, P_1, P_2, P_5, P_6, and P_7;
(3) ,, P_5P_6 ,, ,, P_1, P_2, P_3, P_4, and P_7.

In order that the quartic may pass through any point, P, three conditions must be satisfied—for P must lie upon each of the three 'cones'. So, in general, a finite number (possibly zero—in a poristic case) of quartics may be found to satisfy $3 \times 7 = 21$ conditions[*].

A line which meets the quartic satisfies two conditions—any point of the line has coordinates proportional to

$$X_i + \lambda Y_i \quad (i = 1, 2, \dots 5),$$

where the X_i and Y_i are fixed while λ varies. Substituting these coordinates in the equations of the three 'cones' and eliminating λ from two pairs of the resulting equations in λ we obtain the two conditions for the quartic and line to have a common point. If the line is a chord of the quartic (that is, meets the curve twice), four conditions are satisfied. Thus to make five lines chords we need only impose twenty conditions on the quartic, which then still contains one free parameter.

It thus appears that any five lines of S_4 are chords of an infinite number of normal quartics and so, by Theorem I, any five lines of S_4 lie upon a quadric threefold! In the general case, however, all these quartics may be degenerate in such a manner as to make the reasoning of Theorem I invalid. Two such sets of degenerate curves are easily found as follows:

(*a*) Five lines in S_4 are met by a singly infinite system of planes, (S_2), forming an aggregate of the fifth order[†]. In each of these planes we have a conic determined by the five points where the five lines meet it. Any such conic counted twice forms a quartic curve having the five lines as chords. Theorem I does *not* apply to this case.

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[*] Or, the equations of a normal quartic of S_4 contain 25 homogeneously entering coefficients of which 3 may be removed by linear transformation of the parameter—not affecting the curve—so the equations contain 21 *effective* constants. And, generally, the equations for the normal n-ic contain $(n+1)^2 - 4 = n^2 + 2n - 3$ *effective* constants.

[†] See Prof. C. Segre, *Rendiconti del Circolo Matematico di Palermo*, Tomo II (1888), p. 45.

(β) Through any point of one of the lines we can draw two planes to meet the other four lines (obtainable from the dual (in S_4) of the well-known result that just two lines meet four given lines in S_3)*. As before, we have a conic in each of the two planes meeting all five lines. Such pairs of conics with a common point form quartic curves which are degenerate forms of the normal quartic and to which the argument of Theorem I does not apply†.

§ 2. The fact that a line in S_4 requires three equations in point coordinates, or ten homogeneous coordinates connected by three fundamental quadratic relations, to fix it, makes a more accurate investigation tedious. There is, however, a simple correspondence between the problem and another concerning lines in ordinary space, (S_3), which we proceed to explain.

A Q_3 of S_4 may be considered as a section of a Q_4 of S_5, and using Klein's‡ (1, 1) correspondence between the points of such a Q_4 and the lines of an S_3, it is clear that the points of the Q_3 correspond to lines of a certain linear complex in the S_3. A line on the Q_4 corresponds to a pencil of lines in the S_3 and, in particular, a line of the Q_3 corresponds to all the lines of the linear complex lying in a certain plane.

Corresponding to a normal quartic curve on Q_3 we have a quartic scroll (rational) of the S_3. Through a point on such a scroll passes, in general, one generator, but if the point lie on the nodal line there are two generators of the scroll through it. For the quartic scroll here considered this curve is a twisted cubic whose points correspond to chords (of the normal quartic) lying on the Q_3. To prove this we must show that *any plane* meets the curve in three points. In S_4 *any plane* is represented by a plane of the Q_4 meeting the Q_3 in a line l, so we must find how many chords of the normal quartic meet l. Now if the quartic be projected from l on to a plane we obtain a rational plane quartic which must have three nodes. Therefore three chords of the S_4 quartic meet l.

Clearly the nodal curve is a non-degenerate twisted cubic when the normal quartic from which it is obtained is non-degenerate, and the nine various types of special rational quartic scroll discussed by Cayley, Cremona and others will correspond to cases obtained when the normal quartic degenerates. For a genuine* normal quartic, then, we have a corresponding quartic scroll which is generated by those chords of a twisted cubic that belong to a certain linear complex. In the reverse order it is clear that the chords of any twisted cubic which belong to the linear complex correspond to the points of a quartic curve on the Q_3.

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* See Prof. H. F. Baker's paper: "On a proof of the Theorem of a Double Six of lines by projection from Space of Four Dimensions," *Proc. Camb. Phil. Soc.* vol. xx (1920), p. 136, § 3, or Prof. Segre's paper cited above.

† To make it quite clear that two conics with a common point form a degenerate normal quartic we may add a construction which obtains such a pair as the residual intersection of three Q_3's through three lines, and their transversal. For, join three points of one conic to three points of the other by three lines: then there are three linearly independent quadric threefolds through

(1) the common point of the two conics,

(2) four further points on each of the conics,

and (3) three further points, one on each of the three lines.

Each of these quadrics meets the conics in five points each and the lines in three points each; therefore contains both conics, the three lines—and their transversal.

‡ *Math. Annalen*, Bd. v (1872), p. 261, or *Gesammelte Mathematische Abhandlungen*, Bd. I, p. 111, also p. 153.

§ 3. Given five lines on the Q_3 we shall now enquire how many quartic curves have them as chords.

Corresponding to the Q_3 we have in S_3 a linear complex, L; to each point of Q_3 corresponds one line of L and *vice versa*. To the five lines on Q_3 correspond five pencils of lines of L and *any* genuine twisted cubic through the five centres of these pencils determines a quartic scroll of the type considered above. Moreover, each pencil contains two generators of a quartic scroll so obtained: hence the quartic scroll in S_3 determines a corresponding genuine normal quartic curve having the five lines of Q_3 as chords.

It thus appears that there are as many normal quartics having five lines of Q_3 for chords as there are twisted cubics through five points in S_3: that is, a doubly infinite system. Also, all such quartics lie upon the Q_3 since they meet it in ten points (intersections with the five lines).

§ 4. Since five lines of perfectly arbitrary position in S_4 do not, in general, lie upon a quadric, it must follow from Theorem I that there is no genuine quartic having them as chords in general. We shall examine this by the correspondence and see what happens when the five arbitrary lines are chords of a genuine quartic.

Take the Q_3 through the first four lines, it meets the fifth in a pair of points, and using the same correspondence, the corresponding problem is:

Given four points and two lines, all arbitrarily placed in S_3, can there be a twisted cubic passing through the four points and having the two lines as chords?

Suppose there is such a cubic. Then the cross ratio of the four planes through one chord and the four points must be equal to the cross ratio of the four planes through the other chord and the four points, each ratio being equal to the cross ratio of the parameters of the four points. But this is not true in general, so there can be no such cubic except for special positions of the points and lines.

[Otherwise† if such a cubic exist there is a quadric, (Q_2), containing both lines and passing through the four points (make it pass through the cubic and two further points one on each line). But in general there is no such quadric—therefore, etc.]

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* I.e., non-degenerate.

† See Prof. H. F. Baker, *Principles of Geometry*, vol. III, p. 139, Ex. 19.

If there is one twisted cubic through the four points having the two lines as chords, there must be a doubly infinite number. For, taking the one quadric containing this cubic and the two lines we may cut out other such cubics upon it by taking all quadrics through the four points and one of the lines—a doubly infinite system. Here the two lines are supposed skew, but the case is not essentially different from that in which they intersect, giving § 3. For, since there is a twisted cubic through the four points having the two lines as chords there must be a Q_3 through the four lines in S_4 containing the fifth line (Theorem I). Between the points of this Q_3 and the one already taken we have a (1, 1) linear correspondence which is represented in the S_3 by a (1, 1) linear correspondence of lines such that the two skew lines already considered correspond to two intersecting lines.

Putting these results together with Theorem I we obtain

Theorem II. *If five lines, a, b, c, d, e, are chords of one normal quartic curve, they are chords of a doubly infinite number of such curves.*

We shall now prove three important properties of this family of quartics as corollaries.

Cor. 1. *The whole family of quartics lies upon the unique quadric threefold containing the five lines.*

This follows at once from Theorem I.

Cor. 2. *Through any two points taken one on each of two of the five lines there passes one member of the family of quartics having the five lines as chords.*

Consider the corresponding problem in S_3: we have to show that there is a unique twisted cubic through the centres A, B, C, D, E of five pencils of lines having δ, ϵ, arbitrary members of the last two pencils respectively, as chords. This follows immediately, for the two quadric cones

(1) vertex E containing ϵ, ED, EC, EB, EA,

and (2) vertex D containing δ, DE, DC, DB, DA,

have DE as a common generator and meet besides in a twisted cubic curve passing through A, B, C, D, E and having δ, ϵ as chords. Moreover any twisted cubic having these properties must lie on (1) and (2), so there is only one such cubic.

Cor. 3. *If a sixth line, f, lie on the same Q_3 as a, b, c, d, e, there is a unique quartic curve having all six lines as chords.*

For, in this case, the corresponding problem in S_3 is to find a twisted cubic passing through six points (the vertices of six pencils of lines), and it is well known that six points in S_3 define a unique twisted cubic passing through them. The chords of this cubic which belong to the fixed linear complex L generate a quartic scroll which corresponds to a normal quartic (non-degenerate) having a, b, ... f for chords.

It is assumed, of course, that the six points in S_3 are arbitrarily placed, and this assumption is justified since a, b, ... f may be *any* six lines upon Q_3 so the six pencils are *any* six pencils of L and their vertices are therefore of completely arbitrary position.

§ 5. It is a well-known fact (first pointed out by W. K. Clifford in his paper on Classification of Loci) that any property proved for a normal rational curve establishes corresponding properties for all rational curves of the same, or lower, orders in spaces of lower dimensions. Moreover 'curve' can be interpreted here in an extended sense to include a one-dimensional aggregate in which the element may be a point, line, plane, or any other geometrical entity which is defined by a set of coordinates expressed as rational functions of one parameter.

The foregoing paragraphs afford an interesting illustration of this interdependence of rational 'curves', whereby we are enabled to choose the easier or more familiar of two corresponding configurations and use its properties to establish results about the other. The problem in question in this paper admits, however, of another solution which does not depend on a correspondence argument, and it would seem worth while to give some account of this method, since it leads naturally to a new theorem for which the particular correspondence method used above is not so suitable.

§ 6. Take Theorem II. We shall need the following:

Lemma 1. *If, in space of four dimensions, three quadrics* pass through three lines (and therefore also their transversal), their residual intersection, if a genuine quartic, has the three lines for chords.*

For, if we project the residual intersection from the transversal into a plane, we obtain a plane quartic having three nodes (since it must be of zero genus). Hence there must be three chords of the S_4 quartic meeting the transversal and therefore lying entirely on all three quadrics. Since the quartic is not degenerate these three chords are identical with the three given lines. Q.E.D.

Let a, b, c, d, e denote five lines lying on a quadric Q''. Then there are three linearly independent quadrics containing a, b, c, d: so that any quadric containing these four lines is included in the net:

$$\lambda Q + \lambda' Q' + \lambda'' Q'' = 0.$$

This net cuts out pairs of points in involution on e, since Q'' contains e. The involution on e is in fact defined by the pencil

$$\lambda Q + \lambda' Q' = 0$$

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* In the sequel 'quadric' will be written for 'quadric threefold'.

and through any particular pair E_1, E_2, of the involution we have one quadric of the pencil, Q_E say.

Similarly we may define a quadratic involution on d (by means of quadrics containing a, b, c, e), and a quadric, Q_D, containing these four lines and a typical pair, (D_1, D_2), of the involution on d.

Consider the curve of intersection of Q'', Q_E and Q_D.

It consists of a, b, c, the transversal (abc) and a quartic curve which passes through D_1, D_2, E_1 and E_2, that is, has d and e for chords.

This quartic is not degenerate in general*, so has a, b, c for chords also (by the lemma). Moreover it has been determined by the assignment of one of the pair of points E_1, E_2 and one of the pair D_1, D_2: hence

a, b, c, d, e, five lines of a Q_3, are chords of a doubly infinite system of quartics, one being determined by the assignment of a point of intersection with each of two of the lines.

§ 7. We have proved incidentally in the last paragraph the necessary part of the following:

Theorem A. *The necessary and sufficient condition for five lines a, b, c, d, e of S_4 to lie upon a quadric is that the pairs of spaces (ab), (cd); (ca), (bd); (bc), (ad) meet e in pairs of points in involution*†, for, the pairs of spaces are particular quadrics through a, b, c, d. The condition is clearly sufficient, since a net of quadrics cannot cut a line in pairs of points in involution unless one quadric of the net contain the line entirely.

From Theorems I and II it follows that the same condition is necessary and sufficient for five lines to be chords of a genuine normal quartic, i.e.,

Theorem B. *The necessary and sufficient condition for five lines a, b, c, d, e in S_4 to be chords of a genuine normal quartic curve is that the pairs of spaces (ab), (cd); (bc), (ad); (ca), (bd) meet e in pairs of points in involution.*

§ 8. Turning now to six lines in S_4, we have

Theorem III. *If there are three genuine quartics having a, b, c, d, e; a, b, c, e, f; a, b, c, f, d, respectively for chords, then there is one genuine quartic having all six for chords, and so, any five lines of the six lie upon a quadric.*

By Theorem I we have three quadrics F, D, E containing all six lines except f, d, e respectively. These quadrics intersect in a, b, c and their transversal (abc). The residual intersection is a normal quartic q of which a, b, c are chords (Lemma 1).

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* It is not difficult to show that, if the quartic is not genuine the transversal lines and planes of a, b, c, d, e possess special properties.

† This is stated by Mr H. W. Richmond, *Proc. Camb. Phil. Soc.* vol. x (1899), p. 212.

q also has d, e, f for chords (since F and E contain d and D meets it in two points which are not on a, b, c nor—in general—on their transversal (abc); therefore d is a chord. Similarly e and f are chords).

Hence q is the required quartic, and by Theorem I any five lines of the six lie upon a quadric.

[By the correspondence used in the earlier portion of the paper we deduce the following theorem in ordinary space:

A, B, C, D, E are five points and ϕ_1, ϕ_2, two lines, in S_3. If there are two twisted cubics Γ_1, Γ_2 having ϕ_1, ϕ_2 for chords and passing respectively, through A, B, C, D and A, B, C, E, then there is a twisted cubic through A, B, C, D, E having ϕ_1, ϕ_2 for chords.

An independent proof is quite simple.

Γ_1, ϕ_1, ϕ_2 lie on a unique ordinary quadric Q (alternative proof of the theorem of § 4).

Q is met by Γ_2 in seven points, viz. A, B, C and two on each of ϕ_1, ϕ_2: therefore Γ_2 lies on Q and, in particular, E lies on Q.

Through the five points A, B, C, D, E on Q there passes one twisted cubic lying on Q and having the generators ϕ_1, ϕ_2 for chords.]

We have assumed here that a, b, c, d, e, f do not lie all on one and the same quadric. This is in general true and the special case in which the assumption is not permissible has already been dealt with in Theorem II, Cor. 3. There it was shown that there is a unique normal quartic having the six lines as chords. So the Theorem III is true whether the six lines lie on a quadric or not.

§ 9. It may be remarked that Theorem III resembles another in plane geometry which has been made the basis of a simple proof of the existence of a double six of lines in S_3*. The theorem in question may be stated as follows:

"*Denoting the conics touching fives of six lines in a plane by A', B', C', D', E', F'; any conic k triangularly circumscribed*† *to A', B', C' is also triangularly circumscribed to D', E', F'.*"

Now if a quartic q in S_4 be projected from one of its chords on to a plane Π we obtain a conic k. Moreover, the chords of k (i.e. the lines of Π) are in $(1, 1)$ correspondence with the chords of q. Using this correspondence and the theorem just quoted we shall establish the following property of chords of a normal quartic:

Theorem IV. *a, b, c, d, e, f are six chords of a normal quartic curve q, and A, B, C, D, E, F the quadrics which contain b, c, d, e, f, ..., a, b, c, d, e, respectively. If each of A, B, C has a node then the same is true of D, E, F.*

---

* See E. K. Wakeford, *Proc. Lond. Math. Soc.* Ser. 2, vol. xv (1916), p. 341. The proof there given uses Miquel's Theorem: also Mr F. P. White's recent paper in *Proc. Camb. Phil. Soc.* vol. XXII (1924), p. 11.

† I.e. there exist triangles whose vertices lie on k and whose sides touch A' or B' or C', etc., as the case may be.

The proof depends on

Lemma 2. *The projections (from an arbitrary fixed chord on to a plane* Π*) of the chords of a normal quartic curve,* q*, which lie on a quadric,* Q*, containing the curve, touch a conic;* conversely

The chords of a normal quartic curve corresponding to tangents of any conic in a plane Π *(when the quartic is projected from an arbitrary chord on to* Π*) lie on a quadric,* Q*, containing the quartic.*

Using the notation already explained, take P any point on q. The lines of Q which pass through P all lie on the tangent S_3 to Q at P. This S_3 meets q in four points of which two coincide at P; denoting the remaining pair by R, S it is clear that PR, PS are the chords of q through P which lie on Q. [PR, for example, meets Q twice at P and again at R (since q lies on Q) and therefore lies entirely on Q.]

So the chords of q which lie on Q determine a (2, 2) correspondence on the curve; moreover, this correspondence is symmetric (i.e. if P corresponds to R then R corresponds to P). In the projection we have a symmetric (2, 2) correspondence between P', R' points on k: so $P'R'$ envelopes a conic.

Conversely, the tangents of any conic in Π determine on k a symmetric (2, 2) correspondence which is defined by five pairs (since the conic is defined by five tangents)*. These five pairs are projections of five pairs of points on q which determine on that curve a symmetric (2, 2) correspondence. Through the joins of these five pairs there passes one quadric (Q)—Theorem I.

Q is such that chords of q which lie upon it cut out a symmetric (2, 2) correspondence on q, and this correspondence must be identical with the one defined by the five pairs. So the tangents of any conic in Π are projections of chords of q lying on a quadric. Q.E.D.

The conic envelope in Π is triangularly inscribed in k when the three points P', R', S' (R', S' correspond to P') are so related that the same triad is obtained whatever point of the three we start with (i.e. S', P' correspond to R' and P', R' to S'). In S_4 this means that RS, as well as PR, PS, must lie on the quadric Q determining the (2, 2) correspondence. In this case the whole plane PRS must lie on Q which is therefore a 'cone' (i.e. Q has a double point). Conversely if Q, the quadric containing five chords of q, has a node, then these five chords project into the tangents of a conic triangularly inscribed in k.

The use of the theorem quoted at the beginning of the present paragraph (p. 196) clearly completes the proof of Theorem IV.

---

* The equation of a symmetric (2, 2) correspondence is of the form

$$\alpha t^2 s^2 + \beta(t^2 s + t s^2) + \gamma(s^2 + t^2) + \delta st + \epsilon(t + s) + \phi = 0,$$

and the assignment of five pairs of corresponding points determines $\alpha : \beta : \ldots : \phi$.

The Double-Six.

Take the cone A of Theorem IV and suppose V is its vertex. Each of the planes (Vb), (Vc), (Vd), (Ve), (Vf) belongs to a system of generators of A each of which meets q in three points. Any generator of the second system on A meets q in one point only and meets each generator of the first system in a line. In particular, any generator of the second system meets each of the lines b, c, d, e, f. Moreover, there is one generator of each system through any point O of q; hence through O there passes one plane meeting b, c, d, e, f.

Theorem IV therefore states that

"*If through any point of a quartic q having a, b, c, d, e, f as chords there pass three planes meeting b, c, d, e, f; a, c, d, e, f; a, b, d, e, f, respectively, then there are three other planes through O meeting a, b, c, d, e; a, b, c, e, f; a, b, c, f, d, respectively.*"

Replacing the six lines by the six planes (Oa), (Ob), (Oc), (Od), (Oe), (Of) we have the dual, in S_4, of the theorem of a double-six of lines in S_3, and the actual three-dimensional configuration is obtained by cutting the S_4 figure with any S_3 not passing through O.

§ 10. A brief sketch of other connexions is added.

Six lines of S_4 are met, in general, by five planes which are *associated*—i.e. any plane meeting four of the five meets the fifth*.

This result remains true when the six lines are chords of a normal quartic and this case forms an important part of Wakeford's proof of Mr Grace's Theorem on six lines meeting a line (in S_3)†.

Five associated planes determine a Segre cubic variety‡, Ω, generated by all the lines which meet the five planes. These lines form one of six doubly infinite sets lying on the variety. The coordinates of a point on the variety may be expressed as quadratic functions of four homogeneous variables which afford a representation in ordinary space, Σ. Space (S_3) sections of Ω correspond to quadrics through five fixed points A, B, C, D, E in Σ. Five of the sets of lines on Ω correspond to all the lines through each of the five points A, B, ..., E and the sixth set to twisted cubics through the same five points.

Suppose five of the lines of one set on Ω (say those represented in Σ by lines through A) are chords of a normal quartic. Then they lie upon a Q_3 (Theorem I) which meets Ω in a sextic surface represented in Σ by a quartic surface through the five fixed points $A, B, \ldots, E$ and containing five lines through A. We may consider how such a quartic surface can contain five lines through a point or, again, we may project from A into a plane Π. By this latter method we obtain a plane sextic having nine nodes (projections of B, C, D, E and the five lines through A which lie on the surface). Now at first sight it would seem that any nine points in a plane determine a sextic of which they are nodes (for a given point to be a node requires three conditions and $3 \times 9 = 27$ the number of effective constants in the equation of a plane sextic).

---

* First discussed by C. Stephanos, *Comptes Rendus*, t. 93 (1881), pp. 578–580 and 633–636; see also Prof. C. Segre already quoted.

† E. K. Wakeford, "Chords of Twisted Cubics," *Proc. Lond. Math. Soc.* Ser. 2, vol. XXI, p. 98.

‡ Prof. C. Segre, *Memorie di Torino*, XXXIX (1888), p. 3; *Atti di Torino*, XXII (1887), p. 791; also Mr H. W. Richmond, *Quart. Journ. Math.* XXXI (1899), p. 125, and XXXIV (1902), p. 117.

But on more careful examination it will appear that in the general case the sextic so obtained is the cubic through the nine points counted twice. For genuine sextics having nine nodes the following theorem is true:

If a sextic having double points at eight given points have a ninth double point, then this ninth lies upon a curve of the ninth degree having triple points at each of the eight.*

The connexion between five lines on a quadric threefold and nine double points of a plane sextic here described I owe to Mr H. W. Richmond.

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* See Mr J. Hodgkinson, *Proc. Lond. Math. Soc.* Ser. 2, vol. XV (1916), p. 343.

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CONTENTS.

PAGE

On Graeffe's method for complex roots of algebraic equations. By Mr S. **Brodetsky**, Trinity College, and G. **Smeal** 83

On the problem of four particles. By Mr J. **Brill**, St John's College . 88

Canonical forms of the quaternary cubic associated with arbitrary quadrics. By Mr H. W. **Turnbull**, Trinity College 92

On protecting and sensitising colloidal sols. By Mr **Eric K. Rideal** . 101

The equation of state of a gas. By J. E. **Jones**, 1851 Exhibition Senior Research Student, Trinity College, Cambridge. (Communicated by Mr R. H. **Fowler**) 105

On the differential invariants of the Laguerre group. By **Tadahiko Kubota**. (Communicated by Professor H. F. **Baker**) . . 113

On the penetration of waves and their rays into a medium of gradually varying properties. By J. A. **Wilcken**, B.Sc. (Copenhagen), Ph.D. (La Plata); late of Christ's College. (Communicated by Professor Sir **Joseph Larmor**) 124

Dissociation under a temperature gradient. By P. A. M. **Dirac**, St John's College, Cambridge. (Communicated by Mr R. H. **Fowler**) . 132

The electromagnetic equations as basis of Einstein's quadratic form. By Mr R. **Hargreaves**, St John's College. (With One Text-figure) . 138

On the polygons inscribed in one conic and circumscribed to another. By Dr W. **Burnside**, Pembroke College, Cambridge 163

The problems of random flight and conduction of heat. By Dr W. **Burnside**, Pembroke College, Cambridge 167

The integral expansions of arbitrary functions connected with integral equations. By J. **Hyslop**, St John's College. (Communicated by Professor E. W. **Hobson**) 169

On the angle of incidence in soaring flight. By Dr E. H. **Hankin**, St John's College, and J. D. **North**. (With One Text-figure) . 186

On five lines, in space of four dimensions, which lie upon a quadric threefold and the normal quartic curves of which they are chords. By F. **Bath**, King's College. (Communicated by Mr H. W. **Richmond**, King's College) 189

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Complexes of Conics and the Weddle Surface. By Mr C. G. F. JAMES, Trinity College.

[*Received* 18 January 1924.]

A complex or system ∞^3 of conics in space of four dimensions is such that a finite number of conics pass through an arbitrary point. Linear complexes are those for which this number is unity, and are such that their curves are defined by conditions of incidence with fixed surfaces, curves and points. In this paper are discussed briefly the linear complexes defined by the condition that their curves meet an irreducible curve in four points. Denoting by $c_m{}^p$ a curve of order m and genus p it is found that the curves in question are $c_5{}^0$, $c_6{}^2$, $c_7{}^3$, $c_8{}^3$. The complex associated with $c_6{}^2$ is considered in greater detail, since it is found to have an interesting connection with the well-known Weddle quartic surface of ordinary space. In fact the conics of the system touching a space (of three dimensions) do so in the points of such a surface. The main properties of this surface can be thence deduced. In addition we discuss certain results in connection with this curve $c_6{}^2$. The paper closes with certain enumerative results which were obtained in the course of the researches giving the results recorded and which we believe are worth record.

Surfaces and forms are denoted by such symbols as $V_2{}^m$ and $V_3{}^m$ respectively, the lower suffixes giving the dimensions and the upper suffixes the orders of the constructs.

§ 1. If a linear complex is defined entirely by incidence conditions with fixed curves it is seen at once that the system of planes must be ∞^2 only, and that the conics in each form a pencil the base points being points of intersection with these curves. Further, the system of planes must be linear in the sense that one only passes through an arbitrary point of space. Since the section of such a system is a linear congruence of lines it is seen that such systems are of three types.

(1) The [1]-star of planes, or the system of planes passing through a fixed line,

(2) the system of planes cutting in conics a normal cubic scroll (say the system of conic-planes of a cubic scroll),

(3) systems which pass the generators of each of two [0]-cones of lines with a common vertex.

If the directrix curve of the complex of conics is irreducible it is seen that the system of planes can only be the system (2).

§ 2. We shall therefore determine exhaustively all such complexes* when we have found all curves on the cubic scroll which cut the conic-planes in four points, for it should be observed that the curve must lie on the scroll, since through each of its points pass ∞^1 conic-planes. Conversely a quadrisecant plane of the curve is necessarily a conic-plane of the scroll. Generally let us denote by γ the number of points of intersection of a curve with the conic-planes and by α those with the generators. Since a conic-plane is formed by the plane of a generator and the directrix of the scroll the curve must meet the directrix in $\gamma - \alpha$ points so that $\gamma \geqslant \alpha$. Now it is well known that the scroll can be represented on a plane by projection from a line meeting two generators. The plane contains two simple principal points, A_1, and A_1', arising from these generators and a double principal point, A_2, arising from the section of the conic-plane through the line of projection. If, following the usual notation, we denote by h the number of chords of the curve meeting a line, and by m its order, we have almost immediately

$$m = \alpha + \gamma,$$

$$h = \binom{\gamma}{2} + 2\binom{\alpha}{2}.$$

The same result can be obtained by projection from a point on to an arbitrary space, and using the geometry of the general cubic scroll of ordinary space.

It may be noted for future reference that the conics on the surface are represented by conics passing through A_2, A_1, A_1', the directrix by the line A_1A_1', and space sections by cubics through A_1, A_1' having a double point at A_2. Also that the curves (γ, α), (γ', α') intersect in $(\alpha\gamma' + \alpha'\gamma - \alpha\alpha')$ points†.

In our case we are to take $\gamma = 4$, and remembering the condition $\gamma \geqslant \alpha$ it is seen that *linear complexes of conics arise from the curves*

$$c_5^{\,0},\ c_6^{\,2},\ c_7^{\,3},\ c_8^{\,3},$$

provided these curves lie on a cubic scroll, which, as is seen later, is a restrictive condition only in the last two cases. In fact the number of quadrisecant conics through a general point of space is equal to the number of quadrisecant planes through such a point, and therefore by projection to the number of quadrisecants of a three dimensional curve of the same order and rank. Applying a known formula*, we find that in the last two cases the numbers in question are 5 and 23 respectively for the general curve of these characteristics. The cause of the reduction in these special cases is seen without difficulty. The complex associated with the curve c_5^0 is simply a special case of that associated with c_6^2 (namely the quintic curve forms with the directrix of the scroll a degenerate c_6^2 which replaces the singular curve in the other case). The complexes for c_7^3 and c_8^3 can be discussed in the same manner as that about to be used for c_6^2, but the results are not of particular interest.

---

* Excluding the complex defined by a cubic curve on the scroll for which the pencil of conics in each plane have double contact with the conic on the scroll at the points of the cubic curve, and similar cases.

† Using these results it can be shown by exhaustion that if a complex is defined by intersections with a set of curves on a cubic scroll these curves taken together form a degenerate curve of one of the four cases, with the exception of the case of the complex defined by four generators, distinct or coincident.

§ 3. *The curve c_6^2 on the cubic scroll is a general curve of these characteristics.* For such a curve contains an involution g_2^1, and if we select an arbitrary plane and associate pairs of spaces through this plane, when they contain respectively the points of a pair of the involution, the resulting correspondence is (3, 3), and the united spaces are the spaces through a united point of the involution, and spaces containing a generator of the scroll of lines joining the pairs of the involution. The scroll in question easily results to be cubic. Further, the curve lies on a system of ∞^3 quadric forms. In fact the postulation† of a general curve c_m^p on a M-tic form is always $(m\,.\,M - p + 1)$ whatever be the dimension of the space, and the system of quadric forms in [4] is ∞^{14}. Conversely the section of a cubic scroll by a quadric form is a sextic curve, c_6^2, for which $\gamma = 4$, and $a = 2$, since the points in a conic-plane are the points where the conic of that plane cuts the form, and so also for the points on a generator.

§ 4. *A note on the theory of complexes of conics in space of four dimensions.*

It can be shown that the orders of the principal constructs associated with a complex can be expressed in terms of the following characteristics:

m, the number of conics in an arbitrary space,

N, the order of the complex section of the system of planes (in general),

θ_1, the first class, or the number of conics meeting a line and a plane through it elsewhere,

θ_2, the second class, or the number of conics meeting an arbitrary line and an arbitrary plane,

n, the order, in the present case unity,

---

* Salmon-Rogers, *Geometry of three dimensions*, ed. 5, vol. 2, § 474.

† For example, the method of Picard-Simart, *Fonctions algébraiques de deux variables indépendants...*, t. 1, p. 223 (Paris, 1897) shows that it is at least this, and it cannot be greater in the present case, for then six points of an ordinary space would lie on more than ∞^3 quadrics.

v, the order of the surface locus of vertices of degenerate conics of the system, or as an alternative to the last number,

w, the number of lines of degenerate conics in an arbitrary space.

In the present case m and N are zero, and it remains to find θ_2, θ_1, and v. We shall find the last two directly, and the value of θ_2 is known, when we know the order of the surface locus of points of contact of conics touching a space, say x; for a theorem of Pieri[*] can be applied to the correspondence cut on a space by the conics of the system, and gives at once:

$$x = \theta_2 - \theta_1 - n.$$

The number of conics touching a plane can be shown to be $2(\theta_1 - N)$, for ∞^1 conics meet the plane twice, and if we join their intersections to a fixed point, Q, we obtain in the pencil a correspondence of index θ_1 between the lines through points of section of the same conic; and $2N$ of the united lines arise from the N chords of conics, passing through Q (each counted twice since the correspondence is involutory).

§ 5. *Principal properties of the complex defined by the curve c_6^2.*

The singular points of the system (points through which more than a finite number of conics pass) are the points of c_6^2, through each of which pass ∞^2 conics, namely ∞^1 for each conic-plane passing through the point, and the points of the scroll V_2^3 itself, through each of which pass ∞^1 conics. The complex contains the conics on the scroll. The conics through a point of c_6^2 generate the cone projecting V_2^3 from the point, those through a point of V_2^3 the scroll itself.

From the nature of the system of planes, and the fact that the conics in each plane form a pencil whose base points are the intersections with c_6^2, we have seen that m and N are zero, and it is seen further that θ_1 is equal to three and therefore (by § 4) v to six.

The conics whose planes pass through a point of space generate the conic-plane through that point and the associated constructs (curve locus of points of contact of tangents passing through a point, etc.) are given by elementary plane results.

The conics whose planes meet a line and the form of planes of conics meeting the same line are the same quartic form having the cubic scroll as nodal construct (§ 4), namely that formed by conic-planes meeting the line. It is in fact the most general quartic form of this nature.

Consider a quartic form having a nodal scroll V_2^3. It contains ∞^3 lines since the section by any space is a quartic surface with a nodal cubic curve, and must therefore be described by a series of ∞^1 planes, since it is well known that either this is the case, or the form must be a quadric. In fact, if a point on the form outside V_2^3 is selected, then the form must contain the conic-plane through it, since the plane cuts the form in a repeated conic and in an external point. Now the system of conic-planes can be represented on a plane by selecting an arbitrary plane and associating with each plane of the system the point in which it cuts the fixed plane. The representative plane has three principal points, and further to the points of each line joining two principal points corresponds one conic-plane only, namely that one passing through the line in question. If then we apply to the plane a suitable quadratic transformation the system of conic-planes is represented on a plane without principal points*. Now the planes generating our quartic form cut the old plane in a trinodal quartic curve, and are therefore represented on the new plane by the points of a conic, and it can be seen without difficulty that any conic in the new representative plane gives rise to such a quartic form. (To a line in the plane evidently corresponds the planes of a quadric [0]-cone.)

---

[*] "Formule di coincidenza per le serie algebriche...," *Rend. Palermo*, 5, 252 (1891), formula a_{n-1}.

Now the conic-planes meeting a line certainly give such a quartic (as in fact again appears from the above representations), but the system of lines is ∞^6, and we have just seen that the system of quartic forms is ∞^5 only. Hence each can be generated in ∞^1 ways at least as the form of conic-planes meeting a line. Consider any such line. The system of planes meeting the line can be regarded as the system meeting four fixed lines of the scroll and the extra line, and is ∞^1. Now it is known that such a system of planes meets ∞^1 lines of a *quintic scroll*†, and generates a quintic form.

It follows that the ∞^1 directrix lines of the form belong to a regulus, the conjugate regulus being given by the sections of the ∞^1 conic-planes. The residual section of the form by the space of the regulus is the pair of planes through the directrix and the two generators of V_2^3 the space contains. The quintic form of the general case is made up of the quartic form and the space of the quadric, as locus of planes through the directrix of V_2^3.

Finally there exists on V_2^3 a curve locus of cuspidal points of the form. The tangent spaces at a general point of V_2^3 are the spaces containing the tangent plane to V_2^3, and the respective conic-planes through the point meeting one of the directrices of the scroll. These spaces coincide if either

(1) the two conic-planes and the tangent plane are cospacial, though it is seen at once that there are no such cases, or

(2) the two conic-planes coincide. The order of the curve locus of points for which this is the case is seen to be equal to the rank of a space cubic curve (namely four), by considering a space through one of the directrices. The conic-planes through the directrix of V_2^3 generate a quadric [1]-cone, and its planes meeting a line do not in general coincide. Now the [0]-cones of conic-planes through points of V_2^3 in the neighbourhood of the directrix tend to the [1]-cone in the limit. It follows that the cuspidal curve does not meet the directrix of V_2^3 (or, in the notation of § 3, $\gamma = 2$, $a = 2$). The quartic form has no points of higher singularity.

This form has been rather fully discussed since all the quartic scrolls of ordinary space with a nodal curve of total order three can be obtained from it as the section by a suitably chosen space.

A Digression. It may perhaps be remarked that the elliptic quartic scroll of ordinary space (with two skew nodal lines) cannot be so obtained as the arbitrary section of a quartic form in [4] other than as a cone of the same abstract nature. For the form in question must have two nodal planes, which can be assumed in arbitrary position. Taking any point on the form we consider

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[*] The representation on the new plane is in fact the representation given by the canonical equations

$$\Sigma a_i a_{ix} = \Sigma a_i b_{ix} = 0,$$

for the system of conic-planes, and our results can be verified readily by algebraic arguments.

[†] Segre, "Sull' incidenza di retta e piani...," *Rend. Palermo* (1888), p. 45, n. 8.

the plane through it which has line intersection with each of the nodal planes, and which therefore passes through their point of intersection. This plane meets the form in two lines, each counted twice, and in an exterior point. It is thus a generating plane of the form, which is therefore a cone having the above point as vertex. The same result and proof apply to the scroll of order $(\mu+\nu)$, generated by the joins of points on two skew lines in (μ, ν) correspondence.

Except for the dimension of the spaces involved this property is shared by a number of constructs of which the Surface of Veronese is a well-known case*. Another set of examples are the constructs represented by the vanishing of matrices of two rows and r columns (excluding the case of r equal to two), in space of dimension $2r-1$. This is indicated, though not proved, by the fact that the $2r$ elements can be taken as independent coordinates, so that if the result is not true, we have at least the fact that the construct in $[2r]$, from which the former is derivable by section, cannot be represented by a matrix of this type. We have a V_r^r of $[2r-1]$ described by ∞^1 spaces $[r-1]$. If there exists a V_{r+1}^r of $[2r]$, of which the general section is such a V_r^r, then either:

(i) V_{r+1}^r contains ∞^1 spaces $[r]$, or

(ii) it contains ∞^{r+1} spaces $[r-1]$, for this is the dimension of the system of $[r-1]$ of $[2r]$, such that an arbitrary $[2r-1]$ contains ∞^1.

In the former case the $[r]$ either pass through the same linear space, or there exists a surface of points through which two (or some other finite number) of the generating spaces pass, and there would be a curve of points with this property in the space $[2r-1]$ of the original figure, which is not the case. Turning to case (ii) the V_{r+1}^r can be projected into a V_{r+1}^r of $[r+2]$ with ∞^{r+1} $[r-1]$, and section gives a V_3^r of $[4]$ with ∞^{r+1} lines, and this is known to be possible only for r equal to one or two.

If there are s rows the construct contains ∞^{s-1} linear spaces $[rs-r-1]$ and is of dimension $(rs-r+s-2)$, and a similar argument shows that it can at most be derived from a conical construct or from one of the same order, and containing linear spaces of dimension one higher than those on the first, namely $[rs-r]$.

A final example is given by Steiner's Quartic Surface†. The form of $[4]$ from which it is derived as an arbitrary section must possess three double planes concurrent in a triple line. Any plane through this line meets the form in one further line, of which there are therefore ∞^2 on the form. The form is either a cone, or else ∞^1 of these lines must pass through a given point of this triple line. In the latter case Steiner's Surface would contain simple lines through its triple point and this is not the case.

§ 5 *a*. Passing on to determine the second class θ_2 we assume a result to be proved immediately, namely that x (§ 4) is equal to four and hence by the formulae quoted θ_2 is equal to eight. It follows that the conics of the complex meeting a line l describe an 8-tic surface passing once through l and twice through c_6^2, since two conics through a point of this curve meet l. It cannot meet V_2^3 elsewhere than in c_6^2, since the conics through a point of V_2^3 lie on it. In fact any surface of conics of the complex meets the scroll in c_6^2 and in conics or lines only. The genus of the space sections of the present surface is three (and therefore their "h" is equal to 12), and it contains eight degenerate conics.

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* Bertini, *Geometria Proiettiva degli Iperspazi*, ed. 2 (Messina, 1923), p. 402.

† Jessop, *Quartic Surfaces* (Cambridge, 1916), p. 133 *et seq.*

Similarly the conics meeting a plane describe a form of the eighth order. Since a conic-plane cuts it in one conic, and in the conic on V_2^3 counted with suitable multiplicity, it follows that V_2^3 is three-fold; and since a chord of c_6^2 can meet it on c_6^2 only, this curve is four-fold on the form. In addition the conic-planes which meet it in lines are nodal on the form, etc.

Its section by an arbitrary space is an interesting surface of the eighth order having a triple cubic curve; three nodal lines, chords of the cubic; and a simple line, meeting the nodal lines but not the cubic. It has six four-fold points on the cubic curve. The surface possesses a (1, 1) transformation into itself, for which the locus of united points is the section by the surface locus of points of contact of conics of the complex touching the space, which, as we shall see, is the Weddle surface having the six four-fold points as nodes. It is therefore a curve of order 23, etc.

The degenerate conics of the complex are formed by pairs of intersecting chords of c_6^2, and each such chord can belong to one conic only, since through it passes one conic-plane only. They therefore describe the chord-form of c_6^2, of order eight (= "h" of the curve), passing simply through V_2^3 (in virtue of the generators), and five times through c_6^2. The surface locus of vertices (below) is nodal on the form, which has no other multiple points. By § 8 it can be shown that the vertex locus is of the seventh order, V_2^7. This surface is also the locus of vertices of self-conjugate triangles for the pencils of the complex, and therefore cuts each conic-plane in three points and on c_6^2, so that it passes simply through the curve. $\binom{6}{2} = 15$ lines of degenerate conics lie in a space.

We have seen that c_6^2 is the base curve of a linear system ∞^3 of quadric forms, and the surface we are dealing with is included amongst the locus of vertices of cones of the system. Now in the general case this locus is easily shown to be of order 10 and genus (of space section) 11. In the present case the locus is completed by V_2^3 itself since the cone projecting V_2^3 from one of its points contains* c_6^2. The two partial surfaces cut in c_6^2 only, and from this it is seen that the genus of V_2^7 is six, since the surface has no multiple points (cf. § 9).

§ 6. ∞^2 conics of the surface touch a space **H**, and hence the locus of such points of contact is a surface, which we proceed to determine. Since there is a conic of the system touching each tangent to V_2^3 at the point of contact it follows that the section c_3^0 of V_2^3 lies on the surface and simply. The remaining pair of points on the line of section of any conic plane are the points where conics of the pencil in that plane touch the line, namely they are the united points of the involution cut by the conics on the line. The surface sought is thus of the fourth order, and possesses a (1, 1) transformation into itself, in which the points of contact with these lines of section are interchanged. (These lines are rays of the chord congruence (1, 3) of c_3^0.) Now the conic-planes have an important property in connection with c_6^2, namely they are cut in pencils by the linear ∞^3 of quadric forms[*] circumscribing c_6^2. Hence the surface, F^4, is the locus of united points of the correspondence of pairs of points in which the quadrics of the ∞^3 through six points cut the lines of the chord congruence of the cubic curve through the six points ($Q_1 \ldots Q_6$), namely a Weddle surface[†]. The present construction enables us to verify certain known properties of this surface in a new way. Thus we have at once that the four points on any of the lines of section above are harmonic, since to the involution on that line belong the two points where V_2^3 cuts it. The points Q_i are nodes on the surface, the tangent cone at each being the cone standing on c_3^0. For the involution on the line joining Q_i to a point on c_3^0 is degenerate, and the two united points coincide.

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* It may be noted that V_2^3 is the locus of points from which c_6^2 is projected into a sextic curve of genus two, having one actual node and lying on a quadric. From a point on the directrix it is projected into a sextic with two actual nodes ($h = 6$).

The surface F^4 can be generated as the locus of vertices of quadric cones through $\{Q_i\}$. For taking any point, P, in the four dimensional space, through it passes a system ∞^2 of quadric forms the base consisting of c_6^2 and the conic of the complex through P. The tangent, g, to the conic at P is touched by ∞^2 of the forms, and one form touches an arbitrary space through g. Taking this space as the space **H**, the result follows at once. It then follows as in the usual theory that F^4 contains the lines Q_iQ_j (or from the fact that they form part of degenerate conics of the complex) and the lines $(Q_iQ_jQ_k)\,(Q_lQ_mQ_n)$.

Notes on the above generation of the Weddle Surface.

(1) The surface constructed in the same way for the complex associated with the curve c_5^0 is also a Weddle Surface, the sixth node being the intersection with the directrix of V_2^3.

(2) It is known that the Weddle surface is intimately connected with curves of genus two, and it might be thought that the class moduli of the curve associated with F^4, say the curve in (y, θ),

$$y^2 = \prod_1^6 (\theta - \theta_i),$$

where (θ_i) are parameters of the points Q_i on c_3^0, are the same as for c_6^2. However this is not the case for, for a given c_6^2, any four of the points Q_i can be arbitrarily selected.

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[*] Jessop, *Quartic Surfaces* (Cambridge, 1916), p. 173.

[†] A similar result is: For an ∞^2 of quadric forms in [4], the base being a c_8^5, a c_7^3 + its 3-secant, etc., there exist ∞^2 planes, the quadrisecant planes, on each of which the system determines a pencil of conics only. 10 such planes pass through a point and 30 meet a fixed plane in lines (§ 9).

(3) It is known that the surface is uniquely determined by the points $\{Q_i\}$. For a given space through the space of F^4 there is a system of c_6^2 giving the same F^4. What is the dimension of this system? For a given scroll through c_3^0 the dimension of the system of suitable sextics thereon is three less than the dimension of quadric forms through the points Q_i, and the conditions to contain these points being independent, the dimension sought is therefore five. The dimension of suitable sextics on all possible cubic scrolls is therefore given by adding the dimension of the system of scrolls. We can obtain a cubic scroll through c_3^0 by selecting any line meeting it as the directrix, and any (1, 1) correspondence between the points of the curve, and the line, in which the point of intersection is self-corresponding, and generating the scroll as the locus of joining lines. The dimension of the system is therefore eight. A given F^4 in a space of the [4] can thus be generated in ∞^{13} ways as the locus of points of contact of the conics of a certain complex.

(4) It is evident that the section of V_2^7 by $\mathbf{H}$ lies on F^4. A Weddle surface therefore contains a system of ∞^{13} curves c_7^6, with no actual nodes, passing through the points Q_i, and not meeting c_3^0 elsewhere.

(5) The curve on F^4 described by the pairs of the transformation whose joining lines meet a line, l, is the section of the locus of points of contact of tangents to conics of the complex which meet a line. This (§ 4) is an 11-tic surface passing through l simply, and the curve is therefore of order ten, meeting the line in four points (viz. where it meets F^4). The pairs on the curve given by corresponding points coincide only at the points Q_i, where the curve has actual nodes, and a coincidence occurs on each branch at the node. (The tangents to the branch meet l.) The curve is thus of genus five. It is cut on F^4 by a quartic scroll passing twice through c_3^0, generated by the joining lines in question.

(6) *A corollary of the uniqueness of the Weddle surface associated with six given points.* It has already been seen that on any cubic scroll passing through c_3^0 there are ∞^5 sextics all giving rise to the same surface F^4. Now each sextic cuts a given conic plane in four points, and the sets of four points so given are such that through each pass two conics touching a given line in given points. It would appear that the sets of four points describe a series (not in the first place necessarily linear) of freedom two. The series, however, is of freedom three, since the two prescribed points of contact are harmonic with respect to the points of intersection with the conic in the plane on V_2^3. For if any conic touching the line at the first of the given points cuts the fixed conic in four points (P_i) then through these points necessarily passes a conic touching the line

at the second given point. The series is linear (using elementary geometry), and the points of intersection with the conic of V_2^3 form a *neutral pair*, namely they can be associated with two arbitrary points of the conic to form a group of g_4^3. We can thus formulate the theorem:

All the ∞^1 sextics of genus two on a cubic scroll in space of four dimensions passing through six cospacial points and three points on a given conic-plane pass through a fourth fixed point on the plane.

Similarly all sextics through the six points and two others cut the conic-plane through the points in an ordinary involution. Consider a c_6^2 composed of two cubic curves, for which the complex acquires certain specialities which we shall not describe. Associated with a given F^4 there are 20 sets of ∞^2 such sextics, namely we associated the section by a space through $Q_iQ_jQ_k$ with that by a space through $Q_lQ_mQ_n$, $i, j, k; l, m, n$, being any partition of $1 \ldots 6$. Two curves for each set pass through two fixed points P_1, P_2, namely those formed by the sections

$$\{P_1Q_iQ_jQ_k,\ P_2Q_lQ_mQ_n\},\ \{P_1Q_lQ_mQ_n,\ P_2Q_iQ_jQ_k\}.$$

The residual points of intersection of each curve with the conic in the conic-plane through P_1P_2 are the points where the first pair of spaces cut the conic (which by the lemma below are the same as the points of intersection of the second pair of spaces).

Lemma. Given an involution on a fixed line and a conic through a pair of the involution then the correspondence given by projecting the first point of each pair on to the conic from a point P_1 thereon and the second from a fixed point P_2 thereon is itself an involution, so that the same involution is obtained by projecting the pairs from P_2, P_1.

(Proof: The correspondences obtained in this way from P_1, P_2, and P_2, P_1, share three pairs, namely a pair arising from each of the united points of the involution and the pair through which the given conic passes, and they are therefore identical, and the result is involutory.)

Considering the figure in the conic-plane through P_1P_2, the pair of points in which the two cubics (respectively through $P_1Q_iQ_jQ_k$ and $P_2Q_lQ_mQ_n$) cut the conic belong to the involution mentioned above, and they are the projection of the points in which the planes $Q_iQ_jQ_k$, $Q_lQ_mQ_n$ cut the line of section g with **H**, from the points P_1, P_2, respectively. Now each of these pairs of points on the line g belongs to the involution defined on that line by the cubic c_3^0 or by F^4. We thus have

The involution given on a conic of V_2^3 by the c_6^2 through six cospacial points and two fixed points on the conic is the projection from the fixed points of the involution defined by the cubic curve through the six points on the line of intersection with its space of the plane of the *conic, the order in which the points P_1, P_2 are taken being immaterial by the lemma.* (The involution on the line is the involution of harmonic pairs with respect to the points of intersection with c_3^0.)

The results in this section give results for the cubic scroll of ordinary space, and the c_6^2 which lie on it; and for c_6^2 in a plane with reference to a fixed conic. The centres and united points of the involutions indicated can in each case be found by linear constructions. It may also be noted that a c_6^2 on V_2^3 is also formed by its conic through Q_iQ_j and a quartic curve through $Q_kQ_lQ_mQ_n$ ($\gamma = 3, \alpha = 1$). There are 15 sets of ∞^2 such c_6^2.

These theorems acquire special features, and give rise to some interesting corollaries when applied to the conic-planes each passing through the directrix and a generator of V_2^3.

§ 7. After these digressions we return to the complex of conics defined by c_6^2, to consider the form of conics touching a space **H**. We have seen that it touches the space along a Weddle surface, and it is therefore of order eight. It passes twice through V_2^3, and four times through c_6^2 since a chord of this curve meets the form on the curve only. It contains a scroll of lines of degenerate conics of order 29 passing six times through c_6^2 and containing therefore five generators of V_2^3. It meets **H** in the 15 lines Q_iQ_j and in the vertex locus a c_7^6. All these results follow without difficulty.

If the ∞^2 conics in question are projected on to **H** we obtain a congruence of conics of order eight (number of curves through a point), and class six (number having an arbitrary line as chord). The conics meet the projection of c_6^2 four times and touch F^4 and a second focal surface, namely the apparent contour of the above V_3^8 from the point of projection, a surface which is therefore in (1, 1) correspondence with the Weddle surface.

§ 8. *Appendices. On a birational transformation of the four dimensional space defined by* c_6^2, and on some associated problems.

In this section are considered certain results in association with the curve c_6^2. Consider in the first place the involution defined as follows: the point A' associated with a point A is that point in the conic-plane through A in which concur its polars for the conics of the complex in that plane. This involution is in fact the involution of pairs of points similarly obtained from the ∞^3 of quadric forms circumscribed to c_6^2, and the results obtained may be verified algebraically using this fact.

The only united points of the involution are the points of c_6^2. The singular points are the points of V_2^3 and the vertex locus V_2^7. The following results are proved without difficulty: the two columns give pairs of corresponding constructs.

A chord of c_6^2:	itself.
A conic-plane:	itself.
A line in a conic-plane:	a conic through the points of V_2^7 in the plane.
An arbitrary line:	a normal quartic curve, meeting V_2^7 15 times and V_2^3 not at all.
An arbitrary plane:	a sextic surface, meeting V_2^7 in a 12-tic curve, and V_2^3 in an 8-tic.
An arbitrary space:	a quartic form passing once through V_2^7 and V_2^3.
A conic of the complex:	a quartic curve, nodes at the points of V_2^7 in its plane.

To a point, O, of V_2^3 correspond the points of a curve lying in the tangent plane at that point, and cutting a line through the point in that plane in one point only (since through such a line passes a conic-plane). Since this curve does not pass through O it is a straight line, o. It cuts the generator through O in the harmonic conjugate with respect to the points of c_6^2 thereon, and is, in fact, the locus of points of contact (other than at O) of conics of the complex with such a plane, each such plane being singular in that it meets twice ∞^2 conics of the system. To a point P of V_2^7 corresponds the point of a line, namely the opposite side of the self-conjugate triangle for the pencil of conics in the conic-plane through P. The locus of such lines is a form of order 15, since the number of its lines meeting a given line is equal to the number of times the correspondent of the line meets V_2^7. This form passes six times through V_2^3. To a point O on the directrix d corresponds a line o in the plane through d and the generator g through O, and joining the harmonic points of O with respect to the pairs of points of c_6^2 on d and g respectively. The locus of these lines o as O describes d is a scroll passing through d, and such that a space through d contains a generator for each generator of c_6^2 in the space. It is therefore a second cubic scroll $V_2'^3$ (note: one generator passes through each point of d). The two cubic scrolls cut in d and in the locus of harmonic points of points O of d with respect to the pairs of points of c_6^2 on the generators of V_2^3 through them. Two points of this curve lie on d and two in each point space through d (one on each generator in the space), so that the curve is a quartic. Consider finally the locus of the residual vertices of the self-conjugate triangles in the planes through d, a curve on $V_2'^3$ cutting each of its generators in two points, and meeting d in the same two points as c_6^2 meets it. It is therefore also a sextic of genus two, and we reach the interesting result: *A c_6^2 of space of four dimensions determines uniquely a second such curve. The relation of the two curves is reciprocal, and the* *associated scrolls have the same directrix and meet again in a normal quartic curve. The sextics cut in two points on the directrix and in the six points where a generator of* V_2^3 *touches* c_6^2, *these being also points where a generator of* $V_2'^3$ *touches* $c_6'^2$, *and they lie on the above quartic curve. The curves are in* (1, 1) *correspondence in two ways, namely by projection from either of the common points on d.*

§ 9. *Some enumerative results.* These results were obtained in the course of the researches leading to the above results, and we believe it useful to record them. For the most part they are deduced by the Severi functional method (see the lemma below), and on account of the facility with which this method can be used, we give details only when such appear to be necessary.

Lemma. Integration of a class of functional relations. If there exists a function satisfying a functional relation of the type

$$\phi(x_1 + x_1', \dots x_n + x_n') \\ = \phi(x_1 \dots x_n) + \phi(x_1' \dots x_n') + f(x_1, \dots x_n; x_1', \dots x_n'),$$

where f is the symbol of a given function of the arguments x_1, $x_2, \dots x_n$; $x_1', x_2', \dots x_n'$, which vary in the field of integral and positive numbers (zero included), it is of the form

$$\phi(x_1 \dots x_n) = \sum_{1}^{n}{}_k c_k x_k \\ + \sum_{1}^{n}{}_k \sum_{1}^{x_k - 1}{}_i f(0, 0, \dots 0, 1_k, 0 \dots 0; 0 \dots 0, x_k - i, 0, \dots 0) \\ + \sum_{2}^{n}{}_i f(0, \dots 0, x_i, x_{i+1} \dots x_n; 0, \dots 0, x_{i-1}, 0 \dots 0),$$

where the c are arbitrary constants and with the symbol 1_k is expressed that x_k has been put equal to unity*.

(1) *On systems of* ∞^2 *planes defined by a curve or curves.* The order N of the system is the number of planes through a point; the class M the number of planes meeting a fixed plane in a line (the order and class of the congruence of section by an arbitrary space).

Planes meeting c_n^R, $c_{n'}^{R'}$ and two lines†:

$$N = 2nn'; \; M = 3nn'.$$

In this and the following formulae the result for N is seen on projection to arise from certain well-known formulae for ordinary space. To replace a line in any case by a curve it is necessary to multiply by the order of the curve.

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* Quoted from Severi, "Sopra alcune singolarità delle curve di un iperspazio," *Mem. Torino*, Ser. 2, t. 51 (1902), p. 81; § 9, n. 22, p. 107.

† In § 9 the symbol c_n^R denotes a curve of order n and rank R, n and R being the variables of the functional relations. The curves are supposed to be general.

Planes meeting c_n^R twice, $c_{n'}^{R'}$ once, and a line:

$$N = n'\{n(n-1) - \tfrac{1}{2}R\},$$

$$M = n'\{\tfrac{3}{2}n(n-1) - \tfrac{1}{2}R\}.$$

Planes meeting c_n^R thrice, and a line once:

$$N = 2\binom{n}{3} - \tfrac{1}{2}R(n-2),$$

$$M = (n-2)\,h;\ \text{where } h = \binom{n}{2} - \tfrac{1}{2}R.$$

Planes meeting c_n^R and $c_{n'}^{R'}$ each twice:

$$N = 2\binom{n}{2}\binom{n'}{2} - \tfrac{1}{2}R\binom{n'}{2} - \tfrac{1}{2}R'\binom{n}{2} + \tfrac{1}{4}RR',$$

$$M = 3\binom{n}{2}\binom{n'}{2} - \tfrac{1}{2}R\binom{n'}{2} - \tfrac{1}{2}R'\binom{n}{2}.$$

Planes meeting c_n^R four times:

$$N = \tfrac{1}{24}\{-n^4 + 18n^3 - 71n^2 + 78n - 48nh + 132h + 12h^2\},$$

$$M = 3\binom{n}{4} - \tfrac{1}{2}R\binom{n-2}{2}^{*}.$$

(2) *On the form of chords of a curve c_n^R in space of four dimensions.* The form in question is of order h (the number of apparent nodes of the curve), passing $(n-2)$ times through the curve in question. We propose to determine the order of its double surface, locus of points of concurrence of pairs of chords, and its triple curve, locus of concurrences of triads of chords. There is also in general an isolated set of concurrences of sets of four chords, but this number has not been determined. It is necessary to calculate as well similar formulae for the form of lines joining points of a curve c_n^R to those of a curve $c_{n'}^{R'}$, a form of order nn' passing n' times through c_n^R and n times through $c_{n'}^{R'}$. Suppose in the first place that the second curve is a line. Then the double surface in question consists of the h planes through the line containing chords of the curve. Now allow $c_{n'}^{R'}$ to break up into n' lines with $\tfrac{1}{2}R'$ intersections. Then the double surface breaks up into n' sets of h planes, and the intersection of the n-tic forms for the n' lines taken in pairs, excluding the cones whose vertices are at the $\tfrac{1}{2}R'$ points and which stand on c_n^R. This gives for the order of the double surface on the form of lines joining points of c_n^R and $c_{n'}^{R'}$:

$$\binom{nn'}{2} - \tfrac{1}{2}n'R - \tfrac{1}{2}nR'.$$

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[*] Generally in space of m dimensions the number of $[m-2]$, m-secant to a c_n^R, and meeting a $[m-2]$ in a $[m-3]$, is $(m-1)\binom{n}{m} - \tfrac{1}{2}R\binom{n-2}{m-2}$.

The same result is found by the use of the functional method, keeping c_n^R fixed and replacing $c_{n'}^{R'}$ by two curves $c_{n'}^{R'}$ and $c_{n''}^{R''}$ in the usual way, bearing in mind that the result must involve the two original curves symmetrically. The surface passes h' times through c_n^R and h times through $c_{n'}^{R'}$, as is again seen by allowing $c_{n'}^{R'}$ to degenerate. In fact the sheets of the surface at a given point of $c_{n'}^{R'}$ arise from the h chords of c_n^R meeting the tangent at the point.

Passing to the form of chords of c_n^R, we replace c_n^R by two curves c_n^R and $c_{n'}^{R'}$. The double surface consists of the surfaces for c_n^R and $c_{n'}^{R'}$ separately, the intersections of the two forms of chords for c_n^R and $c_{n'}^{R'}$ respectively, and the surface of intersection of pairs of common secants of the curves, as just determined. We can then form a functional equation, and the solution is

$$\tfrac{1}{2}\binom{n}{2}\left\{\binom{n}{2}-1\right\}+\tfrac{1}{4}\binom{R}{2}-\tfrac{1}{2}R\binom{n+1}{2}+\tfrac{15}{8}R,$$

remembering that the value for c_3^4 and c_4^6 must be zero. This gives the order of the double surface on the form of chords of c_n^R. To find the multiplicity of c_n^R thereon we replace c_n^R by a c_{n-1}^{R-2} and one of its unisecants. The multiplicity of the line is equal to the number of chords o c_{n-1}^{R-2} meeting it or

$$\tfrac{1}{2}(n^2-5n+10)-\tfrac{1}{2}R,$$

which is the multiplicity of c_n^R sought. (Checks are given by c_4^6 and c_6^{14}.) This number is also the number of chords of c_n^R meeting the tangent at P other than at P.

A precisely similar process is employed to give the order of the triple curve locus in each case of points of triune concurrence of common secants, or chords (respectively). In the second case, however, the result as first obtained contains redundantly the number of trisecants* of the curve.

Triple curve on the form of common secants of c_n^R and $c_{n'}^{R'}$.

$$\text{Order:}\quad RR'-R'n'\binom{n}{2}-Rn\binom{n'}{2}$$
$$+\tfrac{1}{6}nn'(n-1)(n'-1)(nn'+n+n'-2).$$

It meets c_n^R in those points through which pass chords of c_n^R meeting $c_{n'}^{R'}$ and in general in these points only. Each point arises in virtue of this chord and the lines joining consecutive points of intersection of the tangent to the points of intersection with $c_{n'}^{R'}$ of its chords meeting the tangent. There are thus $2 \, . \, n' \, . \, h$ such points, and each is h'-ple on the triple curve.

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* In number $\binom{n}{3}-n+2R-\tfrac{1}{2}Rn$. See Severi, *loc. cit.*, § 8.

The order of the triple curve on the form of chords of $c_n{}^R$ is:

$$15\binom{n}{6}+30\binom{n}{5}+12\binom{n}{4}+3\binom{n}{3}-6\binom{n}{2}+16n$$
$$-\tfrac{1}{8}\binom{R}{3}+\tfrac{1}{8}\binom{R}{2}\left\{2\binom{n}{2}+4n-7\right\}$$
$$-\tfrac{1}{16}R\left\{24\binom{n}{4}+48\binom{n}{3}-30\binom{n}{2}-4n+217\right\}.$$

It meets the original curve in the $3\,.\,t$ intersections with a trisecant of the curve, each counted with multiplicity equal to the number of chords of the curve meeting such a tangent, namely (as above):

$$\tfrac{1}{2}(n^2-5n+10)-\tfrac{1}{2}R-1.$$

§ 8·1. In space of dimension greater than six (exclusive) the construct of chords of a curve, or of common secants of two curves, possesses no multiple points, nor any points in which two chords meet. We have therefore completed the results of this section by a consideration of the cases in five or six dimensions.

In space of five dimensions there exists in general a curve locus of concurrences of pairs of chords or common secants, and this curve is determined by the same process as we used in § 8. The results obtained are:

Lines meeting two curves $c_n{}^R$ and $c_{n'}{}^{R'}$, describing a construct of order $n\,.\,n'$ passing n' times through $c_n{}^R$ and n times through $c_{n'}{}^{R'}$. The curve locus of intersections of pairs of chords is of order $2\binom{n}{2}\binom{n'}{2}-\tfrac{1}{2}RR'$, and meets $c_n{}^R$ at those points where the tangent meets a chord of $c_{n'}{}^{R'}$, namely in $R\,.\,h'$ points. Similarly it meets $c_{n'}{}^{R'}$ in $R'\,.\,h$ points.

Chords of $c_n{}^R$, describing a construct of order h passing $(n-2)$ times through the curve. The locus of intersections of pairs of chords is of order

$$\tfrac{1}{2}\binom{n}{2}\left\{\binom{n}{2}-2n-R+7\right\}-\tfrac{1}{4}\binom{R}{2}-9n+\tfrac{57}{8}R$$

and meets $c_n{}^R$ in the points of contact of tangent planes which are elsewhere bisecant or in

$$\theta=6\binom{n-3}{3}+p\,(n-5)\,(n-12)-4\binom{p}{2}$$

points*.

In space of six dimensions the construct of common secants of two curves contains a finite number of points in which concur two chords, namely twice the number of common bisecant planes, or $2\,.\,h\,.\,h'$.

The number of such points on the construct of chords of a curve $c_n{}^R$ is three times the number of quadrisecant planes, or thrice

$$\binom{n}{4}+2\,.\,n\,.\,R-2\binom{n}{2}-\tfrac{1}{2}R\binom{n}{2}+\tfrac{1}{8}R^2+5n-\tfrac{29}{4}R.$$

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* Severi, *loc. cit.*, § 10.

Trajectories of small horizontal velocity in a resisting medium.

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1. *Introduction.*

The results of the present paper were developed some years ago for quite trivial purposes. It seems possible, however, to judge by a recent publication[*], that they may sooner or later have scientific utility. They have also a certain slight intrinsic interest.

We are concerned with trajectories of a particle under gravity in a medium, in the first instance homogeneous, which produces a retardation μV^2 when the velocity is V. We consider here only the downward motion, reckoned from the vertex as origin, where the velocity is U in a horizontal direction[†]. The general numerical solution of this problem requires tables of double entry. If, however, we suppose that $a = \mu U^2/g$ is a small number, it will be found that the problem becomes soluble by approximation, in terms of functions requiring only tables of single entry.

Our formulae are arranged to give the elements x, t, u, v of the trajectory for a given y: this seems to be the form most likely to be required in practice. It is, however, as easy to develop a parallel set giving x, y, u, v for a given t, and the method to be adopted for this end will be clear after the work of §§ 2.2 to 2.6.

It should be emphasized that we assume merely that a is small, not that μ, or more strictly μy[‡], is small; in fact the formulae are valid for $y = \infty$ $(\mu y = \infty)$.

The vertex is a singularity of the motion, and all our arguments are a little more subtle than would be expected. Pedantic rigour is out of place in the paper, and I hesitated between the method actually adopted, and another which promises greater simplicity at the outset. I judged in the end that certain real difficulties connected with the vertex make the rejected (unrigorous) method less transparent than the rigorous one[§]. The arguments, then, of section 2 are perfectly rigorous down to the end of § 2.6, after which they are merely sketched.

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[*] L. F. Richardson, "Theory of the measurement of wind by shooting spheres upwards," *Phil. Trans. Roy. Soc.*, A, 223 (1923), 345–382.

[†] The corresponding problem for the ascending branch can be dealt with by a parallel argument involving only systematic changes throughout (and of course new tables).

[‡] μy is a "pure number."

[§] Also the parallelism of argument in obtaining formulae in terms of a given t would have been less apparent.

Section 3 of the paper deals with the problem of variable atmospheric density. This is a fruitful source of fallacy, and I have discussed it for some other ballistic questions, to which the answers are sometimes unexpected.

2. *Homogeneous medium.*

2.1. We take the origin at the vertex and the axis of y downwards. Let U be the initial (horizontal) velocity, x, y the coordinates of the particle at time t, $u = \dot{x}$, $v = \dot{y}$, $V^2 = u^2 + v^2$. Let θ be the inclination of the tangent to the path, and let

$$a = \frac{\mu U^2}{g}, \quad k = (\mu g)^{\frac{1}{2}}, \quad l = \log \frac{4}{ea}.$$

The following formulae are well known*:

$$(2.111) \qquad gx = \int_0^\theta u^2 d \tan\theta.$$

$$(2.112) \qquad gy = \int_0^\theta u^2 \tan\theta d \tan\theta.$$

$$(2.113) \qquad gt = \int_0^\theta u d \tan\theta.$$

$$(2.114) \quad \frac{1}{u^2} - \frac{1}{U^2} = \frac{2a}{U^2}\int_0^\theta \sec^3\theta d\theta = \frac{a}{U^2}(\tan^2\theta + \phi).$$

$$(2.115) \quad \phi = \log(\sec\theta + \tan\theta) + \tan\theta(\sec\theta - \tan\theta)$$

$$= 2\int_0^\theta (\sec\theta - \tan\theta)\, d\tan\theta.$$

Substituting from (2.114) in (2.112) we obtain

$$\frac{gy}{U^2} = \int_0^\theta \frac{\tan\theta d\tan\theta}{1 + a\tan^2\theta} - a\int_0^\theta \frac{\phi\tan\theta d\tan\theta}{(1 + a\tan^2\theta)^2}$$

$$+ a^2 \int_0^\theta \frac{\phi^2 \tan\theta d\tan\theta}{(1 + a\tan^2\theta)^2(1 + a\tan^2\theta + a\phi)}.$$

We multiply this by a and write

$$(2.12) \qquad a\tan^2\theta = z^2,$$

obtaining

$$(2.13) \quad \mu y = \int_0^z \frac{z\,dz}{1 + z^2} - a\int_0^z \frac{\phi z\,dz}{(1 + z^2)^2} + a^2\int_0^z \frac{\phi^2 z\,dz}{(1 + z^2)^2(1 + z^2 + a\phi)}$$

$$= \tfrac{1}{2}\log(1 + z^2) - aI + a^2 R.$$

---

* We have $\frac{1}{\dot{\theta}} = \frac{V}{V^2/\rho} = \frac{u\sec\theta}{g\cos\theta} = \frac{u}{g}\sec^2\theta$; $\frac{dx}{d\theta} = \frac{u}{\dot{\theta}} = \frac{u^2}{g}\sec^2\theta$;

and $\frac{d}{d\theta}\left(\frac{1}{u^2}\right) = -\frac{2}{u^3}\frac{\dot{u}}{\dot{\theta}} = -\frac{2}{u^3}\mu(u\sec\theta)^2\cos\theta \,.\, \frac{u}{g}\sec^2\theta = -2\mu\sec^3\theta.$

Similarly we have from (2.111), (2.114), (2.12) (*not* multiplying by a)

$$(2.14)\quad \frac{kx}{U} = \int_0^z \frac{dz}{1+z^2} - a\int_0^z \frac{\phi\,dz}{(1+z^2)^2} + a^2\int_0^z \frac{\phi^2\,dz}{(1+z^2)^2(1+z^2+a\phi)}$$

$$= \arctan z - aI_1 + a^2R_1.$$

For u, v, t there is a slight complication in the R-term. We have

$$\frac{u}{U} = (1+z^2+a\phi)^{-\frac{1}{2}},$$

and so

$$(2.15)\quad \frac{u}{U} = \frac{1}{(1+z^2)^{\frac{1}{2}}} - \tfrac{1}{2}a\frac{\phi}{(1+z^2)^{\frac{3}{2}}} + \tfrac{3}{8}a^2\frac{\phi^2}{(1+z^2+\vartheta a\phi)^{\frac{5}{2}}},$$

where ϑ (as also $\vartheta_1 \ldots$) is a function of z and a, but lies between 0 and 1. Since $v = u\tan\theta$ we obtain from (2.15)

$$(2.16)\quad \frac{kv}{g} = \frac{z}{(1+z^2)^{\frac{1}{2}}} - \tfrac{1}{2}a\frac{\phi z}{(1+z^2)^{\frac{3}{2}}} + \tfrac{3}{8}a^2\frac{\phi^2 z}{(1+z^2+\vartheta a\phi)^{\frac{5}{2}}};$$

and since

$$k\frac{dt}{dz} = (\mu g)^{\frac{1}{2}}a^{-\frac{1}{2}}u = \frac{u}{U}$$

we have finally*

$$(2.17)\quad kt = \int_0^z \frac{dz}{(1+z^2)^{\frac{1}{2}}} - \tfrac{1}{2}a\int_0^z \frac{\phi\,dz}{(1+z^2)^{\frac{3}{2}}} + \tfrac{3}{8}a^2\int_0^z \frac{\phi^2\,dz}{(1+z^2+\vartheta a\phi)^{\frac{5}{2}}}.$$

Our method is now to approximate for z as a function of y from (2.13), and to substitute the result in (2.14)–(2.17).

2.2. A word is necessary about notation. We suppose for definiteness that $a \leqslant 1$. Our final formulae are valid for large y, and in fact for $y = \infty$; on the other hand, we have to distinguish a special case when μy is small compared with a. We obtain, in point of fact, one set of formulae valid for all μy, small, "finite," or large, and another (better) set valid for "finite" or large values, and some, but not all, small values. The error in every case is expressed in the form $O(f(a, \mu y))$, the function f depending only on a and μy (or ψ). The statement $E = O(f)$ will mean that $|E|/f$ is less than an *absolute* constant for all values of μy and a under consideration; i.e. for all $a \leqslant 1$ and all μy in one case, and for all $a \leqslant 1$ and all $\mu y \geqslant a$ in the other. Finally, we use the symbol A always for a positive absolute constant, not always the same at different occurrences.

2.3. Let $$\sec\psi = e^{\mu y},$$

$$(2.31)\quad p = \tan\psi = (e^{2\mu y} - 1)^{\frac{1}{2}}.$$

---

* It must be remembered that in these formulae ϕ is a function of z and a.

We wish to approximate for x, t, u, v in terms of ψ, or p, and a. From (2.13) and (2.31) we have

$$p = z\left\{1 - \frac{1+z^2}{z}\left(1 - e^{-2a(I-aR)}\right)\right\}^{\frac{1}{2}}. \tag{2.32}$$

Now as θ increases from 0 to $\frac{1}{2}\pi$, ϕ increases from 0 to ∞. It vanishes like $\tan\theta$ at $\theta = 0$ and is infinite like $\log\tan\theta$ at $\theta = \frac{1}{2}\pi$. Thus

$$\begin{aligned} |\phi| < A\log(\tan\theta + 2) &< A\left\{\log(z+2) + \log\frac{1}{a}\right\} \\ &< A\{\log(z+2) + l\}. \end{aligned} \tag{2.33}$$

We require upper bounds for R, $I - aR$, etc. Now (for $z > 0$)

$$|R| < \int_0^z \frac{\phi^2 z\,dz}{(1+z^2)^2} < \int_0^z \frac{A\{\log(z+2)+l\}^2 z\,dz}{(1+z^2)^2} < \frac{Al^2z^2}{1+z^2}; \tag{2.34}$$

this may be verified by considering the cases $z \leqslant 1$, $z > 1$ separately. Similarly

$$|R_1| < Al^2. \tag{2.35}$$

Again

$$\begin{aligned} |I - aR| = I - aR &= \int_0^z \frac{\phi z\,dz}{(1+z^2)(1+z^2+a\phi)} \\ &< A\int_0^z \frac{z\{\log(z+2)+l\}}{(1+z^2)^2}dz < Al\frac{z^2}{1+z^2}. \end{aligned} \tag{2.36}$$

Combining (2.34) and (2.36) we have also

$$|I| < Al\frac{z^2}{1+z^2}. \tag{2.37}$$

Next, from (2.32) and (2.36)

$$\begin{aligned} p &= z\left\{1 - \frac{1+z^2}{z^2}\left[2a(I-aR) + O\left(al\frac{z^2}{1+z^2}\right)^2\right]\right\}^{\frac{1}{2}} \\ &= z\left\{1 - \frac{1+z^2}{z^2}a(I-aR) + O(a^2l^2)\right\} \\ &= z\left\{1 - a\frac{1+z^2}{z^2}I + O(a^2l^2)\right\}. \end{aligned} \tag{2.38}$$

From (2.37) and (2.38) it follows, first that $p = O(z)$, next that

$$z - p = O(zal) = O(pal), \tag{2.381}$$

and so lastly that

(2.39)

$$\begin{aligned} \operatorname{arc\,tan} z - \psi = \operatorname{arc\,tan} z - \operatorname{arc\,tan} p &= \frac{z-p}{1+\{z+\vartheta_1(p-z)\}^2} \\ &= \frac{a\dfrac{1+z^2}{z}I + O(za^2l^2)}{(1+z^2)\{1+O(al)\}} = \frac{aI(z)}{z} + O(a^2l^2). \end{aligned}$$

2.4. From (2.14), (2.35), and (2.39) it follows that

$$\frac{kx}{U} = \psi + af(z) + O(a^2l^2), \tag{2.41}$$

where

$$f(z) = \frac{I(z)}{z} - I_1(z) = \frac{1}{z}\int_0^z \frac{\phi z\,dz}{(1+z^2)^2} - \int_0^z \frac{\phi\,dz}{(1+z^2)^2}. \tag{2.42}$$

Then

$$f'(z) = -\frac{I(z)}{z^2} = O\left(\frac{l}{1+z^2}\right) \tag{2.43}$$

by (2.37). From (2.43), (2.381), and the mean value theorem

$$f(z) - f(p) = O(al^2),$$

and so, from (2.41),

$$\frac{kx}{U} = \psi + af(p) + O(a^2l^2). \tag{2.44}$$

This is an approximation in terms of a and ψ, but the variables are not "separated" in $f(p)$, and it remains to effect this by further approximation. We have from (2.115), after a little rearrangement*,

$$\begin{aligned}\phi = \phi(z) = \phi(z, a) = {}& \left\{\log\left(\frac{2z}{a^{\frac12}}\right) + \frac{1}{2}\right\} \\ & + \left\{\log\left[\frac{1 + (1 + az^{-2})^{\frac12}}{2}\right] + \frac{z^2}{a}\left[\left(1 + \frac{a}{z^2}\right)^{\frac12} - 1 - \frac{1}{2}\frac{a}{z^2}\right]\right\} \\ = {}& \phi_1 + \phi_2,\end{aligned} \tag{2.45}$$

say, and so

$$\begin{aligned}f(p) = {}& \left\{\frac{1}{p}\int_0^p \frac{\phi_1 z\,dz}{(1+z^2)^2} - \int_0^p \frac{\phi_1\,dz}{(1+z^2)^2}\right\} \\ & + \left\{\frac{1}{p}\int_0^p \frac{\phi_2 z\,dz}{(1+z^2)^2} - \int_0^p \frac{\phi_2\,dz}{(1+z^2)^2}\right\} \\ = {}& f_1 + f_2.\end{aligned} \tag{2.46}$$

By straightforward calculation it is found that

$$\frac{1}{p}\int_0^p \frac{z\phi_1\,dz}{(1+z^2)^2} = \cot\psi\int_0^\psi \sin\psi\cos\psi\,(\tfrac12 l + 1 + \log\tan\psi)\,d\psi,$$

$$\int_0^p \frac{\phi_1\,dz}{(1+z^2)^2} = \int_0^\psi (\tfrac12 l + 1 + \log\tan\psi)\cos^2\psi\,d\psi,$$

$$f_1(p) = -\tfrac14 l\psi + F(\psi), \tag{2.47}$$

where

$$F(\psi) = \frac{1}{2}\int_0^\psi \log\cot\psi\,d\psi - \tfrac12\cot\psi\log\sec\psi. \tag{2.48}$$

---

[*] The number z, from now on, only occurs as the apparent variable in an integration from 0 to p.

2.5. It remains to discuss $f_2(p)$. There are two cases.

Case (i). $\mu y \leqslant a$. Then $p^2 \leqslant e^{2a} - 1 < 8a$. It follows easily that*

$$|\phi_2| < A \log\left(\frac{9a}{z^2}\right) + A < A \log\left(\frac{9a}{z^2}\right),$$

$$(2.51) \qquad |f_2| < \frac{A}{p}\int_0^p \log\left(\frac{9a}{z^2}\right) z\,dz + A\int_0^p \log\left(\frac{9a}{z^2}\right) dz$$

$$< \frac{A}{p}\int_0^p \log\left(\frac{p^2}{z^2}\right) z\,dz + \frac{A}{p}\int_0^p \log\left(\frac{9a}{p^2}\right) z\,dz + A\int_0^{3a^{\frac{1}{2}}} \log\left(\frac{9a}{z^2}\right) dz$$

$$= Ap + Ap\log\frac{9a}{p^2} + Aa^{\frac{1}{2}} < Aa^{\frac{1}{2}}\dagger.$$

From (2.51) we obtain the first result contained in (2.71) below.

2.6. *Case* (ii). $\mu y \geqslant a$. Here $p^2 \geqslant e^{2a} - 1 > 2a$. We calculate first

$$f_2(\infty) = -\int_0^\infty \frac{\phi_2\,dz}{(1+z^2)^2}.$$

If we write $z = a^{\frac{1}{2}} x$ and substitute for ϕ_2 from (2.45), this becomes

$$(2.61) \qquad -f_2(\infty) = a^{\frac{1}{2}}\int_0^\infty \frac{g(x)\,dx}{(1+ax^2)^2}$$

$$= a^{\frac{1}{2}}\int_0^\infty g(x)\,dx - a^{\frac{1}{2}}\int_0^\infty g(x)\,\frac{2ax^2 + a^2x^4}{(1+ax^2)^2}\,dx = a^{\frac{1}{2}}(P - Q),$$

where
$$g(x) = g_1(x) + g_2(x),$$
$$g_1(x) = \log\left[1 + \tfrac{1}{2}\{(1+x^{-2})^{\frac{1}{2}} - 1\}\right],$$
$$g_2(x) = x^2\left[(1+x^{-2})^{\frac{1}{2}} - 1 - \tfrac{1}{2}x^{-2}\right].$$

Now $g(x)$, a function of x alone, is $O(x^{-2})$ as $x \to \infty$, $O\left(\log\frac{2}{x}\right)$ as $x \to 0$. Hence

$$(2.62) \qquad Q = \int_0^\infty = \int_0^1 + \int_1^{a^{-\frac{1}{2}}} + \int_{a^{-\frac{1}{2}}}^\infty$$

$$= O\left(\int_0^1 \log\frac{2}{x}\frac{a}{1}\,dx\right) + O\left(\int_1^{a^{-\frac{1}{2}}} x^{-2}\frac{ax^2}{1}\,dx\right) + O\left(\int_{a^{-\frac{1}{2}}}^\infty x^{-2}\frac{a^2x^4}{(ax^2)^2}\,dx\right)$$

$$= O(a) + O(a^{\frac{1}{2}}) + O(a^{\frac{1}{2}}) = O(a^{\frac{1}{2}}).$$

---

* For the logarithm in ϕ_2 is greater than $\log\frac{1}{2}$ (algebraically), and less than

$$\log\left(\frac{2 + az^{-2}}{2}\right) < \log\left(\frac{16az^{-2} + az^{-2}}{2}\right);$$

while
$$\left|\left(1 + \frac{a}{z^2}\right)^{\frac{1}{2}} - 1 - \frac{1}{2}\frac{a}{z^2}\right| < A\frac{a}{z^2} \text{ for } \frac{a}{z^2} > \frac{1}{8}.$$

† The term $p\log\frac{9a}{p^2}$ is a maximum when $p = 3e^{-1}a^{\frac{1}{2}}$, when it is $O(a^{\frac{1}{2}})$.

In the definite integral P we write $1 + x^{-2} = y^{-2}$, and then, in the term involving a logarithm, $2y = u(1 + y)$. This reduces P to definite integrals of known types, and we find

$$P = \tfrac{2}{3}. \tag{2.63}$$

From (2.61), (2.62), (2.63)

$$f_2(\infty) = -\tfrac{2}{3}a^{\frac{1}{2}} + O(a). \tag{2.64}$$

Now in

$$f_2(p) - f_2(\infty) = \frac{1}{p}\int_0^p \frac{z\phi_2\,dz}{(1+z^2)^2} - \int_p^\infty \frac{\phi_2\,dz}{(1+z^2)^2}, \tag{2.65}$$

we have

$$\int_0^p \frac{z\phi_2\,dz}{(1+z^2)^2} = a\int_0^\varpi \frac{xg(x)\,dx}{(1+ax^2)^2} \quad (\varpi = pa^{-\frac{1}{2}} > \sqrt{2}) \tag{2.66}$$

$$= O(a)\int_0^1 |g(x)|\,dx + O(a)\int_1^\varpi \frac{x^{-1}\,dx}{1}$$

$$= O(a) + O(a\log\varpi) = O\left(a\log\frac{p^2}{a}\right),$$

$$\int_p^\infty \frac{\phi_2\,dz}{(1+z^2)^2} = a^{\frac{1}{2}}\int_\varpi^\infty \frac{g(x)\,dx}{(1+ax^2)^2} = O(a^{\frac{1}{2}})\int_\varpi^\infty x^{-2}\,dx \tag{2.67}$$

$$= O\left(\frac{a}{p}\right) = O\left(\frac{a}{p}\log\frac{p^2}{a}\right).$$

From (2.64) to (2.67)

$$af_2(p) = -\tfrac{2}{3}a^{\frac{3}{2}} + O\left(\frac{a^2}{p}\log\frac{p^2}{a}\right) = -\tfrac{2}{3}a^{\frac{3}{2}} + O(a^2l^2) + O\left(\frac{a^2}{p}l\right)^*, \tag{2.68}$$

and (2.44), (2.46), (2.47), (2.68) give (2.71) below.

2.7. This completes the discussion of x, with errors

$$O(a^{\frac{3}{2}}), \quad O\left(\frac{a^2}{p}l\right) + O(a^2l^2)$$

in the two cases. The calculations for t, u, v are similar, and I give only the final results, which are as follows:

$$e^{\mu v} = \sec\psi, \quad p = \tan\psi.$$

$$\frac{kx}{U} = \psi(1 - \tfrac{1}{4}al) + aF(\psi) - \tfrac{2}{3}a^{\frac{3}{2}} + R\dagger, \tag{2.71}$$

$$F(\psi) = \tfrac{1}{2}\int_0^\psi \log\cot\psi\,d\psi - \tfrac{1}{2}\cot\psi\log\sec\psi,$$

$$R = \begin{cases} O(a^{\frac{3}{2}}) & (\text{all } y) \\ O(a^2l^2) + O(a^2l\cot\psi) & (\mu y \geqslant a). \end{cases}$$

---

[*] For if $\sqrt{2}a^{\frac{1}{2}} < p \leqslant \frac{1}{2}a^{-\frac{1}{2}}$, then $\frac{1}{p}\log\frac{p^2}{a} < \frac{Al}{p}$; and if $p > \frac{1}{2}a^{-\frac{1}{2}}$, then $\frac{1}{p}\log\frac{p^2}{a} < A$.

[†] The approximation $kx = U\psi$ is given by L. F. Richardson, *loc. cit.*

(2.72) $$kt = \log(\sec\psi + \tan\psi) + a\Phi(\psi) - \tfrac{1}{3}a^{\frac{3}{2}} + R',$$

$$\Phi(\psi) = \tfrac{1}{2}\log(\sec\psi + \tan\psi) - \tfrac{1}{2}\operatorname{cosec}\psi \log\sec\psi,$$

$$R' = \begin{cases} O(a^{\frac{3}{2}}) & (\text{all } y) \\ O(a^2 l) + O(a^2 l\cot\psi) & (\mu y \geqslant a). \end{cases}$$

(2.73) $$\frac{u}{U\cos\psi} = \begin{cases} 1 - \tfrac{1}{4}al + O(a) \quad (\text{all } y) \\ 1 - \tfrac{1}{4}al - \tfrac{1}{2}a(1 - \log\operatorname{cosec}\psi) + O(a^2 l) + O(a^2\cot^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mu y \geqslant a). \end{cases}$$

(2.74) $$v = \left(\frac{g}{\mu}\right)^{\frac{1}{2}} \sin\psi\{1 - \tfrac{1}{2}a\cot^2\psi\log\sec\psi + R''\},$$

$$R'' = \begin{cases} O(a) & (\text{all } y) \\ O(a^2 l\operatorname{cosec}^2\psi) & (\mu y \geqslant a). \end{cases}$$

2.8. The approximations can be carried further. I give the results for x, t, when $\mu y > a$, sketching the proof for x, and omitting the estimation of the order of the error-term.

From (2.13) we have, approximately,

$$\mu y = \tfrac{1}{2}\log(1 + z^2) - aI + a^2 J,$$

$$J = J(z) = \int_0^z \frac{z\phi^2 dz}{(1 + z^2)^3},$$

whence, solving approximately for z in terms of y, or ψ,

$$\frac{p}{z} = 1 - a\frac{1 + p^2}{p^2}I + a^2\frac{1 + p^2}{p^2}\left\{J + I^2 - pI\frac{d}{dp}\left(\frac{1 + p^2}{p^2}I\right) - \frac{1}{2}\frac{1 + p^2}{p^2}I^2\right\},$$

in which $I = I(p)$, $J = J(p)$. Substituting for p/z from this in

$$\frac{kx}{U} = \frac{k}{U}\int\frac{dy}{z} = \int_0^p \left(\frac{pdp}{1 + p^2}\cdot\frac{1}{z}\right),$$

we obtain

(2.81) $$\frac{kx}{U} = \psi - a\int_0^p I\frac{dp}{p^2} + a^2 K,$$

(2.82) $$K = \int_0^p \frac{dp}{p^2}\left\{J + I^2 - pI\frac{d}{dp}\left(\frac{1 + p^2}{p^2}I\right) - \frac{1}{2}\frac{1 + p^2}{p^2}I^2\right\}.$$

Now $$I(z) = \int_0^z \frac{z\phi\, dz}{(1 + z^2)^2} = -\frac{\phi}{2(1 + z^2)} + \int_0^z \frac{dz}{1 + z^2}\,\frac{\sqrt{(1 + \tan^2\theta)} - \tan\theta}{a^{\frac{1}{2}}}.$$

Hence, replacing $\tan^2\theta$ by $\dfrac{z^2}{a}$, and taking $z = \infty$,

$$I(\infty) = \frac{1}{a}\int_0^\infty \frac{zdz}{1 + z^2}\left\{\left(1 + \frac{a}{z^2}\right)^{\frac{1}{2}} - 1\right\}.$$

In this we write $1 + az^{-2} = y^2$, and obtain

(2.83) $$I(\infty) = \int_1^\infty \frac{y(y - 1)\,dy}{\{y^2 - (1 - a)\}(y^2 - 1)} = \tfrac{1}{4}l + \tfrac{1}{2} + \tfrac{1}{16}al + \tfrac{1}{32}a + (\text{higher terms})^*.$$

* The integral can be found in finite terms by known methods.

Next, we have

$$\int_0^\infty \frac{I(z)\,dz}{z^2} = \left[-\frac{I(z)}{z}\right]_0^\infty + \int_0^\infty \frac{\phi\,dz}{(1+z^2)^2} = \int_0^\infty \frac{\phi\,dz}{(1+z^2)^2} = I_1(\infty).$$

This can be evaluated by similar arguments; we obtain, neglecting terms in a^2,

$$\int_0^\infty \frac{\phi(z)\,dz}{(1+z^2)^2} = C = \tfrac{1}{8}\pi l + \tfrac{2}{3}a^{\frac{1}{2}} - \tfrac{3}{32}\pi a. \tag{2.84}$$

Now neglecting terms in a^2

$$\phi(z) = \log(ez) + \tfrac{1}{2}l + \tfrac{1}{8}az^{-2}, \tag{2.85}$$

and, substituting (2.85) in

$$\int_0^z \frac{I(z)\,dz}{z^2} = C - \int_z^\infty \frac{I(z)\,dz}{z^2} = C + \left[\frac{I(z)}{z}\right]_z^\infty - \int_z^\infty \frac{\phi\,dz}{(1+z^2)^2},$$

we obtain

$$-\int_0^z \frac{I(z)\,dz}{z^2} = \frac{I(z)}{z} - C + \mathscr{I}, \tag{2.86}$$

where $\mathscr{I}$ is a sum of various integrals in z alone with coefficients involving a, but independent of z. To complete the discussion of the left-hand side of (2.86) we must approximate for $I(z)$ itself, or, since $I(\infty)$ is known, for

$$\int_z^\infty \frac{z\phi\,dz}{(1+z^2)^2}.$$

To do this we have only to substitute for ϕ from (2.85), and the approximate calculation of the coefficient of a in (2.81) is completed.

It remains to approximate for K, and in this we may neglect all *small* terms in a. Thus where $I(z)$ occurs in K we may take

$$I(z) = I(\infty) - \int_z^\infty \frac{z\phi\,dz}{(1+z^2)^2},$$

and then $\phi = \log(ez) + \tfrac{1}{2}l$, whereupon $I(z)$ is expressed in finite terms. It follows that K is a sum of *simple* (as opposed to double) integrals except for a term

$$\int_0^z \frac{J(z)\,dz}{z^2}.$$

But this is

$$-\frac{J(z)}{z} + \int_0^z \frac{\phi^2\,dz}{(1+z^2)^3},$$

and we can (in point of fact) ignore the terms $\tfrac{1}{8}az^{-2} + \dots$ in

$$\phi = \log(ez) + \tfrac{1}{2}l + \tfrac{1}{8}az^{-2} + \dots,$$

so that the integral involving ϕ^2 is a quadratic function of l with simple integrals for coefficients; and finally we can approximate for J itself on lines that are now familiar.

2.9. The extended formulae for x, t are:

$$\frac{kx}{U} = (1 - \tfrac{1}{4}al)\psi + aF(\psi) - \tfrac{2}{3}a^{\frac{3}{2}} + (\tfrac{3}{32}\psi l^2 + X_1 l + X_2)\,a^2, \tag{2.91}$$

$$kt = \log(\sec\psi + \tan\psi) \tag{2.92}$$
$$+ \tfrac{1}{2}a\{\log(\tan\psi + \sec\psi) - \operatorname{cosec}\psi \log\sec\psi\} - \tfrac{1}{3}a^{\frac{3}{2}} + (T_1 l + T_2)\,a^2,$$

$$X_1 = -\tfrac{1}{8}\psi + \tfrac{1}{16}\cot\psi + \tfrac{3}{8}\cot\psi\log\sec\psi + \tfrac{3}{8}\int_0^\psi \log\tan\psi\,d\psi. \tag{2.93}$$

$$(2.94)\quad X_2 = -\tfrac{3}{16}\psi + \tfrac{7}{32}\cot\psi - \tfrac{1}{8}\cot\psi\log\operatorname{cosec}\psi$$
$$+\tfrac{1}{2}\cot\psi\log\sec\psi + \tfrac{3}{4}\cot\psi\log\tan\psi\log\sec\psi$$
$$-\tfrac{1}{8}\cot\psi\operatorname{cosec}^2\psi(\log\sec\psi)^2 - \tfrac{1}{4}\int_0^\psi \log\tan\psi\,d\psi$$
$$-\tfrac{1}{2}\cot\psi\int_0^\psi \cot\psi\sec^2\psi\log\sec\psi\,d\psi + \tfrac{3}{8}\int_0^\psi(\log\tan\psi)^2\,d\psi,$$

$$(2.95)\quad T_1 = \tfrac{1}{16}\operatorname{cosec}\psi(4\log\sec\psi + 1) - \tfrac{1}{4}\log(\sec\psi + \tan\psi),$$

$$(2.96)\quad T_2 = \tfrac{1}{2}\int_0^\psi \sec\psi\log\cot\psi\,d\psi - \tfrac{1}{4}\log(\sec\psi+\tan\psi)$$
$$+\operatorname{cosec}\psi\left\{\tfrac{5}{32} + \tfrac{1}{4}\log\sec\psi - \tfrac{1}{8}\log\operatorname{cosec}\psi\right.$$
$$\left.-\tfrac{1}{8}\cot^2\psi(\log\sec\psi)^2 - \tfrac{1}{2}\int_0^\psi \tan\psi\log\cot\psi\,d\psi\right\}.$$

The accuracy of the formulae for x and t has been tested in a case $a = \cdot 03$ (about), $U = 120$ f.s., $\mu = 7 \,.\, 10^{-5}$ (about). For $y = 15{,}000$ (feet) x is about 3,000. The error in x in the formula

$$\frac{kx}{U} = \psi$$

is about 70, that of the next approximation* about 3, and the error of the formula (2·91) is a matter of decimals of a foot, for which the comparison calculation is insufficiently refined. The intermediate approximation gives an accuracy in x for values of y between 1,000 and 15,000 much the same as that for $y = 15{,}000$, and the accuracy of the corresponding formula for t is again about the same as this. Probably the most elaborate formulae give reasonably good results for much larger values of a.

The following tables are available:

ψ	X_1	X_2	T_2	ψ	X_1	X_2	T_2
0°	$+\infty$	$-\infty$	$-\infty$	50°	−·2624	1·030	·37
5	+·5505			55	−·2657	1·037	·37
10	+·1575	+·548	−·25	60	−·2648	1·046	·36
15	+·0015	·817	+·10	65	−·2601	1·057	·35
20	−·0883	·922	·23	70	−·2522	1·072	·34
25	−·1479	·972	·30	75	−·2414	1·089	·34
30	−·1896	·996	·34	80	−·2281	1·110	·33
35	−·2193	1·010	·36	85	−·2129	1·134	·32
40	−·2402	1·018	·37	90	−·1963	(1·159)†	·32
45	−·2542	1·024	·37				

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* That is, ignoring the term $-\tfrac{2}{3}a^{\frac{3}{2}}$ (which corresponds to about −4 feet in x).

† The X_2 table was computed by a step-by-step process. It can be checked at $\psi = \tfrac{1}{4}\pi$, where it is correct, and at $\psi = \tfrac{1}{2}\pi$, where $X_2(\psi) = \tfrac{3}{64}(\pi^3 - 2\pi) = 1{\cdot}1589$. The computation process gave 1·162, and the results of the table may be wrong by +·003 between 45° and 90°.

ψ	$F(\psi)$	$\Phi(\psi)$	ψ	$F(\psi)$	$\Phi(\psi)$	ψ	$F(\psi)$	$\Phi(\psi)$
0°	0	0	31°	·29926	·13516	62°	·21156	·26630
1	·03969	·00436	32	·29990	·13950	63	·20571	·27026
2	·06728	·00873	33	·30026	·14384	64	·19970	·27419
3	·09031	·01309	34	·30033	·14818	65	·19354	·27807
4	·11037	·01745	35	·30014	·15251	66	·18724	·28192
5	·12822	·02182	36	·29968	·15684	67	·18078	·28571
6	·14431	·02618	37	·29897	·16116	68	·17418	·28948
7	·15894	·03054	38	·29801	·16549	69	·16745	·29319
8	·17231	·03491	39	·29680	·16982	70	·16057	·29684
9	·18460	·03927	40	·29535	·17414	71	·15356	·30044
10	·19589	·04363	41	·29367	·17845	72	·14642	·30397
11	·20631	·04800	42	·29176	·18276	73	·13915	·30744
12	·21594	·05236	43	·28962	·18705	74	·13176	·31083
13	·22483	·05672	44	·28726	·19134	75	·12424	·31415
14	·23305	·06109	45	·28469	·19562	76	·11660	·31736
15	·24063	·06545	46	·28191	·19990	77	·10885	·32050
16	·24767	·06981	47	·27892	·20416	78	·10099	·32353
17	·25406	·07418	48	·27573	·20841	79	·09302	·32646
18	·25998	·07854	49	·27233	·21266	80	·08495	·32926
19	·26541	·08290	50	·26874	·21688	81	·07679	·33193
20	·27037	·08727	51	·26496	·22110	82	·06853	·33445
21	·27489	·09163	52	·26100	·22531	83	·06019	·33680
22	·27898	·09599	53	·25684	·22949	84	·05176	·33898
23	·28266	·10036	54	·25250	·23367	85	·04327	·34096
24	·28595	·10472	55	·24798	·23782	86	·03471	·34271
25	·28888	·10908	56	·24329	·24197	87	·02609	·34417
26	·29144	·11343	57	·23842	·24608	88	·01742	·34541
27	·29365	·11778	58	·23338	·25017	89	·00872	·34623
28	·29552	·12213	59	·22817	·25425	90	0	·34657
29	·29708	·12648	60	·22279	·25829			
30	·29832	·13082	61	·21726	·26230			

3. *Non-homogeneous medium.*

3.1. We suppose in this section that the resistance varies as the density, and that the density $\rho\ (y)$ at depth y is $e^{\lambda y}$, where λ is a constant.

Suppose that p, q are two elements of the trajectory: to fix ideas suppose p and q are any two of x, y, t, u, v, θ. We ask the question: in what homogeneous atmosphere, of density $\bar{\rho}$, will q have the same value, for given p, that it has in the given atmosphere? Let $\bar{\rho} = e^{\beta_{p,q}\lambda y}$, so that $\bar{\rho}$ is the density at depth $\beta_{p,q}y$: $\beta_{p,q}$ is a function of p, U, μ, λ, and our object is to find approximations for it in as many cases as possible. What happens, in point of fact, is that $\beta_{p,q}$ tends to an absolute constant when U, μy, λy tend to zero, but the constant is not the same for all pairs p, q. We can obtain second and higher approximations for β, and there are results available when only U and μy, or only U and λy, are small.

3.2. We begin with the simplest problem, that of finding v given y, supposing $U = 0$ and λy, μy small. We have

$$\frac{d}{dy}(v^2) = 2g - 2\mu e^{\lambda y} v^2,$$

whence

$$(3.21) \qquad Pv^2 = 2gP_1,$$

$$(3.22) \quad P = P(y, \lambda, \mu) = \exp\left(2\mu \int_0^y e^{\lambda y}\, dy\right), \quad P_1 = \int_0^y P dy.$$

Let us write

$$\bar{P} = \bar{P}(y, \bar{\mu}) = P(y, 0, \bar{\mu}) = e^{2\bar{\mu} y}, \quad \bar{P}_1 = \int_0^y \bar{P} dy = (e^{2\bar{\mu} y} - 1)/2\bar{\mu}.$$

It is evident from (3.21) that the number $\beta = \beta_{y,v}$ is given by

$$(3.23) \qquad P\bar{P}_1 - \bar{P}P_1 = 0, \quad \bar{\mu} = e^{\beta\lambda y},$$

and so far our results are *exact*. It is a simple matter to approximate for β as a function of λy and μy when these are small: we have only to expand everything in powers of λy, μy, and employ the method of successive approximation. We find

$$(3.24) \quad \beta_{y,v} = \tfrac{2}{3} + \tfrac{1}{36}(2\mu + \lambda)\, y + (\tfrac{1}{135}\mu^2 + \tfrac{7}{270}\mu\lambda - \tfrac{1}{810}\lambda^2)\, y^2 + O\{(\lambda^3 + \mu^3)\, y^3\}.$$

It is, of course, physically evident that $0 \leqslant \beta \leqslant 1$, and the explicit terms in (3.24) suggest that the approximation is likely to be good when λy and μy are less than, say, unity.

It will be instructive in this connection to consider two further cases, the values of $\beta_1 = \lim\limits_{\lambda \to 0} \beta$ when μy is not small, and $\beta_2 = \lim\limits_{\mu \to 0} \beta$ when λy is not small. In the first case we may take

$$P = e^{2\mu y + \lambda \mu y^2} = e^{2\mu y}(1 + \lambda \mu y^2),$$

$$\bar{\mu} y = \mu y e^{\beta \lambda y} = \mu y (1 + \beta \lambda y).$$

Then (3.23) leads easily to

$$(3.25) \quad \beta_1 = \lim_{\lambda \to 0} \beta = \{1 - \tfrac{1}{2}\eta e^{-\eta} - \eta^{-1}(1 - e^{-\eta})\}/(1 - e^{-\eta} - \eta e^{-\eta}),$$

where $\eta = 2\mu y.$

It is not difficult to verify that β_1 lies between $\frac{2}{3}$ and 1*, and that $\beta_1 = 1$ when $y = \infty$. The radius of convergence of the expansion (3.24) in powers of μy (when $\lambda = 0$) is $\frac{1}{2}k$, where k is the least modulus of a non-zero root of the denominator in (3.25). The value of k is 7.62....

Let us denote by β' the sum of the first two terms in (3.24). Then numerical calculation (from (3.25)) shows that when $\lambda = 0$ and $\mu y = 1$, β' differs from the true value β by about $-\cdot 0056$.

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* Probably $\frac{2}{3} \leqslant \beta \leqslant 1$ in the general case.

In the second case, $\mu \to 0$, we may take

$$P = 1 + 2\mu\lambda^{-1}(e^{\lambda y} - 1), \quad \bar{P} = 1 + 2\mu y e^{\beta\lambda y},$$

and obtain

$$(3.26) \quad \beta_2 = \lim_{\mu \to 0} \beta = \frac{1}{\theta} \log \left\{ \frac{2(\theta - 1)e^{\theta} + 2}{\theta^2} \right\}, \quad \theta = \lambda y.$$

Here also $\frac{2}{3} \leqslant \beta_2 \leqslant 1$, and $\beta_2 = 1$ for $y = \infty$. The radius of convergence of the expansion of β in powers of λy is k. The error $\beta' - \beta_2$ committed by taking the first two terms in (3.24) is $+ \cdot 001$.

These results suggest that the approximation $\beta = \beta'$ will in general be satisfactory.

3.3. We take up now the problem of correcting for the variation of ρ in finding x, t, u, v, given y, when the number a is small. We shall ignore, for the present, the terms in a in the approximate formula of 2·7, and correct only the first approximation. This amounts to correcting for v and t on the assumption that U is actually zero, and for x/U, u/U in the limit when U or a tends to 0. The problem for v is therefore solved already. It will be enough, for the rest, to discuss x/U in detail, and merely to state the remaining results.

3.4. We suppose first μy, λy small, and approximate for $\beta_{v,x}$. We have

$$\frac{d}{dy} \log \frac{u}{U} = - \mu e^{\lambda y} \left(1 + \frac{u^2}{v^2}\right)^{\frac{1}{2}}.$$

When $U \to 0$ the right-hand side becomes $- \mu e^{\lambda y}$, and we obtain

$$(3.41) \qquad \frac{u}{U} = P^{-\frac{1}{2}},$$

where P is given by (3.22). Now if in

$$\frac{x}{U} = \int_0^v \frac{u}{U} \frac{dy}{v}$$

we make $U = 0$ in the right-hand side, we obtain from (3.21) and (3.41)

$$\frac{x}{U} = (2g)^{-\frac{1}{2}} \int_0^v \{P_1(y)\}^{-\frac{1}{2}} dy = Q(y) = Q(y, \lambda, \mu).$$

The value of $\beta = \beta_{v,x}$ is given by

$$Q(y, 0, \bar{\mu}) = Q(y, \lambda, \mu), \quad \bar{\mu} = e^{\beta\lambda y},$$

and the process of approximation gives

$$(3.42) \quad \beta_{v,x} = \tfrac{1}{5} + \tfrac{11}{700}\lambda y + \tfrac{1}{20}\mu y + O\{(\lambda^2 + \mu^2) y^2\}.$$

Similarly we obtain

$$(3.43) \quad \beta_{v,t} = \tfrac{2}{5} + \tfrac{19}{700}\lambda y + \tfrac{11}{350}\mu y + O\{(\lambda^2 + \mu^2) y^2\},$$

$$(3.44) \quad \beta_{v,u} = \tfrac{1}{2} + \tfrac{1}{24}\lambda y + O\{(\lambda^2 + \mu^2) y^2\},$$

and we recall the value of $\beta_{y,v}$, viz.

$$(3.45) \qquad \beta_{y,v} = \tfrac{2}{3} + \tfrac{1}{36}(2\mu + \lambda)\, y + O\{(\lambda^2 + \mu^2)\, y^2\}.$$

3.5. There is another soluble case, viz. that of x given y when λ is small compared with μ, but μy is not necessarily small. Here, in fact, y may be ∞; we are then finding a "λ-correction" for the position of the asymptote of the trajectory, always ignoring λ-corrections of "a-terms."

We have

$$\frac{\sqrt{2g}\,x}{U} = \int_0^y P_1{}^{-\frac{1}{2}}\, dy.$$

For $P = \exp\left(2\mu \int_0^y e^{\lambda y}\, dy\right)$ we may write

$$P = \exp(2\mu y + \lambda\mu y^2) = e^{2\mu y}(1 + \lambda\mu y^2),$$

whence, retaining always only the first power of λ,

$$P_1{}^{-\frac{1}{2}} =$$

$$\sqrt{(2\mu)}\,(e^{2\mu y} - 1)^{-\frac{1}{2}}\,[1 - \tfrac{1}{4}\lambda\mu^{-1}(e^{2\mu y} - 1)^{-1}\{(2\mu^2y^2 - 2\mu y + 1)\, e^{2\mu y} - 1\}]$$

$$= F(y, \lambda, \mu),$$

say. $\bar{\mu}$, the corrected μ, is given by

$$\int_0^y F(y, \lambda, \mu)\, dy = \int_0^y F(y, 0, \bar{\mu})\, dy,$$

$\bar{\mu}$ being treated as constant in the integration, or

$$\frac{\mu - \mu}{\lambda}\int_0^y \frac{\partial F(y, 0, \mu)}{\partial \mu}\, dy = \int_0^y \left(\frac{\partial F(y, \lambda, \mu)}{\partial \lambda}\right)_{\lambda=0} dy.$$

This leads finally to

$$\frac{\bar{\mu} - \mu}{\lambda} = (\psi - 2\cot\psi \log\sec\psi)\Big/\left\{\tfrac{1}{2}\psi + \int_0^\psi [(\log\sec\psi)^2 - \log\sec\psi]\operatorname{cosec}^2\psi\, d\psi\right\}.$$

For $y = \infty$ in particular

$$\bar{\mu} = \mu + (2\log 2 - \tfrac{1}{2})\,\lambda.$$

3.6. I add a short statement about the λ-correction of a-terms. If we suppose, to fix ideas, that we know *exactly* the numbers β of 3.4, which correct the terms independent of a, the question arises: what value $\bar{a} = \bar{\mu}_1 U^2/g$ is the best value of a to take in the a-terms? The answer is that, to a first approximation as $\lambda y \to 0$ and $a \to 0$, we must take $\bar{\mu}_1 = \mu$, i.e. we must "take a at the vertex." Let us suppose, for simplicity, that μy is small. Then the error committed in x is approximately Δx, where

$$\frac{\Delta x}{x} = -\tfrac{1}{24}\, a\lambda y.$$

I omit the rather delicate discussion of this formula, as hardly worth while in the circumstances. Assuming its truth, it is seen that to make the correction *by replacing a by*

$$\bar{a} = a + \delta a = (\mu + \delta\mu)\, U^2/g,$$

we must have

$$\tfrac{1}{24}\, a\lambda y = -\frac{\Delta x}{x} = \delta a \frac{d}{da}\left\{\tfrac{1}{4}\, a \log\frac{4}{ea} - aF(\psi)\right\}$$

$$= \delta a \left\{\tfrac{1}{4}\log\frac{4}{ea} - 1 - F(\psi)\right\} \sim \tfrac{1}{4}\delta a \log\frac{1}{a} = \tfrac{1}{4} a\, \frac{\delta\mu}{\mu}\log\frac{1}{a}.$$

If then $\delta\mu/\mu = \beta''\lambda y,$

we have $6\beta'' \sim 1/\log\frac{1}{a} \to 0,$

as $a \to 0$, in accordance with what was said above.

Since the vertex is a singularity of the formulae of section 2 the result $\lim \beta'' = 0$ is not very surprising. A paradox of the same kind which is really striking appears in (3.72) below.

3.7. It is perhaps worth while to record some other results on λ-corrections. These fall into two classes, in each of which we suppose the retardation due to air-resistance to be the density ρ multiplied by $\mu f(V)$, an arbitrary function of the velocity.

Suppose first that we have a trajectory of small time of flight on a plane inclined to the horizontal at an angle $S \neq 0$. Let E be the elevation (i.e. the angle that the direction of projection makes with the *plane*), t the time of flight, and R the range on the plane; $h = R\sin S$ is the height of the point of arrival on the plane. When E and t are small it is a straightforward exercise in successive differentiation and approximation to expand any one of E, R, t in powers of any other[*]. (The initial velocity V is supposed given.)

We denote by $\beta_{p,q}$ the proportion of the total height h above O, the origin of the trajectory, at which the equivalent homogeneous atmosphere must be taken to make "q given p" have the same value that it has in the actual atmosphere. We find

(3.71) $\lim \beta_{R,\phi} = \frac{3}{8}, \quad \lim \beta_{R,t} = \frac{1}{3},$

(3.72) $\lim \beta_{\phi,t} = 0.$

The last result is surprising: we must suppose the density to be that at the point of projection, and ignore its variation with height[†].

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[*] For example, to obtain E in powers of t we may work out, in powers of E and t', the height y of the particle above the plane at *any* time t'. Expressing that this is zero we have an equation connecting E and t, soluble by successive approximation.

Doing this both for constants (μ, λ) and $(\bar{\mu}, 0)$ we obtain by comparison a formula for $\bar{\mu}$ or β. The method is similar in the other problems considered. The more elaborate cases are exceedingly laborious, though the results are simple.

[†] To avoid possible misunderstanding it should perhaps be said that the value of β is not *indifferent* to a first approximation; any constant value other than 0 gives an error of lower order (larger) than that resulting from $\beta = 0$.

3.8. The second class of problems concerns trajectories on the horizontal plane. Here y_0/R, where y_0 is the maximum height of the trajectory, is of the second order of small quantities, and the theory is more complicated in detail. We take an equivalent homogeneous atmosphere at height βy_0, and the results are as follows: ϕ is the elevation, R the range, t the time of flight, v the velocity at the point of arrival, ω the angle of arrival (reckoned positive), x_0, y_0 the coordinates of the vertex. Then the limiting value of $\beta_{p,\,q}$ is given by the entry corresponding to (p, q) in the following table:

	ϕ	R	t	ω	v	x_0	y_0
ϕ	—	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
R	$\frac{3}{5}$	—	$\frac{2}{3}$	$\frac{7}{10}$	$\frac{2}{3}$	—	—
t	$\frac{2}{5}$	$\frac{2}{3}$	—	$\frac{18}{25}$	$\frac{2}{3}$	—	—

The average height of the trajectory, in any ordinary sense, is in the limit $\frac{2}{3}y_0$. The usual assumption of ballisticians, that $\lim \beta = \frac{2}{3}$ in all cases, loses its plausibility on a sufficiently close analysis of the problem. But the table shows that wherever a velocity is a member of the pair p, q the average height gives the value of βy. This is a general rule in all problems; for example, in the problem of an *inclined* plane $\lim \beta_{p,\,v}$ is always $\frac{1}{2}$.

3.9. I note finally the approximate value of $\beta_{\phi,\,R}$ (for a horizontal trajectory) carried to terms of the second order. The law of resistance is supposed to be $\rho\mu V^n$, and $\rho\,(y)$, the density at height y, to be $e^{-\lambda y}$*. We have then

$$\beta_{\phi,\,R} = \tfrac{3}{5} - \tfrac{1}{20}(n-2)\,\tau - \tfrac{17}{350}\,\lambda y_0 - \tfrac{17}{350}(n-1)\tan^2\phi - N\tau^2 + O(\phi^3),$$

where $$\tau = (\tan\omega - \tan\phi)/\tan\phi,$$

$$N = \tfrac{1}{16800}(144n^2 - 1023n + 1578) = (\cdot00857)(n - 2\cdot264)(n - 4\cdot340),$$

and it is supposed that the pure numbers λR, ϕ are comparable in magnitude† (and constitute first order terms). The error in R for given ϕ due to the approximate β is, on this understanding, of order $R\phi^7$, a *proportionate* error of the seventh order.

It may be observed in conclusion that approximate formulae in ballistics are often accurate over a much larger range than would

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[*] In §§ 3.7, 3.8 we have only considered limiting values of the β's, where only the first power of λ is relevant, and the law of density might equally have been taken as $1 - \lambda y$.

As regards the law of resistance $f(v) = \mu v^n$ the only change in the general case is an additional term in the formula for N below, n being interpreted to mean $Vf'(V)/f(V)$.

[†] This is to suppose that $\lambda V^2/g$ is comparable with unity. In the actual atmosphere $\lambda V^2/g$ is about unity if $V = 1000$ f.s.

be expected: in some circumstances 30° is a "small" angle*. But if we are to use formulae in such a case a good deal depends on a suitable selection between forms of the same theoretical "order." For example, in the problem last considered τ/ϕ is a measure of the initial retardation r, but no formula involving r explicitly would be a good selection for practical purposes. And in the last resort some judicious empirical adjustment is necessary to deal with a law of resistance other than μv^n.

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* To take an analogous example: certain formulae give the range of trajectories of $\phi = 15°$ with a proportionate error less than 10^{-4}, although the velocity may fall to about $\frac{2}{3}$ of its original value.

Two notes on the Riemann Zeta-function[*]. By Mr J. E. LITTLEWOOD, Trinity College, Cayley Lecturer.

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1. *On the explicit formula for* $\psi(x)$.

1.1. Let $\Lambda(n)$ be the arithmetic function usually so denoted, which is zero unless n is a prime power p^m $(m \geqslant 1)$, when it is $\log p$. We write as usual

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n),$$

and

$$\bar{\psi}(x) = \sum_{n \leqslant x}{}' \Lambda(n),$$

where the dash denotes that if x is an integer the last term $\Lambda(x)$ of the sum is to be taken with a factor $\frac{1}{2}$. We write further

$$(1.11) \quad \bar{\psi}(x) = x - \tfrac{1}{2}\log(1 - x^{-2}) - \log 2\pi - \chi(x) \quad (x > 1).$$

Then the explicit formula for $\psi(x)$ becomes

$$(1.12) \qquad \chi(x) = \Sigma \frac{x^\rho}{\rho},$$

the summation being taken over the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$.

In what follows A's denote positive absolute constants, not always the same at different occurrences. The O's which occur are also absolute, i.e. the constants they imply are A's.

It has been shown (among other things) by Landau[†] that

$$(1.13) \quad \chi(x) = \sum_{|\gamma| < y} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{y}\right) + O\left(\frac{x \log y}{y}\right) + O(\log x).$$
$$(x \geqslant 3,\ y \geqslant 3).$$

It follows at once that

$$(1.14) \quad \chi(x) = \sum_{|\gamma| < y} \frac{x^\rho}{\rho} + O(\sqrt{x} \log^2 x) \quad (x \geqslant 3,\ y \geqslant \sqrt{x}\ddagger),$$

and

$$(1.15) \quad \chi(x) = \sum_{|\gamma| < y} \frac{x^\rho}{\rho} + O(\sqrt{x} \log x) \quad (x \geqslant 3,\ y \geqslant \sqrt{x} \log x).$$

We recall that the order of $\chi(x)$ itself is, under the Riemann hypothesis, known to lie between $\sqrt{x} \log\log\log x$ and $\sqrt{x} \log^2 x$, and that the latter bound is presumably too high. A certain interest therefore attaches to approximations of type (1.15) as against (1.14). I prove here, *employing the Riemann hypothesis*, that

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[*] The results, among others, were stated without proofs in the *Records of Proceedings of the London Mathematical Society*, Feb. 10, 1921.

[†] E. Landau, "Über einige Summen, die von den Nullstellen der Riemannschen Zetafunktion abhängen," *Acta Math.*, 35 (1911), pp. 271–294.

[‡] Our convention as to the use of O implies, of course, that the formula holds "uniformly for $y \geqslant \sqrt{x}$."

$$(1.16)\quad \chi(x) = \sum_{|\gamma|<y} \frac{x^\rho}{\rho} + O(\sqrt{x}\log x) + O\left(\frac{x\log x}{y}\right) \quad (x \geqslant 4,\ y \geqslant 2),$$

which gives in particular

$$(1.17)\quad \chi(x) = \sum_{|\gamma|<y} \frac{x^\rho}{\rho} + O(\sqrt{x}\log x) \quad (x \geqslant 4,\ y \geqslant \sqrt{x}).$$

The improvement on (1.15) is, of course, rather slight. The reader familiar with the subject will, however, remember that the problem of reducing the first O term in (1.13) (by a direct argument) is very intractable, and not apparently assisted by the Riemann hypothesis. A comparison of (1.13) and (1.17), however, provides a proof that when (for example) $y = \sqrt{x}$, the term must actually be of the form $O(x\log x/y)$. This argument is exceedingly indirect, since our proof of (1.17) is entirely different from that of (1.13).

1.2 Let
$$X(x) = \sum \frac{x^{\rho+1}}{\rho(\rho+1)} \quad (x \geqslant 4).$$

Since the series for $\chi(x)$ is boundedly convergent in any finite range $2 \leqslant x \leqslant x_0$ we may integrate term by term, and obtain

$$(1.21)\qquad X(x) = \int_2^x \chi(t)\,dt + A \quad (x \geqslant 4).$$

Lemma 1. *If* $|z| \leqslant \frac{1}{2}$, $|mz| \leqslant 2$, *then*

$$|(1+z)^m - 1 - mz| \leqslant A\,|m|\,(|m|+1)\,|z|^2.$$

We write $|z| = r$, $|m| = \mu$, and may suppose $\mu > 0$, $r > 0$. We have

$$T = \frac{|(1+z)^m - 1 - mz|}{|m|\,(|m|+1)\,|z|^2} \leqslant \sum_{n=2}^{\infty} \frac{|m(m-1)\dots(m-n+1)|}{\mu(\mu+1)\,n!} r^{n-2}$$
$$\leqslant \sum_{n=2}^{\infty} \frac{\mu(\mu+1)\dots(\mu+n-1)}{\mu(\mu+1)\,n!} r^{n-2} = \frac{(1-r)^{-\mu} - 1 - \mu r}{\mu(\mu+1)\,r^2}.$$

The last form but one shows that for fixed r T increases with μ, and is therefore a maximum when $\mu = 2/r$. Thus

$$T \leqslant \frac{(1-r)^{-2/r} - 1 - 2}{2(2+r)} < \tfrac{1}{4}(1-r)^{-2/r} < A,$$

since $r \leqslant \frac{1}{2}$.

1.3. *Lemma* 2. *Let* $x \geqslant 4$, $y = x/\delta$, $0 < \delta \leqslant \frac{1}{2}x$. *Then* (*on the Riemann hypothesis*)

$$\frac{X(x+\delta) - X(x)}{\delta} - \sum_{|\gamma|<y} \frac{x^\rho}{\rho} = O(x^{\frac{1}{2}}\log y).$$

The left side is

$$\sum_{|\gamma|<y} \frac{(x+\delta)^{\rho+1} - x^{\rho+1} - (\rho+1)\,\delta x^{\rho}}{\delta\rho\,(\rho+1)} + \sum_{|\gamma|\geqq y} \frac{(x+\delta)^{\rho+1} - x^{\rho+1}}{\delta\rho\,(\rho+1)} = \sigma_1 + \sigma_2,$$

say. Now in σ_1

$$\left|\frac{(x+\delta)^{\rho+1} - x^{\rho} - (\rho+1)\,\delta x^{\rho}}{\delta\rho\,(\rho+1)}\right|$$

$$< \frac{A}{(|\rho|+1)(|\rho|+2)} \cdot \frac{x^{\frac{3}{2}}}{\delta} \left|\left(1+\frac{1}{y}\right)^{\rho+1} - 1 - (\rho+1)\frac{1}{y}\right|.$$

Since $1/y \leqslant \frac{1}{2}$ and $|(\rho+1)/y| \leqslant |\gamma/y| + \frac{3}{2}/y \leqslant 1 + \frac{3}{4} < 2$, Lemma 1 shows that the last expression does not exceed

$$\frac{A}{(|\rho|+1)(|\rho|+2)}\frac{x^{\frac{3}{2}}}{\delta} \cdot (|\rho|+1)(|\rho|+2)\left(\frac{1}{y}\right)^2 = A\delta x^{-\frac{1}{2}}.$$

Hence

$$(1.31) \quad \sigma_1 < A\delta x^{-\frac{1}{2}} N(y) < A\delta x^{-\frac{1}{2}}\, y \log y = Ax^{\frac{1}{2}} \log y.$$

Next we have

$$|\sigma_2| \leqslant \frac{2\,(x+\delta)^{\frac{3}{2}}}{\delta} \sum_{|\gamma|\geqq y} \frac{1}{|\rho\,(\rho+1)|} < \frac{4\,(2x)^{\frac{3}{2}}}{\delta} \sum_{\gamma \geqq y} \frac{1}{\gamma^2}.$$

Since $\displaystyle\sum_{\gamma\geqq y} \frac{1}{\gamma^2} \leqslant \sum_{n\geqq y} \frac{1}{n^2} \sum_{n-1\leqslant\gamma\leqslant n} 1 < \sum_{n\geqq y} \frac{1}{n^2} A \log n < \frac{A \log y}{y}$,

we have $$|\sigma_2| < \frac{Ax^{\frac{3}{2}} \log y}{\delta y} = Ax^{\frac{1}{2}} \log y.$$

The lemma follows from this and (1.31).

1.4. *Lemma* 3. *We have* (*independently of the Riemann hypothesis*)

$$\chi(x) - A\,(\delta+1) \log x < \frac{X(x+\delta) - X(x)}{\delta} < \chi(x) + \delta + A$$

$$(x \geqslant 4,\ \delta > 0).$$

These results are trivial. Writing u for the middle term we have, from (1.21),

$$(1.41) \quad u = \frac{1}{\delta}\int_x^{x+\delta} \chi(t)\,dt = \frac{1}{\delta}\int_x^{x+\delta} \{t - \bar{\psi}(t)\}\,dt + O(1).$$

Now for $x \leqslant t \leqslant x+\delta$ we have, on the one hand,

$$(1.42) \quad t - \bar{\psi}(t) \geqslant x - \bar{\psi}(x+\delta) \geqslant x - \bar{\psi}(x) - \sum_{x}^{x+\delta} \Lambda(n)$$

$$> x - \bar{\psi}(x) - (\delta+1)\log(x+\delta) > \chi(x) + O(1) - A\,(\delta+1)\log x,$$

and on the other

$$(1.43) \quad t - \bar{\psi}(t) \leqslant x + \delta - \bar{\psi}(x) = \chi(x) + \delta + O(1).$$

From (1.41), (1.42), (1.43) we obtain

$$O(1) - A(\delta + 1)\log x < u - \chi(x) < \delta + O(1),$$

from which the desired result follows immediately.

1.5. We can now prove our main result (1.16). It is true for $y \geqslant \frac{1}{2}x$ in virtue of Landau's result (1.13). If now $2 \leqslant y < \frac{1}{2}x$, we have $\delta = x/y \leqslant \frac{1}{2}x$, and so, from Lemmas 2 and 3

$$\left|\chi(x) - \sum_{|\gamma|<y} \frac{x^\rho}{\rho}\right| \leqslant |u - \chi(x)| + \left|u - \sum_{|\gamma|<y} \frac{x^\rho}{\rho}\right|$$

$$= O\{(\delta + 1)\log x\} + O(x^{\frac{1}{2}}\log y) = O\left(\frac{x+y}{y}\log x\right) + O(x^{\frac{1}{2}}\log x)$$

$$= O\left(\frac{x\log x}{y}\right) + O(x^{\frac{1}{2}}\log x).$$

2. *On the zeros of* $\zeta(s)$.

2.1. It is well known that if $N(T)$ is the number of zeros of $\zeta(s)$ for which $0 < t \leqslant T$, then, as $T \to \infty$,

$$N(T) = P(T) + O(\log T),$$

where
$$P(T) = \frac{1}{2\pi}\{T\log T - (1 + \log 2\pi)T\}.$$

If the roots with positive imaginary parts are $\beta_1 + i\gamma_1, \beta_2 + i\gamma_2, \ldots$, in order of increasing γ_n, it follows at once that an absolute constant A exists, such that

$$\gamma_{n+1} - \gamma_n < A$$

for every n. More than this is not a trivial consequence of any known result. If the Riemann hypothesis is true it can be shown* that

$$N(T) = P(T) + O\left(\frac{\log T}{\log\log T}\right),$$

whence it follows that

$$\gamma_{n+1} - \gamma_n < \frac{A}{\log\log\gamma_n} \quad (\gamma_n > e^2).$$

More than this is quite possibly false; any upper bound for $\gamma_{n+1} - \gamma_n$ that we can hope to establish *without* the Riemann hypothesis is therefore likely to decrease to zero very slowly. I prove here (without any hypothesis) that

$$\gamma_{n+1} - \gamma_n < \frac{A}{\log\log\log\gamma_n} \quad (\gamma_n > e^{e^2}).$$

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* I hope to publish this result shortly in another paper.

I prove in fact:

Theorem. For every large positive T $\zeta(s)$ has a zero $\beta + i\gamma$ satisfying

$$|\gamma - T| < \frac{16}{\log\log\log T}$$

(*and of course* $\frac{1}{2} \leqslant \beta < 1$*).

I have adopted the plan of keeping to numerical constants throughout, purely for the sake of simplicity.

2.2. *Lemma* 4. *Let $z = \phi(\zeta)$ conformally represent the interior of a rectangle of sides a, va in the ζ-plane on the unit circle $|z| < 1$, the centre of the rectangle corresponding to $z = 0$. Let $\frac{1}{4} \leqslant \vartheta \leqslant \frac{1}{2}$, and let Γ be the (boundary of the) rectangle obtained from the original one by a dilatation ϑ about the centre: to Γ corresponds a z-contour C. Then if $v > 100$ the distance of C from $z = 0$ is at least $\frac{1}{10}$, and the distance of C from $|z| = 1$ is at least $100e^{-v}$.*

It is evident that if the result holds for any ϕ it holds for $e^{i\alpha}\phi$, the most general form possible; also that we may suppose a to have any value we choose, and the rectangle to have any position and orientation. We suppose then that the rectangle is bounded by the four lines

$$\mathfrak{R}\zeta = \pm K, \qquad \mathfrak{I}\zeta = \pm \tfrac{1}{2}iK',$$

where $v = K'/2K$, and the notation is that of elliptic function theory. We may then take

$$\phi(\zeta) = \frac{s - s_0}{s + s_0},$$

where $s = \operatorname{sn}\zeta' = \operatorname{sn}(\zeta + \frac{1}{2}iK')$, $s_0 = \operatorname{sn}\zeta_0' = \operatorname{sn}(\frac{1}{2}iK')$†.

We write $q = e^{-\pi K'/K}$ (as usual), and $p = q^{\frac{1}{8}}$. Since $v > 100$, we have

(2.21) $$p < 10^{-6}.$$

We prove now that, for ζ of Γ,

(2.22) $$\frac{s}{s_0} = e^{-\frac{1}{2}i\pi\zeta/K}(1 + t),$$

where

(2.23) $$|t| < \tfrac{1}{10}p.$$

Let $\zeta = \lambda K + \frac{1}{2}\mu iK'$, so that $|\lambda| \leqslant \vartheta$, $|\mu| \leqslant \vartheta$. Then

$$|e^{i\pi\zeta'/K}| = q^{\frac{1}{2}(1+\mu)} \leqslant q^{\frac{1}{2}(1-\vartheta)} \leqslant q^{\frac{1}{4}},$$

$$|e^{i\pi\zeta_0'/K}| = q^{\frac{1}{2}},$$

$$|\cos\zeta'\pi/K| < e^{|\mathfrak{I}\zeta'\pi/K|} = q^{-\frac{1}{2}(1+\mu)} \leqslant q^{-\frac{3}{4}},$$

$$|\cos\zeta_0'\pi/K| < q^{-\frac{1}{2}}.$$

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* Since all roots with $\gamma > 0$ lie in $0 < \sigma < 1$, and are symmetrically disposed about the line $\sigma = \frac{1}{2}$.

† s_0 is a pure imaginary, and s describes the real axis as ζ describes the large rectangle.

Hence

$$\frac{s}{s_0} = \frac{\operatorname{sn}\zeta'}{\operatorname{sn}\zeta_0'} = \frac{\sin(\pi\zeta'/2K)}{\sin(\pi\zeta_0'/2K)} \times$$

$$\prod_{n=1}^{\infty}\left(\frac{1-2q^{2n}\cos(\zeta'\pi/K)+q^{4n}}{1-2q^{2n-1}\cos(\zeta'\pi/K)+q^{4n-2}}\right) \Big/ \prod_{n=1}^{\infty}\left(\frac{1-2q^{2n}\cos(\zeta_0'\pi/K)+q^{4n}}{1-2q^{2n-1}\cos(\zeta_0'\pi/K)+q^{4n-2}}\right)$$

$$= e^{-\frac{1}{2}i\pi\zeta/K} \cdot \frac{1-e^{i\pi\zeta'/K}}{1-e^{i\pi\zeta_0'/K}} \cdot \Pi\left(\frac{1-2q^{2n}\cos(\zeta'\pi/K)+q^{4n}}{1-2q^{2n}\cos(\zeta_0'\pi/K)+q^{4n}}\right) \times$$

$$\Pi\left(\frac{1-2q^{2n-1}\cos(\zeta_0'\pi/K)+q^{4n-2}}{1-2q^{2n-1}\cos(\zeta'\pi/K)+q^{4n-2}}\right)$$

$$= e^{-\frac{1}{2}i\pi\zeta/K}\, T_1 T_2 T_3,$$

say. Now

$$|\log T_1| \leqslant -\log(1-|e^{i\pi\zeta'/K}|) - \log(1-|e^{i\pi\zeta_0'/K}|)$$

$$\leqslant 2\,.\,q^{\frac{1}{2}} + 2\,.\,q^{\frac{1}{2}} < 4p^2 < \tfrac{1}{100}p.$$ [*]

$$|\log T_3| \leqslant -\sum_{n=1}^{\infty}\log\{1-(2q^{2n-2}\,.\,q\,|\cos(\zeta'\pi/K)| + q^{2n-2}\,.\,q^{2n})\}$$

$$-\sum_{n=1}^{\infty}\log\{1-(2q^{2n-2}\,.\,q\,|\cos(\zeta_0'\pi/K)| + q^{2n-2}\,.\,q^{2n})\}$$

$$< \sum_{1}^{\infty} 2\,(2q^{2n-2}\,q^{\frac{1}{2}} + q^{2n-2}\,q^{2n}) + \sum_{1}^{\infty} 2\,(2q^{2n-2}\,q^{\frac{1}{2}} + q^{2n-2}\,q^{2n})$$

$$< \sum_{1}^{\infty} 12q^{2(n-1)}\,q^{\frac{1}{2}} = \frac{12q^{\frac{1}{2}}}{1-q^2} < 13p^2 < \tfrac{1}{50}p,$$

and a similar argument shows (indeed *a fortiori*) that

$$|\log T_2| < \tfrac{1}{50}p.$$

It follows easily that $|T_1T_2T_3 - 1| < \tfrac{1}{10}p$,
and this is equivalent to (2.23).

2.3. Let us write $\dfrac{s}{s_0} = \rho e^{i\psi}.$

Then, from (2.22),

(2.31) $\operatorname{Max}(\rho, \rho^{-1}) \leqslant e^{\frac{1}{2}\vartheta\pi K'/K} \operatorname{Max}(|1+t|, |1+t|^{-1}) < 2q^{-\frac{1}{4}} = 2p^{-1}.$

Further

$$\psi - \tfrac{1}{2}\lambda\pi| = |\arg(1+t)| \leqslant \arctan\frac{|t|}{1-|t|} \leqslant \arctan\frac{\frac{1}{10}p}{1-\frac{1}{10}p} < \tfrac{1}{9}p < \tfrac{1}{20},$$

and so

(2.32) $|\psi| \leqslant \tfrac{1}{2}\vartheta\pi + \tfrac{1}{20} \leqslant \tfrac{1}{4}\pi + \tfrac{1}{20} < \tfrac{1}{3}\pi$

on the one hand, and on the other

(2.33) $|\psi| \geqslant \tfrac{1}{2}|\lambda|\pi - \tfrac{1}{20}.$

[*] Here and in the argument below we use the inequality

$$|\log(1-z)| \leqslant -\log(1-|z|) \leqslant 2|z| \quad (|z| < \tfrac{1}{2}).$$

A glance at the inequalities immediately following verifies the condition $|z| < \frac{1}{2}$.

If now $r = |z|$, where z is any point of C, we have

$$r^2 = \frac{1 - 2\rho\cos\psi + \rho^2}{1 + 2\rho\cos\psi + \rho^2}.$$

Hence, from (2.32) and (2.31)

$$1 - r^2 = \frac{4\rho\cos\psi}{1 + 2\rho\cos\psi + \rho^2} > \frac{2\rho}{(1+\rho)^2} = \frac{2}{2 + \rho + \rho^{-1}} > \frac{2}{2 + 2p^{-1} + 2p^{-1}} > \tfrac{1}{3}p,$$

$$1 - r > \tfrac{1}{6}p > p^{\frac{7}{6}} = e^{-\frac{7}{24}\pi v} > 100e^{-v}.$$

This proves the second part of the lemma. Consider now the first part. If ζ lies on a parallel to the imaginary axis, $|\lambda| = \vartheta \geqslant \frac{1}{4}$, and

$$\tfrac{1}{3}\pi \geqslant |\psi| \geqslant \tfrac{1}{8}\pi - \tfrac{1}{20} > \tfrac{1}{10}\pi,$$

by (2.33). Hence

$$(2{\cdot}34)\quad r^2 > \frac{1 - 2\rho\cos\psi + \rho^2}{(1+\rho)^2} \geqslant \frac{(1 - \cos\psi)(1+\rho^2)}{2(1+\rho^2)} \geqslant \sin^2 \tfrac{1}{20}\pi \geqslant (\tfrac{1}{10})^2.$$

If, on the other hand, ζ lies on a parallel to the real axis, we have

$$|\mu| = \vartheta,$$

$$\rho + \rho^{-1} \geqslant \tfrac{1}{2}e^{\frac{1}{2}\vartheta\pi K'/K} \geqslant \tfrac{1}{2}q^{-\frac{1}{16}} > 4,$$

$$(2.35)\quad r^2 \geqslant \frac{(1-\rho)^2}{(1+\rho)^2} = \frac{(\rho + \rho^{-1}) - 2}{(\rho + \rho^{-1}) + 2} > \frac{4 - 2}{4 + 2} > \left(\frac{1}{10}\right)^2.$$

It follows from (2.34) and (2.35) that in any case $r > \frac{1}{10}$, and this completes the proof of the lemma.

2.4. *Lemma* 5. *Let Γ_1, Γ_2, Γ_4 be three concentric, similar, and similarly oriented rectangles, the ratio of the sides of any one of them being v, and the dimensions of Γ_2 and Γ_4 being respectively* 2 *and* 4 *times those of Γ_1. Suppose now that a function $f(\zeta)$ is regular, and satisfies the inequality $\mathfrak{R}f \leqslant P$, in the interior of Γ_4, and satisfies $|f| \leqslant M_1$ on the boundary of Γ_1. Then, if $v > 100$,*

$$|f| \leqslant (M_1 + P)\,e^v \left(\frac{M_1}{M_1 + P}\right)^{e^{-v}}$$

on the boundary of Γ_2.

We employ the conformal representation $z = \phi(\zeta)$ of Lemma 4. The function $f(\zeta)$ becomes $F(z)$, satisfying $\mathfrak{R}F \leqslant P$ in the domain $|z| < 1 = r_4$, whose boundary we denote by C_4', and satisfying $|F| \leqslant M_1$ on the contour C_1 corresponding to Γ_1. Let now C_1' denote $|z| = \frac{1}{10} = r_1$, C_2' denote $|z| = 1 - 100e^{-v} = r_2$, and C_3' denote $|z| = 1 - 50e^{-v} = r_3$. Lemma 4 shows that C_1' is interior to C_1, and C_2' exterior to C_2. Let

$$\mu_1 = \operatorname*{Max}_{C_1'} |F| \leqslant \operatorname*{Max}_{C_1} |F| \leqslant M_1,$$

$$\mu_2 = \operatorname*{Max}_{C_2'} |F| \geqslant \operatorname*{Max}_{C_2} |F|,$$

$$\mu_3 = \operatorname*{Max}_{C_3'} |F|.$$

Since $|F(0)| \leqslant M_1$, Carathéodory's theorem, applied to the circles C_3' and C_4', gives

$$(2.41) \quad \mu_3 \leqslant (2\,|F(0)| + 2P)\,\frac{r_4}{r_4 - r_3} \leqslant (M_1 + P)\,e^v.$$

By Hadamard's ("three circles") theorem, applied to C_1', C_2' and C_3', we have, writing

$$\xi = \log\frac{r_3}{r_2}\Big/\log\frac{r_3}{r_1} < 1,$$

$$\mu_2 \leqslant \mu_3\left(\frac{\mu_1}{\mu_3}\right)^{\xi} \leqslant (M_1 + P)\,e^v\left(\frac{\mu_1}{M_1 + P}\right)^{\xi} \leqslant (M_1 + P)\,e^v\left(\frac{M_1}{M_1 + P}\right)^{\xi}.$$

Since the index ξ is

$$\log\left(\frac{1 - 50e^{-v}}{1 - 100e^{-v}}\right)\Big/\log\left(\frac{1 - 50e^{-v}}{\frac{1}{10}}\right) \geqslant 50e^{-v}/\log 10 > e^{-v},$$

we have finally

$$\operatorname*{Max}_{\Gamma_2}|f| = \operatorname*{Max}_{C_2}|F| \leqslant \operatorname*{Max}_{C_2'}|F| = \mu_2 \leqslant (M_1 + P)\,e^v\left(\frac{M_1}{M_1 + P}\right)^{e^{-v}},$$

the desired result.

2.5. We are now in a position to prove the theorem. Suppose it false. Let Γ_1, Γ_2, Γ_4 be rectangles in the s-plane, with sides parallel to the real and imaginary axes, having a common centre at the point $5 + it$, Γ_1 having a corner at $2 + iT + ih$, where $h = 4/\log\log\log T$, and Γ_2, Γ_4 being obtained from Γ_1 by dilatations 2 and 4 about the centre. Then there must exist arbitrarily large T for which Γ_4 contains no zero of $\zeta(s)$. The function $f(s) = \log\zeta(s)$ is then regular in Γ_4: moreover, since

$$|\zeta(s)| < At^{\frac{1}{2}+19} \quad (\sigma \geqslant -19,\ t > 1),$$

we have, for large enough T,

$$|\zeta(s)| < T^{20}$$

on the boundary of Γ_4[*], and so

$$\mathfrak{R} f(s) < \log(T^{20}) = 20\log T = P.$$

Since $v = 3/h > 100$ for large T, and since

$$|f| = \left|\sum_{p,m}\frac{1}{m}p^{-m(\sigma+iT)}\right|^{\dagger} \leqslant \sum_2^{\infty} n^{-2} < 1 = M_1$$

on the boundary of Γ_1, it follows by Lemma 5 that on Γ_2

$$(2.51) \qquad |f| < (P + 1)\,e^v\left(\frac{1}{P + 1}\right)^{e^{-v}}.$$

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[*] Γ_4 extends to the left (only) as far as $\sigma = -19$.

[†] We naturally choose the determination of $\log\zeta$ defined by the Dirichlet series for $\sigma > 1$.

Now

$$\left(\frac{1}{P+1}\right)^{e^{-v}} < \exp\left(-\log\log T e^{-\frac{3}{4}\log\log\log T}\right) = \exp\{-(\log\log T)^{\frac{1}{4}}\}$$
$$= o\{\exp(-\tfrac{3}{4}\log\log\log T)\} = o(e^{-v}).$$

Hence (2·51) gives $|f| = o(P+1) = o(\log T)$ on Γ_2, and in particular at $s = -1 + iT$. Then

$$\zeta(-1+iT) = e^{f(-1+iT)} = e^{o(\log T)} = T^{o(1)}.$$

Since this is known to be false the theorem is established.

On the Applicability and Deformation of Surfaces. By B. M. SEN, King's College. (Communicated by Professor H. F. BAKER.)

[*Received* 29 April 1924.]

Preface.

Beltrami's discovery* that corresponding to any ruled surface there is another applicable on it having corresponding generators parallel but with the parameter of distribution of opposite signs, has brought into prominence the distinction between the applicability of two surfaces and the continuous deformation of one into the other. It is pointed out that since the parameter of distribution differs in sign, the tangent plane rotates in opposite directions as the point of contact proceeds along the two corresponding generators on the respective surfaces. One surface cannot, therefore, be deformed into the other. It is proposed in the present paper to investigate the distinction between applicability and deformability of two surfaces in general, and to connect this isolated fact with the general theory of deformation.

The notation used is that of Bianchi and Eisenhart.

1. We define two surfaces to be applicable when the first quadratic expression (the expression for the square of the arc-element) can be represented as the same function of two Gaussian parameters and their differentials. With the usual notation, they are applicable if the formula

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

gives the arc-element for both.

One surface is said to be continuously deformable into the other if we can transform it supposed flexible but inextensible into the other, without tearing or joining.

It may be pointed out that Beltrami's ruled surface is not the only case of applicability without deformation. A standard illustration of conditional deformation is that of a surface with one curve on it deformed into a given curve†. It is shown that, if C be the given curve traced on the surface which is to be deformed into a given curve C',

$$\sin \varpi'/\rho' = 1/\gamma,$$

where $1/\rho$ is the circular curvature of C' and ϖ' the angle between the normal to the deformed surface at any point and the principal normal to C' at that point, and $1/\gamma$ the geodesic curvature of the given surface at the corresponding point. Since ρ' and γ are given, ϖ' can have two supplementary values. It is, therefore, concluded that the given surface can be deformed in the prescribed manner in two different ways.

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* *Ann. di Mat.*, t. 7 (1865), pp. 139–150, or Forsyth, *Differential Geometry*, p. 387.

† See, for instance, Forsyth, *loc. cit.*, p. 376.

But since the principal normal of the curve C' is determined at every point and ϖ' has two supplementary values, the two normals to the respective surfaces rotate in opposite directions as the point moves away along the curve C'. The tangent planes, therefore, rotate in opposite directions. By Beltrami's criterion, these surfaces though possessing the same arc-element are not deformable.

Another simple example may be found in the mirror-image of any surface on any plane, one of the coordinate planes for example. It is obvious that along two corresponding curves on the two surfaces respectively, the tangent planes rotate in opposite directions. The two surfaces are not, therefore, mutually deformable. As a particular case, we take the ruled surface

$$x = x_0 + lu,\ \ y = y_0 + mu,\ \ z = z_0 + nu, \qquad \text{......(1)}$$

where x_0, y_0, z_0 and l, m, n are functions of v only, and $l^2 + m^2 + n^2 = 1$, and its mirror-image on the z-plane,

$$x = x_0 + lu,\ \ y = y_0 + mu,\ \ z = -(z_0 + nu). \qquad \text{......(2)}$$

The parameter of distribution for the original surface has the expression

$$\beta = -\frac{1}{a^2}\begin{vmatrix} x_0' & y_0' & z_0' \\ l & m & n \\ l' & m' & n' \end{vmatrix}, \qquad \text{......(3)}$$

where $a^2 = l'^2 + m'^2 + n'^2$.

It is clear that β has opposite signs for the two surfaces.

2. It is usually stated in the standard works on Differential Geometry that the Monge-Ampère equation of the second degree governs the deformation of one surface on another. But it is not usually made clear that though necessary, it is not a sufficient condition. Analytically, we enunciate the general problem of deformation as follows*:

Given three functions f, ϕ, ψ of u and v, to determine three functions x, y, z such that

$$\left.\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= E = f_1^2 + \phi_1^2 + \psi_1^2 \\ x_1x_2 + y_1y_2 + z_1z_2 &= F = f_1f_2 + \phi_1\phi_2 + \psi_1\psi_2 \\ x_2^2 + y_2^2 + z_2^2 &= G = f_2^2 + \phi_2^2 + \psi_2^2 \end{aligned}\right\} . \qquad \text{...(4)}$$

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* These commonplaces of Differential Geometry have been reproduced to emphasize their analytical bearing on the problem of deformation.

To solve this problem we introduce three new dependent variables, D, D', D'', given by the equations

$$\left.\begin{aligned} D &= Xx_{11} + Yy_{11} + Zz_{11} \\ D' &= Xx_{12} + Yy_{12} + Zz_{12} \\ D'' &= Xx_{22} + Yy_{22} + Zz_{22} \end{aligned}\right\}, \qquad \ldots\ldots(5)$$

where $$HX = \frac{\partial(y, z)}{\partial(u, v)}, \quad HY = \frac{\partial(z, x)}{\partial(u, v)}, \quad HZ = \frac{\partial(x, y)}{\partial(u, v)}$$

and $$H^2 = EG - F^2.$$

We then get the three equations of Gauss

$$\left.\begin{aligned} x_{11} &= \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} x_1 + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} x_2 + DX \\ x_{12} &= \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} x_1 + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} x_2 + D'X \\ x_{22} &= \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} x_1 + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} x_2 + D''X \end{aligned}\right\}. \qquad \ldots\ldots(6)$$

The conditions of integrability of these three equations give the Gauss expression for curvature

$$\begin{aligned} \frac{DD'' - D'^2}{H^2} = \frac{1}{2H} \Bigg\{ & \frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] \\ & + \frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial E}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right]\Bigg\}, \qquad \ldots\ldots(7) \end{aligned}$$

and the Mainardi-Codazzi relations

$$\left.\begin{aligned} \frac{\partial}{\partial v}\left(\frac{D}{H}\right) - \frac{\partial}{\partial u}\left(\frac{D'}{H}\right) + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}\frac{D}{H} - 2\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}\frac{D'}{H} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\frac{D''}{H} &= 0 \\ \frac{\partial}{\partial u}\left(\frac{D''}{H}\right) - \frac{\partial}{\partial v}\left(\frac{D'}{H}\right) + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\frac{D}{H} - 2\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}\frac{D'}{H} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}\frac{D''}{H} &= 0 \end{aligned}\right\}. \qquad (8)$$

The new dependent variables must, therefore, satisfy the three equations (7) and (8). They are, therefore, necessary.

To prove that they are sufficient, we recall a theorem of Bonnet*.

When the six fundamental magnitudes are given, and when they satisfy the Gauss characteristic equation and the two Mainardi-Codazzi relations, they determine a surface uniquely save as to its position and orientation in space.

The three equations taken together determine the deforms of the given surface.

It may be pointed out that the Monge-Ampère equation of deformation is only another form of equation (7).

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* Forsyth, *loc. cit.*, p. 50.

3. Now from the form of equations (7) and (8), it is obvious that corresponding to any set of solutions of these equations, there is another of opposite sign. If X, X', X'' be solutions of the fundamental equations, $-X$, $-X'$, $-X''$ are also solutions. But by Bonnet's theorem these two sets will give rise to two distinct surfaces, as will appear from the proof.

To examine the geometrical relations of these surfaces, we take the expression for the perpendicular from a point M' with Gaussian coordinates $u+\delta u$, $v+\delta v$ on the tangent plane at M (u, v):

$$p = D du^2 + 2D' du\, dv + D'' dv^2.$$

It is obvious that the perpendiculars on the respective tangent planes at two corresponding points from two consecutive, corresponding points on the two surfaces have opposite signs. The adjacent portions of the two surfaces are, therefore, so to say, mirror-images on the tangent plane. The plane itself rotates in opposite directions as the point of contact moves away along corresponding curves on the respective surfaces. They are, therefore, not mutually deformable.

We can easily verify that for Beltrami's associated ruled surface, D, D', D'' have opposite signs. Consider the ruled surface given by equations (1), where

$$l^2 + m^2 + n^2 = 1, \quad {x_0'}^2 + {y_0'}^2 + {z_0'}^2 = 1.$$

Putting

$$l'^2 + m'^2 + n'^2 = a^2,$$

$$l' x_0' + m' y_0' + n' z_0' = b,$$

$$l x_0' + m y_0' + n z_0' = c,$$

we have $$ds^2 = du^2 + 2c\, du\, dv + (a^2u^2 + 2bu + 1)\, dv^2,$$

and $$D = 0, \quad H^2 = a^2u^2 + 2bu + 1 - c^2.$$

The Christoffel symbols have the values,

$$\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} = 0, \quad \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} = -\frac{1}{H^2}(a^2u + b)\, c,$$

$$\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} = 0, \quad \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = \frac{a^2u + b}{H^2}.$$

The Gauss equation (7) assumes the form

$$-\frac{D'^2}{H^2} = -\frac{a^2 - a^2c^2 - b^2}{H^4}, \qquad \text{......(9)}$$

and the Mainardi-Codazzi equations (8)

$$\left.\begin{aligned} \frac{\partial}{\partial u}\left(\frac{D'}{H}\right) + 2\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}\frac{D'}{H} &= 0 \\ \frac{\partial}{\partial u}\left(\frac{D''}{H}\right) - \frac{\partial}{\partial v}\left(\frac{D'}{H}\right) - 2\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}\frac{D'}{H} &= 0 \end{aligned}\right\}. \qquad \text{......(10)}$$

From equation (9),

$$D'H = (a^2 - a^2c^2 - b^2)^{\frac{1}{2}} = J \qquad \text{......(11)}$$

say, where J is a function of v only; and has either sign of the radical. Solving the first Codazzi equation (10), we have

$$D'H = \phi(v), \qquad \text{......(12)}$$

where $\phi(v)$ is an arbitrary function of v; comparing with equation (11) we put $\phi(v) = J$.

The second Codazzi equation gives D''.

Now Beltrami's associated ruled surface owes its origin to the indeterminateness of the sign of the radical J*. It is clear from the form of the second Codazzi equation (10) and equation (11), that D', D'' change their signs with J.

For any surface and its mirror-image on one of the coordinate planes the verification of the change of signs of D, D', D'' is immediate.

4. It is clear that all surfaces applicable on a given surface are given by the three equations (7) and (8) taken together. And corresponding to any solution there is another giving a surface applicable but not mutually deformable. Beltrami's associated ruled surface is a particular case of a surface of the above-mentioned type. Its peculiarity lies in the easy solution of the three equations under specified conditions. But the solution is impracticable in the general case.

* Forsyth, *loc. cit.*, p. 388.

The Quadratic form for Radial Acceleration, in the Theory of Relativity. By Mr R. Hargreaves, St John's College.

[*Received* 13 May 1924.]

§ 1. In the transformation* corresponding to constant acceleration, and in the resulting quadratic form, there is an arbitrary constant. In Einstein's formula for central acceleration a mass m defining the acceleration appears, but no arbitrary constant. In seeking to account for, or to repair, the deficiency I retained all constants naturally arising in the integration of the differential equations, and it then appeared that the result could be obtained by transformation. It will be convenient to state the transformation at the outset, and then to examine the differential equations with a view to a clear understanding of the assumptions made in reaching the more general result.

If γ is $1 - 2m/r'$, the fundamental form is

$$\gamma^{-1}dr'^2 + r'^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \gamma\, dx_4'^2 \qquad \text{......(1)}$$

and its determinant is $- r'^4 \sin^2\theta$. Transformation by means of

$$r'^3 = ar^3 + b, \text{ and } x_4' = x_4/a, \qquad \text{......(2)}$$

gives

$$\frac{a^2r^4dr^2}{\gamma r'^4} + r'^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \frac{\gamma}{a^2}dx_4^2, \text{ with } \Delta_g = - r^4\sin^2\theta. \qquad \text{......(3)}$$

§ 2. In dealing with the form directly the substantial assumption made is that the determinant has the same value as that of the normal quadratic in polar coordinates. It is further assumed that $g_{33} = g_{22}\sin^2\theta$, that is, transverse coefficients are affected in the same way by any modification. The property of the determinant $g_{11}g_{22}g_{33}g_{44} = - r^4\sin^2\theta$ is then expressed by

$$g_{11}\, g_{22}^2 g_{44} = - r^4; \qquad \text{......(4)}$$

and it is understood that these g's are functions of r only.

Now we find that $B_{33} = B_{22}\sin^2\theta$, so that whether we are dealing with the adjustment of B's so as to vanish or to vary as the corresponding g's, one condition suffices for the pair. In like manner a condition

$$B_{11}/g_{11} - B_{44}/g_{44} = 0 \qquad \text{......(5)}$$

must be satisfied for either type, and will reduce the problem of further determination of g's to one for this pair. If we write dashes for differential coefficients with respect to r, we find

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* *Proceedings Camb. Phil. Soc.*, Vol. 22, p. 138, "The Electromagnetic Equations as Basis of Einstein's Quadratic form."

$$\left.\begin{aligned} B_{11} &= \frac{g_{22}''}{g_{22}} + \frac{g_{44}''}{2g_{44}} - \frac{g_{22}'^2}{2g_{22}^2} - \frac{g_{44}'^2}{4g_{44}^2} - \frac{g_{11}'g_{22}'}{2g_{11}g_{22}} - \frac{g_{11}'g_{44}'}{4g_{11}g_{44}}, \\ B_{22} &= \frac{g_{22}''}{2g_{11}} - \frac{g_{11}'g_{22}'}{4g_{11}^2} + \frac{g_{22}'g_{44}'}{4g_{11}g_{44}} - 1, \quad B_{33} = B_{22}\sin^2\theta, \\ B_{44} &= \frac{g_{44}'}{2g_{11}}\left(\frac{g_{44}''}{g_{44}'} + \frac{g_{22}'}{g_{22}} - \frac{g_{11}'}{2g_{11}} - \frac{g_{44}'}{2g_{44}}\right). \end{aligned}\right\} \quad \ldots(6)$$

Thus

$$\frac{B_{11}}{g_{11}} - \frac{B_{44}}{g_{44}} \text{ is } \frac{g_{22}'}{g_{11}g_{22}}\left(\frac{g_{22}''}{g_{22}'} - \frac{g_{22}'}{2g_{22}} - \frac{g_{11}'}{2g_{11}} - \frac{g_{44}'}{2g_{44}}\right)$$

$$= \frac{g_{22}'}{g_{11}g_{22}}\left(\frac{g_{22}''}{g_{22}'} + \frac{g_{22}'}{2g_{22}} - \frac{2}{r}\right) \quad \ldots\ldots(7)$$

when account is taken of

$$\frac{g_{22}'}{g_{22}} + \frac{g_{11}'}{2g_{11}} + \frac{g_{44}'}{2g_{44}} = \frac{2}{r}, \quad \ldots\ldots(8)$$

the logarithmic derivative of (4). The condition (5) then involves the vanishing of the bracket in (7), which leads to

$$g_{22} = (ar^3 + b)^{\frac{2}{3}} \quad \ldots\ldots(9)$$

as the value of g_{22} applicable to the two problems $B_{pp} = 0$ and $B_{pp} = \lambda g_{pp}$.

Again using (8), the equation $B_{44} = \lambda g_{44}$ is represented by

$$g_{44}'' + 2g_{44}'\left(\frac{g_{22}'}{g_{22}} - \frac{1}{r}\right) = 2\lambda g_{11}g_{44} = -\frac{2\lambda r^4}{g_{22}^2} = -\frac{2\lambda r^4}{(ar^3+b)^{\frac{4}{3}}}. \quad \ldots(10)$$

Without the right-hand member $g_{44}'g_{22}^2 = -Cr^2$, and with it the general solution is

$$g_{44} = A + \frac{C}{a(ar^3+b)^{\frac{1}{3}}} - \frac{\lambda}{3a^2}(ar^3+b)^{\frac{2}{3}}. \quad \ldots\ldots(11)$$

In the same way $B_{22} = \lambda g_{22}$ is represented by

$$-1 + \frac{g_{22}}{r^2}\frac{d}{dr}\left(\frac{r^2 g_{22}'}{2g_{11}g_{22}}\right) = \lambda g_{22},$$

or

$$\lambda r^2 + \frac{r^2}{g_{22}} = \frac{d}{dr}\frac{r^2 g_{22}'}{2g_{11}g_{22}} = -\frac{d}{dr}\frac{g_{22}g_{22}'g_{44}}{2r^2},$$

and

$$-\frac{g_{22}g_{22}'g_{44}}{2r^2} = \frac{\lambda r^3}{3} + \frac{1}{a}(ar^3+b)^{\frac{1}{3}} + D$$

$$= \frac{\lambda}{3a}(ar^3+b) + \frac{1}{a}(ar^3+b)^{\frac{1}{3}} - C,$$

which, since $g_{22}\,g_{22}'/2r^2 = a\,(ar^3+b)^{\frac{1}{3}}$, gives

$$g_{44} = -\frac{1}{a^2} + \frac{C}{a\,(ar^3+b)^{\frac{1}{3}}} - \frac{\lambda}{3a^2}\,(ar^3+b)^{\frac{2}{3}}. \quad \text{......(12)}$$

Thus the constant A left arbitrary in (11) is here given in terms of "a." I interpret the result as an external solution

$$g_{44} = -\frac{1}{a^2} + \frac{C}{a\,(ar^3+b)^{\frac{1}{3}}}, \quad \text{......(13 }a\text{)}$$

and an internal solution

$$g_{44} = -\frac{1}{a^2} - \frac{\lambda}{3a^2}\,(ar^3+b)^{\frac{2}{3}}, \quad \text{......(13 }b\text{)}$$

with (9) applying in each case, and g_{11} given by (4). The more general solution agrees exactly with the transformation of Einstein's simpler form, and is bound up with the recognition of possible modification in the part $r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)$.

§ 3. The change does not affect the deviation of light, which is given by the angle between two asymptotes, so that only infinite values of r are in question. Nor is the recurrence of perihelion (in a planetary orbit with $\theta = \pi/2$) affected in relation to ϕ, though values of r and t may be affected. These two results follow from the fact that the new form is obtainable by transformation.

The law of attraction will be defined by a potential varying as $-g_{44}$, that is, the attraction will vary as $r^2/(ar^3+b)^{\frac{4}{3}}$. This has a maximum value for $ar^3 = b$, and vanishes for $r = 0$; that is, it appears to act as a safeguard against an infinite value occurring for $r = 0$. It is well known that in the theory of attraction no such safeguard is needed so long as the attracting mass has finite density, the infinite value only attaching to a finite mass concentrated at a point. If b is negative the opposite effect is produced, viz. an infinite value is found for $ar^3 + b = 0$, but with no reversal of sign if r is continued below this value. Also if the form is interpreted in relation to the attraction of a finite sphere, b could only be a small fraction of its volume, if there is to be no appreciable change in the gradient near the surface.

With reference to the constant "a," when b is zero it may be applied to make $-g_{44} = 1 - 2m/r + m/a_0$, where $2a_0$ is the major axis of a planetary orbit. In view of $v^2 = 2m/r - m/a_0$ in the normal orbit, this would be equivalent to giving to the final term in the quadratic form a value $-(V^2 - v^2)\,dt^2$, which has a certain plausibility as a correction to $V^2 - 2m/r$. The identification of

$$\frac{1}{a^2} - \frac{C}{ra^{\frac{4}{3}}} \text{ with } 1 - \frac{2m}{r} + \frac{m}{a_0}$$

requires

$$\frac{1}{a^2} = 1 + \frac{m}{a_0} \text{ and } C = 2m\Big/\left(1 + \frac{m}{a_0}\right)^{\frac{2}{3}}.$$

For a hyperbolic orbit the type is $V^2 - \frac{2m}{r} - \frac{m}{a_0}$, but no difficulty is created in respect to light, where the normal orbit is a straight line, and so $1/a_0$ vanishes.

§ 4. For the cylindrical case the more general solution is found to correspond to the use of a transformation $\varpi'^2 = a\varpi^2 + b$. The values of B for a form

$$g_{11}d\varpi^2 + g_{22}d\theta^2 + dz^2 + g_{44}dx_4^2, \qquad \ldots\ldots(14)$$

where g's depend on ϖ only, are

$$\left.\begin{aligned} B_{11} &= -\frac{1}{\varpi^2} - \frac{g_{11}'}{2\varpi g_{11}} - \frac{g_{11}''}{2g_{11}} + \frac{3g_{11}'^2}{4g_{11}^2} + \frac{g_{22}'^2}{4g_{22}^2} + \frac{g_{44}'^2}{4g_{44}^2}, \\ B_{22} &= \frac{g_{22}'}{2g_{11}}\left(\frac{1}{\varpi} + \frac{g_{22}''}{g_{22}'} - \frac{g_{11}'}{g_{11}} - \frac{g_{22}'}{g_{22}}\right), \quad B_{33} = 0, \\ B_{44} &= \frac{g_{44}'}{2\varpi g_{11}} + \frac{g_{44}''}{2g_{11}} - \frac{g_{11}'g_{44}'}{2g_{11}^2} - \frac{g_{44}'^2}{2g_{11}g_{44}}. \end{aligned}\right\} \qquad (15)$$

Giving to the determinant of (14) its normal value we have

$$g_{11}g_{22}g_{44} = -\varpi^2, \qquad \ldots\ldots(16)$$

and it is advisable to use $g_{11}'/g_{11} + g_{22}'/g_{22} + g_{44}'/g_{44} = 2/\varpi$ to clear g_{11}' from the expressions of B_{11} and B_{44} as a preliminary step. This makes

$$\left.\begin{aligned} B_{11} &= \frac{1}{2g_{22}}\left(g_{22}'' - \frac{g_{22}'}{\varpi}\right) + \frac{g_{44}'}{2g_{44}}\left(\frac{g_{44}''}{g_{44}'} + \frac{g_{22}'}{g_{22}} - \frac{1}{\varpi}\right), \\ B_{44} &= \frac{g_{44}'}{2g_{11}}\left(\frac{g_{44}''}{g_{44}'} + \frac{g_{22}'}{g_{22}} - \frac{1}{\varpi}\right). \end{aligned}\right\} \quad \ldots(17)$$

Whether B's are to vanish or to vary as corresponding g's, we must have

$$\varpi g_{22}'' - g_{22}' = 0, \text{ and so } g_{22} = a\varpi^2 + b. \qquad \ldots\ldots(18)$$

The value $g_{44} = A + C\log(a\varpi^2 + b)$ makes B_{11} and B_{44} vanish, but B_{22} has then the constant value $-2a^2C$, a contrast with the spherical case where the condition $B_{22} = 0$ determined the constant A. The second problem is met by the special value $g_{44}' = -\lambda\varpi/a$, which makes B_{22} fall into line; but B_{33} remains zero.

§ 5. To complete the series we take the form

$$dx_1^2 + dx_2^2 + \gamma^{-1}dx_3^2 - \gamma dx_4^2, \qquad \ldots\ldots(19)$$

in which γ is a function of x_3 only. The Christoffel symbols are here

$$\{33, 3\} = -\frac{1}{2\gamma}\frac{d\gamma}{dx_3}, \quad \{34, 4\} = \{43, 4\} = \frac{1}{2\gamma}\frac{d\gamma}{dx_3}, \quad \{44, 3\} = \frac{\gamma}{2}\frac{d\gamma}{dx_3}; \qquad \ldots\ldots(20)$$

and thence

$$B_{33} = -\frac{d}{dx_3}\{33,\, 3\} + \{33,\, 3\}^2 + \{34,\, 4\}^2 = \frac{1}{2\gamma}\frac{d^2\gamma}{dx_3{}^2},$$

$$B_{44} = -\frac{d}{dx_3}\{44,\, 3\} + 2\,\{34,\, 4\}\,\{44,\, 3\} = -\frac{\gamma}{2}\frac{d^2\gamma}{dx_3{}^2},$$

B_{11} and B_{22} being zero. Thus $0 = B_{33} = B_{44}$ makes γ a linear function of x_3 in accordance with the transformation formula, p. 152, *op. cit.* B_{11} and B_{22} cannot be made proportional to g's of like index, but this is possible for $B_{33} = \lambda g_{33}$ and $B_{44} = \lambda g_{44}$, the term added to γ being $\lambda x_3{}^2$. Thus γ with its first differential coefficient can be made continuous for the two types of solution with a transition taking place at the origin chosen for transformation. The new form is not reducible to Maxwell's.

[21 August, 1924. *Addendum* to previous paper, p. 150.

The passage from wave-velocity to ray-velocity and direction may be effected in the general case as follows. Applying $\lambda\omega = \frac{\partial\Omega}{\partial l}$ to (35 *a*) we have

$$\frac{a_{11}'l + a_{12}'m + a_{13}'n}{\sqrt{a_{11}'l^2 + 2a_{12}'lm + \ldots}} \times \frac{\Delta}{\mu} = G_{44}\lambda\omega - G_{41}, \ldots. \qquad \ldots\ldots(a)$$

The use of multipliers g_{11}, g_{12}, g_{13} for the three cases yields

$$\frac{l}{\sqrt{}} \times \frac{\Delta}{\mu} = \omega\,(g_{11}\lambda + g_{12}\mu + g_{13}\nu) + g_{14}, \ldots. \qquad \ldots\ldots(\beta)$$

Form the sum of products of (a) and (β) for the three cases; then *l, m, n* are eliminated and in view of $\Delta^2/\mu^2 = -\Delta g$ the result is

$$\omega^2\,(g_{11}\lambda^2 + 2g_{12}\lambda\mu + \ldots) + 2\omega\,(g_{14}\lambda + g_{24}\mu + g_{34}\nu) + g_{44} = 0. \qquad \ldots\ldots(\gamma)$$

This corresponds to writing $dx = \dot{x}\,dt = \lambda\omega\,dt$ in the quadratic form

$$g_{11}\,dx_1{}^2 + 2g_{12}\,dx_1\,dx_2 + \ldots = 0.$$

Thus the ray-velocity given by the quadratic form is that which corresponds to the wave-velocity given by (35 *a*). The original text shows the correspondence for the special transformation used; but the present method is more general, and the statement just made enhances the significance of the result.

Corrigenda to previous paper. p. 146. In last line of § 9 *read* ϕ^4 *for* ϕ^r. p. 149. After (29) *cancel* $X_1'\sqrt{\Delta_g}$ *and insert* "we have." p. 161. In (69) the second member of each formula should be doubled.]

The Statistical Theory of Dissociation and Ionization by Collision, with applications to the capture and loss of electrons by α-particles.

By Mr R. H. Fowler, Fellow and Lecturer in Trinity College, Cambridge.

[*Read* 19 May 1924.]

§ 1. *Introduction.* In two recent papers in the *Philosophical Magazine*[*] the statistical theory of collisions of electrons with atoms has been considered, and applied to explain the observed features of the capture and loss of electrons by α-particles. The basis of that discussion can however be improved. Firstly, interactions with the core of any atom entered by the α-particle were explicitly ignored. Such interactions—or rather interactions with the general intra-atomic field as distinct from individual electrons—might be expected to be (and are) important, and it has been found possible to include them in this paper. Secondly, the frequency laws for the processes concerned were based on the classical Thomson-Bohr theory of ionization by collision. This theory—it is well known—does not completely reproduce experimental facts at α-particle velocities, and a less restrictive form can be given to the frequency laws which appears to be in accord with all the known facts. The application to the capture and loss of electrons by α-particles proves to be unaffected.

In order to discuss the frequency relations for interactions between α-particles and the general field of an atom we determine in § 2 the general frequency relation for 3-body encounters in a gas which lead to dissociation or recombination. This is an extension of previous work, in that the bodies concerned may all be of comparable masses. We base the deduction of this relation on the principle of detailed balancing of individual processes. This hypothesis, which is wider than that used in the papers quoted, seems likely to prove of general validity. It has already been employed in a similar manner in discussing radiative processes[†].

In § 3 we summarize the evidence from the range, ionization and δ-ray distributions of α- and β-particles in order to specify the most probable form of the frequency coefficient for ionization of a definite type—that is, the production of a δ-ray of given velocity from a given level in a given atom by an α- or β-particle of given velocity. Once this coefficient is determined, the coefficient for the reverse process follows from § 2.

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[*] Fowler, *Phil. Mag.*, 47, 257, 415 (1924).

[†] For example, Pauli, *Zeit. für Phys.*, 18, 272 (1923): Einstein and Ehrenfest, *Zeit. für Phys.*, 19, 301 (1923); and, to a less extent, Einstein, *Phys. Zeit.*, 18, 121 (1917).

In §§ 4–6 we apply these theoretical relations to the desired more complete discussion of the problem of the capture and loss of electrons by α-particles. The agreement between theory and experiment found in the previous paper is preserved—it is in fact more satisfactory in that we now find that an intra-atomic density of lightly bound electrons not much greater than 10^{24} will account for the observations. This is a more reasonable value than the 4×10^{24} of the previous paper.

In particular the exact relationship of Rutherford's V^5-law to the form of the frequency coefficient for ionization has now been elucidated.

§ 2. *General frequency-relations for 2- and 3-body encounters leading to dissociation and recombination.* The number of molecules of mass m_0 and velocity components between u_0, $u_0 + du_0 \ldots w_0$, $w_0 + dw_0$ in a volume do_0 is (Maxwell's Law)

$$\nu_0 \left(\frac{m_0}{2\pi kT}\right)^{\frac{3}{2}} e^{-m_0 c_0^2/2kT} do_0 du_0 dv_0 dw_0 \quad (c_0^2 = u_0^2 + v_0^2 + w_0^2), \quad (2{\cdot}11)$$

where T is the temperature and ν_0 the molecular density. The number of pairs of molecules of masses m_0, m_1, velocity components between u_0, $u_0 + du_0 \ldots w_1$, $w_1 + dw_1$, simultaneously in volume elements do_1 and do_2 exterior to each other's fields of force is likewise

$$\nu_0 \nu_1 \frac{(m_0 m_1)^{\frac{3}{2}}}{(2\pi kT)^3} e^{-(m_0 c_0^2 + m_1 c_1^2)/2kT} do_0 do_1 du_0 \ldots dw_1. \quad \ldots(2{\cdot}12)$$

The number of triple combinations of the same type with masses m_0, m_1, m_2 and resultant velocities c_0, c_1, c_2 is

$$\nu_0 \nu_1 \nu_2 \frac{(m_0 m_1 m_2)^{\frac{3}{2}}}{(2\pi kT)^{\frac{9}{2}}} e^{-(m_0 c_0^2 + m_1 c_1^2 + m_2 c_2^2)/2kT} do_0 do_1 do_2 du_0 \ldots dw_2. \quad (2{\cdot}13)$$

The nature of any encounter is a function of the relative motion of the systems. We must therefore separate out the motion of the centre of gravity of the whole system from the relative motion. The motion of the centre of gravity will be unaltered by any interaction in which the laws of conservation of momentum are preserved. For two bodies the analysis is well known†. If we write u^*, v^*, w^* for the velocity components of the centre of gravity of the system and ξ_1, η_1, ζ_1 for the velocity of the body 1 relative to the body 0, then

$$(m_0 + m_1)\, u^* = m_0 u_0 + m_1 u_1, \quad \xi_1 = u_1 - u_0, \text{ etc.}$$

$$C^2 = \Sigma u^{*2}, \quad V_1^2 = \Sigma \xi_1^2.$$

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† Jeans, *Dynamical theory of gases*, 3rd ed., p. 251, equation (699).

These lead to

$$m_0 c_0^2 + m_1 c_1^2 = (m_0 + m_1) C^2 + \frac{m_0 m_1}{m_0 + m_1} V_1^2, \quad \frac{\partial (u^*, \xi_1)}{\partial (u_0, u_1)} = 1,$$

and so finally to

$$\nu_0 \nu_1 \frac{(m_0 m_1)^{\frac{3}{2}}}{(2\pi kT)^3} e^{-\left\{(m_0+m_1)C^2 + \frac{m_0 m_1}{m_0+m_1} V_1^2\right\}/2kT} do_0 do_1 du^* \dots d\zeta_1, \quad \dots(2\cdot 14)$$

which replaces (2·12). It is easy to extend this analysis to three bodies. We write

$$\left.\begin{aligned} (\Sigma m)\, u^* &= m_0 u_0 + m_1 u_1 + m_2 u_2 \\ \xi_1 &= u_1 - u_0 \\ \xi_2 &= u_2 - u_0 \end{aligned}\right\}, \text{ etc.} \quad \dots\dots(2\cdot 211)$$

These equations can be solved for u_0, u_1, u_2, and we find

$$\frac{\partial (u^*. \xi_1, \xi_2)}{\partial (u_0, u_1, u_2)} = 1, \qquad \dots\dots(2\cdot 213)$$

$$m_0 u_0^2 + m_1 u_1^2 + m_2 u_2^2 = (\Sigma m)\, u^{*2}$$
$$+ \{m_1 (m_0 + m_2)\, \xi_1^2 - 2 m_1 m_2 \xi_1 \xi_2 + m_2 (m_0 + m_1)\, \xi_2^2\}/\Sigma m. \quad \dots(2\cdot 214)$$

If we use (2·213) and (2·214) and similar equations for the other components and write θ for the angle between the relative velocities V_1, (ξ_1, η_1, ζ_1), and V_2, (ξ_2, η_2, ζ_2), we can reduce (2·13) to

$$\nu_0 \nu_1 \nu_2 \frac{(m_0 m_1 m_2)^{\frac{3}{2}}}{(2\pi kT)^{\frac{9}{2}}} \exp \left[- \left\{ (\Sigma m)\, C^2 + \frac{1}{\Sigma m} [m_1 (m_0 + m_2)\, V_1^2 \right.\right.$$
$$\left.\left. - 2 m_1 m_2 V_1 V_2 \cos\theta + m_2 (m_0 + m_1)\, V_2^2] \right\} \Big/ 2kT \right] do_0 do_1 do_2 du^* \dots d\zeta_2.$$
$$\dots\dots(2\cdot 22)$$

Collisions may be classified according to the position of the lines of impact of bodies 1 and 2 on body 0 (asymptotes of orbits) and the time interval between the instants at which the undisturbed relative orbits of (1, 0) and (2, 0) would bring the pairs of bodies closest together. To obtain the number of collisions per unit volume per second of this triple type in which the lines of impact of 1 and 2 on 0 lie between p_1 and $p_1 + dp_1$, p_2 and $p_2 + dp_2$, with a time interval between τ and $\tau + d\tau$, we take

$$do_0 = 1, \quad do_1 = 2\pi p_1 dp_1 V_1, \quad do_2 = 2\pi p_2 dp_2 V_2 d\tau. \quad \dots(2\cdot 221)$$

We next change (2·22) into spherical polar coordinates in the velocities, using the direction of V_1 as the polar axis from which the angles defining V_2 are specified. Then

$$du^* \dots d\zeta_2 = C^2 dC \sin\psi_0 d\psi_0 d\chi_0 V_1^2 dV_1 \sin\psi_1 d\psi_1 d\chi_1 V_2^2 dV_2 \sin\theta d\theta d\phi.$$
$$\dots\dots(2\cdot 222)$$

We are only interested in the relative configurations of the orbits which are specified by p_1, p_2, τ, V_1, V_2 and θ. The other variables are to be eliminated by integration. The number of collisions of type p_1, p_2, τ, V_1, V_2, θ per unit volume per second is found from (2·22), (2·221) and (2·222) after integration to be

$$16\pi^2 \nu_0 \nu_1 \nu_2 \frac{(m_0 m_1 m_2/\Sigma m)^{\frac{3}{2}}}{(2\pi kT)^3}$$
$$\times \exp\left[-\frac{m_1(m_0+m_2)V_1^2 - 2m_1 m_2 V_1 V_2 \cos\theta + m_2(m_0+m_1)V_2^2}{2(\Sigma m)kT}\right]$$
$$\times\ 2\pi p_1 dp_1 2\pi p_2 dp_2 d\tau V_1^3 V_2^3 \sin\theta dV_1 dV_2 d\theta. \qquad ......(2·224)$$

The process of capture to be considered is a triple encounter of this type resulting in a radiationless union of the bodies 0 and 2. The relative motion of (0, 2) and 1 takes off all the superfluous energy. If $\Sigma_2^1(p_1, p_2, \tau; V_2, \theta, V_1)$ is the probability of this event, and

$$S_2^1(V_2, \theta, V_1) = 4\pi^2 \int_0^\infty p_1 \int_0^\infty p_2 \int_{-\infty}^{+\infty} \Sigma_2^1 dp_1 dp_2 d\tau, \quad ...(2·225)$$

then the number of captures per c.c. per second of the (V_2, θ, V_1) type is

$$16\pi^2 \nu_0 \nu_1 \nu_2 \frac{(m_0 m_1 m_2/\Sigma m)^{\frac{3}{2}}}{(2\pi kT)^3} S_2^1(V_2, \theta, V_1)$$
$$\times \exp\{-\epsilon_{\text{rel.}}/kT\} V_1^3 V_2^3 \sin\theta dV_1 dV_2 d\theta. \qquad ......(2·23)$$

In this formula $\epsilon_{\text{rel.}}$ denotes the energy of the relative motion which is given explicitly in the expression in [ ] in (2·224). The energy of the relative motion of the 2-body system (0, 2) and 1 after the event is $\epsilon_0 + \epsilon_{\text{rel.}}$, where ϵ_0 is the energy required to dissociate (0, 2).

To formulate the number of dissociative collisions we must adapt (2·14) as we have adapted (2·22). The mass of the reference body (0, 2) is $m_0 + m_2$. We have to put $do_{02} = 1$, $do_1 = 2\pi p dp V$, express the velocities in polar coordinates and integrate over directions and for all C. This gives a total number of 2-body collisions of type (p, V) per c.c. per second equal to

$$8\pi \nu_{02} \nu_1 \left\{\frac{(m_0+m_2)m_1}{(\Sigma m)2\pi kT}\right\}^{\frac{3}{2}} \exp\left[-\frac{m_1(m_0+m_2)V^2}{2(\Sigma m)kT}\right] V^3 dV 2\pi p dp.$$
$$......(2·31)$$

The suffix 02 refers to the body (0, 2). These collisions are effective if they result in the dissociation of (0, 2) into 0 and 2. There are then three bodies moving with certain relative velocities, and the type of collision depends on the distribution of the available energy. Let*

$$\frac{m_0 m_2}{m_0 + m_2}\Sigma_1^2(p;\ V_2, \theta, V)\ V_2 dV_2 \sin\theta d\theta \quad \text{...(2·321)}$$

be the probability that a body 1 with relative velocity V so breaks up (0, 2) that 2 is thrown off with velocity between V_2 and $V_2 + dV_2$ relative to 0, and in such a direction that the angle between V_2 and the velocity V_1 of body 1 after collision relative to 0 lies between θ and $\theta + d\theta$. If V_2 and θ are arbitrarily specified, then V_1 is determined by the equation of conservation of energy, which is

$$\frac{1}{2}\frac{m_1(m_0 + m_2)}{\Sigma m}V^2$$
$$= \epsilon_0 + \frac{1}{2}\frac{m_1(m_0 + m_2)V_1^2 - 2m_1 m_2 V_1 V_2\cos\theta + m_2(m_0 + m_1)V_2^2}{\Sigma m}. \quad \text{......(2·322)}$$

This may also be written

$$(\epsilon_{\text{rel.}})_1 = \epsilon_0 + (\epsilon_{\text{rel.}})_2, \quad \text{......(2·323)}$$

where $(\epsilon_{\text{rel.}})_1$ and $(\epsilon_{\text{rel.}})_2$ are the energies of the relative motion of the system before and after the process of dissociation. We write

$$S_1^2(V_2, \theta, V) = 2\pi\int_0^\infty p\Sigma_1^2 dp. \quad \text{......(2·324)}$$

The number of dissociations of type (V_2, θ, V) per c.c. per second is therefore

$$8\pi\nu_{02}\nu_1\left\{\frac{(m_0 + m_2)m_1}{(\Sigma m)2\pi kT}\right\}^{\frac{3}{2}}\frac{m_0 m_2}{m_0 + m_2}S_1^2(V_2, \theta, V)$$
$$\times \exp\{-(\epsilon_{\text{rel.}})_1/kT\}\ V^3 V_2 \sin\theta dV dV_2 d\theta. \quad \text{......(2·33)}$$

We are assuming detailed balancing in equilibrium. On this hypothesis we must assert here that if the V's are related by (2·322) then by (2·23) and (2·33)

$$8\pi\nu_{02}\nu_1\left\{\frac{(m_0 + m_2)m_1}{(\Sigma m)2\pi kT}\right\}^{\frac{3}{2}}\frac{m_0 m_2}{m_0 + m_2}S_1^2(V_2, \theta, V)$$
$$\times \exp\{-(\epsilon_{\text{rel.}})_1/kT\}\ V^3 V_2 \sin\theta dV dV_2 d\theta$$
$$= \frac{16\pi^2\nu_0\nu_1\nu_2}{(2\pi kT)^3}\left\{\frac{m_0 m_1 m_2}{\Sigma m}\right\}^{\frac{3}{2}}S_2^1(V_2, \theta, V_1)$$
$$\times \exp\{-(\epsilon_{\text{rel.}})_2/kT\}\ V_1^3 V_2^3 \sin\theta dV_1 dV_2 d\theta. \quad \text{......(2·41)}$$

These frequency coefficients refer only to the process $(0, 2) \rightleftarrows 0 + 2$, each body being in a unique (normal) state internally. To see that the process preserves the equilibrium laws we must consider only the equilibrium laws for just such a reaction. For bodies with more than one internal state—including in this internal vibrations and (strictly) rotations—the equilibria of the separate states can and in fact must be separately discussed with separate coefficients S. For the bodies here considered in single standard internal states the general laws of dissociative equilibrium take the form*

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* This particular specification of Σ_1^2 leads to the simplest forms in applications for $\frac{m_0 m_2}{m_0 + m_2}V_2 dV_2 = d\epsilon_{\text{rel.}}$, where $\epsilon_{\text{rel.}}$ is the energy of the relative motion of the bodies 0 and 2 after dissociation.

$$\frac{\nu_0\nu_2}{\nu_{02}} = e^{-\epsilon_0/kT}\frac{(2\pi kT)^{\frac{3}{2}}}{h^3}\left\{\frac{m_0m_2}{m_0+m_2}\right\}^{\frac{3}{2}}\frac{\sigma}{\varpi}, \quad \text{......(2·421)}$$

where σ is the symmetry number and ϖ the weight of the body (0, 2) relative to the body 0. Inserting this value of $\nu_0\nu_2/\nu_{02}$ in the balance equation (2·41) we find

$$V^3S_1^2(V_2, \theta, V)\,dV = \left(\frac{m_0m_2}{m_0+m_2}\right)^2\frac{2\pi\sigma}{\varpi h^3}V_1^3V_2^2S_2^1(V_2, \theta, V_1)\,dV_1. \quad \text{......(2·422)}$$

On differentiating (2·322) we find

$$V\,dV = V_1\,dV_1\left(1 - \frac{m_2}{m_0+m_2}\frac{V_2}{V_1}\cos\theta\right), \quad \text{...(2·423)}$$

and, combining (2·422) and (2·423),

$$V^2S_1^2(V_2, \theta, V)\left\{1 - \frac{m_2}{m_0+m_2}\frac{V_2}{V_1}\cos\theta\right\}$$
$$= \left(\frac{m_0m_2}{m_0+m_2}\right)^2\frac{2\pi\sigma}{\varpi h^3}V_1^2V_2^2S_2^1(V_2, \theta, V_1). \quad \text{......(2·43)}$$

An important special process of this type is *ionization*, in which m_2 is negligibly small compared with m_0. In that case, when the body 1 is of atomic (not electronic) mass, it is usually presumed that $V_2 < 2V$ and thus that V_2/V_1 is of order unity, since V and V_1 are only slightly different. In this case (2·43) becomes with sufficient accuracy

$$V^2S_1^2(V_2, \theta, V) = \frac{2\pi\sigma m_2^2}{\varpi h^3}V_1^2V_2^2S_2^1(V_2, \theta, V_1), \quad \text{...(2·51)}$$

with the energy condition

$$\frac{1}{2}\frac{m_0m_1}{m_0+m_1}V^2 = \epsilon_0 + \frac{1}{2}\frac{m_0m_1}{m_0+m_1}V_1^2 + \tfrac{1}{2}m_2V_2^2. \quad \text{...(2·52)}$$

These equations are the result of ignoring the momentum of the electron in the equations of conservation of momentum—an approximation which can constantly be made.

When θ has disappeared explicitly from the frequency relations as in (2·51), (2·52), it is possible and often convenient to reformulate these relations more in accordance with the relations of the former paper† where both bodies 1 and 2 were electrons. We recall that $\Sigma_2^1(p_1, p_2, \tau;\ V_2, \theta, V_1)$ is the fraction of $(p_1, p_2, \tau;\ V_2, \theta, V_1)$-collisions that result in capture. The function $S_2^1(V_2, \theta, V_1)$ derived from Σ_2^1 by (2·225) is the fraction of all (V_2, θ, V_1)-collisions distributed at random in the p's and τ which result in capture. The mean value for a random distribution in θ is

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* Fowler, *Phil. Mag.*, 45, 27 (1923), equation (9·82).

$$S_2^{1*}(V_2, V_1) = \frac{1}{2}\int_0^\pi S_2^1(V_2, \theta, V_1)\sin\theta\, d\theta, \quad \ldots(2{\cdot}611)$$

and this is the fraction of all (V_1, V_2)-collisions distributed at random in p_1, p_2, τ and θ which result in capture. It is not always possible in applications to work in sufficient detail to include variations with θ; it is then only possible to assume that the effective fraction for any given distribution is represented sufficiently closely by S_2^{1*}.

In a similar way $\Sigma_1^2(p;\ V_2, \theta, V)\, m_2 V_2 \sin\theta\, dV_2\, d\theta$ denotes that fraction of (p, V)-collisions which result in (V_2, θ, V_1)-relative motions after causing dissociation. Then

$$S_1^2(V_2, \theta, V)\, m_2 V_2 \sin\theta\, dV_2\, d\theta$$

is that fraction of V-collisions distributed at random in p which have the same result. If finally

$$S_1^{2*}(V_2, V) = \int_0^\pi S_1^2(V_2, \theta, V)\sin\theta\, d\theta, \quad \ldots(2{\cdot}612)$$

then $S_1^{2*}(V_2, V)\, m_2 V_2\, dV_2$ is that fraction of V-collisions distributed at random in p which result in (V_2, V_1)-relative velocities after dissociation. The fraction of all V-collisions distributed at random in p which cause dissociation at all is

$$\int_0^{(V_2)_{\text{max.}}} S_1^{2*}(V_2, V)\, m_2 V_2\, dV_2. \quad \ldots\ldots(2{\cdot}62)$$

The relation between S_1^{2*} and S_2^{1*} is

$$V^2 S_1^{2*}(V_2, V) = \frac{4\pi\sigma m_2^2}{\varpi h^3} V_1^2 V_2^2 S_2^{1*}(V_2, V_1), \quad \ldots(2{\cdot}63)$$

with the equation for V^2

$$\frac{1}{2}\frac{m_0 m_1}{m_0 + m_1} V^2 = \epsilon_0 + \frac{1}{2}\frac{m_0 m_1}{m_0 + m_1} V_1^2 + \tfrac{1}{2} m_2 V_2^2. \quad \ldots(2{\cdot}52)$$

In future applications of these frequencies we shall drop the *; its omission will no longer lead to confusion.

We repeat here for completeness the formulae for ionization by *electronic* impact, expressed in the present notation. They are‡

$$V^2 S_1^2(V_2, V) = \frac{4\pi\sigma m_2^2}{\varpi h^3} V_1^2 V_2^2 S_2^1(V_2, V_1),\ (\textit{electrons}) \ldots(2{\cdot}71)$$

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† Fowler, *loc. cit.* (1).

‡ Fowler, *loc. cit.* (1), p. 271, equation C.

with the relation

$$\tfrac{1}{2}m_2 V^2 = \epsilon_0 + \tfrac{1}{2}m_2 V_1^2 + \tfrac{1}{2}m_2 V_2^2. \qquad \text{......(2·72)}$$

Formulae (2·63) and (2·71) are, as we should expect, identical.

Finally, it is well to recall the geometrical meaning of the coefficients which we have introduced. The coefficient $S_1^2(V_2, V)$ may be described as the effective target area of the body 0 for that type of ionization by the body 1 at relative velocity V in which the electron is thrown off with energy between $\tfrac{1}{2}m_2 V_2^2$ and $\tfrac{1}{2}m_2(V_2 + dV_2)^2$. The total effective target area for ionization by the body 1 at velocity V is thus

$$\int_0^{(V_2)_{\max.}} S_1^2(V_2, V)\, m_2 V_2 dV_2.$$

A similar interpretation holds for the more general case of dissociation treated earlier in this section. The coefficient $S_2^1(V_2, V_1)$ which is of dimensions L^4T may be described as a generalized effective target area of the body 0—effective that is in producing combination under bombardment by bodies 1 and 2 with relative velocities V_1 and V_2 distributed at random in time and direction.

§ 3. *Experimental evidence for the form of* $S_1^2(V_2, V)$. Experimental evidence from the phenomena exhibited by α- (and β-) particles throws light on $S_1^2(V_2, V)$, the effective target area for a certain process of ionization. For definiteness we will suppose that the body 1 is a nucleus of charge Ze (atomic mass) and velocity V, the body 2 (of course) an electron and the body 0 any particular atom or molecule, neutral or ionized, at rest, containing a group of electrons σ in number, any one of which may be removed at the cost of energy ϵ_0. Then

(1) The number of *primary* pairs of ions extracted from this group by the body 1 in a length of track dx is by (2·62)

$$\nu_0 dx \int_0^{(V_2)_{\max.}} S_1^2(V_2, V)\, m_2 V_2 dV_2. \qquad \text{......(3·11)}$$

(2) The energy spent on this type of ionization is

$$\nu_0 dx \int_0^{(V_2)_{\max.}} (\epsilon_0 + \tfrac{1}{2}m_2 V_2^2)\, S_1^2(V_2, V)\, m_2 V_2 dV_2. \qquad \text{...(3·12)}$$

(3) The *total* resulting ionization is

$$\nu_0 dx \int_0^{(V_2)_{\max.}} g(V_2)\, S_1^2(V_2, V)\, m_2 V_2 dV_2, \qquad \text{......(3·13)}$$

where $g(V_2)$ is the average number of (pairs of) ions made by a single electron of velocity V_2, which is itself counted as one in $g(V_2)$.

Under certain assumptions, discussed later, it has been shown that

$$g(V_2) = \frac{3}{4}\,\frac{\rho + \tfrac{1}{2}m_2 V_2^2}{\rho}, \qquad \text{......(3·131)}$$

approximately, where ρ is a mean ionization potential which should not differ much from the ionization potential of the most lightly bound electrons.

(4) The ratio of the number of V_2-primaries to V_2'-primaries is

$$S_1^2(V_2, V)\, V_2\, dV_2 / S_1^2(V_2', V)\, V_2'\, dV_2'. \qquad \text{......(3·14)}$$

In the theory of Thomson and Bohr the energy q actually transferred to an electron in the process of ionization is identified with the energy q which would be transferred to a free electron initially at rest in a similar encounter obeying in full the classical laws. If p denotes here the distance of the line of impact from the relevant electron[*],

$$q = \frac{2Z^2e^4}{m_2V^2(p^2 + Z^2e^4/m_2^2V^4)}. \qquad \text{......(3·211)}$$

From this it follows, since $q = \epsilon_0 + \frac{1}{2}m_2V_2^2$, that

$$2p\,dp = -\frac{2Z^2e^4}{V^2}\frac{V_2\,dV_2}{(\epsilon_0 + \frac{1}{2}m_2V_2^2)^2}. \qquad \text{......(3·212)}$$

These equations lead to a formula for $S_1^2(V_2, V)$ which has the exact validity of the Thomson-Bohr theory. For the area $2\pi\sigma p\,dp$ we have $\Sigma_1^2 m_2V_2\,dV_2 = 1$. For any other area $\Sigma_1^2 = 0$. Hence

$$S_1^2(V_2, V) = \frac{2\pi Z^2e^4\sigma}{m_2V^2}\frac{1}{(\epsilon_0 + \frac{1}{2}m_2V_2^2)^2}. \qquad \text{......(3·22)}$$

The maximum energy transferable on this theory is $2m_2V^2$, so that

$$(V_2)^2_{\text{max.}} = 4V^2 - 2\epsilon_0/m_2, \qquad \text{......(3·221)}$$

and $S_1^2 = 0$ when $V_2^2 > (V_2)^2_{\text{max}}$.

The results of the same theory on the form of the function S_1^2 for ionization by *electronic* impact may be quoted for comparison[†]. In place of (3·212) we have

$$2p\,dp = -\frac{2e^4}{V^2}\frac{V_2\,dV_2}{(\epsilon_0 + \frac{1}{2}m_2V_2^2)^2}. \qquad \text{......(3·31)}$$

In place of (3·22) we derive

$$S_1^2(V_2, V) = \frac{2\pi e^4\sigma}{m_2V^2}\frac{1}{(\epsilon_0 + \frac{1}{2}m_2V_2^2)^2}, \quad (\textit{electrons}); \qquad \text{...(3·32)}$$

the maximum transferable energy is now $\frac{1}{2}m_2V^2$ so that

$$(V_2)^2_{\text{max.}} = V^2 - 2\epsilon_0/m_2. \qquad \text{......(3·321)}$$

The only difference in form lies in $(V_2)_{\text{max.}}$.

If the formulae (3·22), (3·321) are used in (3·11) and (3·14), and account is taken of the various groups of electrons in the atom, we obtain the usual formulae for the primary ionization and associated loss of energy per unit length of track. We obtain also an equation for $g(V_2)$, which can be approximately solved*, and hence a formula for the total ionization. These formulae are known to reproduce the facts with varying success over varying ranges of velocities.

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[*] See for example Fowler, *Proc. Camb. Phil. Soc.*, 21, 526 (1923).

[†] Fowler, *loc. cit.* (1), p. 273, transposed to the present notation.

The formula for S_1^2 breaks up into two factors, one depending only on V and the other only on V_2. It is of some importance to consider explicitly which of these factors enters into the interpretation of various groups of facts. Only the second factor

$$(\epsilon_0 + \tfrac{1}{2}m_2 V_2^2)^{-2}$$

affects the relative frequency of ejection of electrons with different velocities. It alone determines also $g(V_2)$, so long as we may assume that practically the whole available energy of the V_2-electron is spent on ionization. This is in fact true in general. If practically the whole energy of any moving body is spent on primary and secondary ionization, then the total ionization produced will depend only on the V_2-factor. It can be maintained with some plausibility that this law of variation with V_2 may be approximately true in general, or at least over a much wider range of velocities than the law of variation with V. The derived form of $g(V_2)$ seems to be in agreement with our information, still unfortunately vague, as to the ratio of the total to the primary ionization along the tracks of α- and β-particles†. The proper evidence on this point would be a count of a large number of secondary tracks of various lengths produced by α- and β-particles of known velocity in photographs by C. T. R. Wilson's method‡. These would provide, as the theory here developed will show, unexpectedly valuable information.

Formulae (3·22) and (3·221) applied to (3·12) give us

$$-m_1 V\left[\frac{dV}{dx}\right]_{\text{ion.}} = \frac{2\pi\nu_0 Z^2 e^4}{m_2 V^2}\,\Sigma_s \sigma_s \log\frac{2m_2 V^2}{\epsilon_s}. \quad \ldots(3{\cdot}41)$$

In this formula $[dV/dx]_{\text{ion.}}$ denotes the rate of loss of velocity by the body 1 due to ionizing the bodies 0, when account is taken of all their groups of electrons. There is a similar formula for β-particles. These are the usual formulae giving the so-called V^4-law for range. They do not, of course, take account of other forms of energy loss (expenditure on transference and molecular dissociation), but with classical laws these may both be expected to possess a variation with V of the same type. The most recent and most general form of the postulates of the quantum theory demands that all atomic systems should be in stationary states after as well as before any interaction§. The only other possible source of energy loss will therefore be a process of direct radiation during the interaction, at the expense of the energy of the body 1. If this can be calculated on the lines of Bohr's* original theory of the α-particle, it also has much the same variation with V.

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* Fowler, *Proc. Camb. Phil. Soc.*, 21, 531 (1923).

† Fowler, *loc. cit.*; C. T. R. Wilson, *Proc. Roy. Soc.*, A, **104**, **1**, **192** (1923), especially p. 200.

‡ Dr J. Chadwick informs me that he has been able to make a rough count of this nature, which, so far as it goes, confirms the classical V_2-factor.

§ Bohr, *The Fundamental Postulates*, Chap. I; *Camb. Phil. Soc. Proc.* (Supplement).

It is well known that the V^4-law even when modified by the logarithmic factor in (3·41) is not correct for α-particles, except possibly at the highest velocities. It seems to be correct for β-particles of medium velocity, equivalent to 10,000–20,000 volts, and may well be true of any other charged moving particles in the same *velocity* region. At ordinary α-particle velocities, 1000–300 volts, it is commonly agreed that a V^3-law holds. This falls to a still lower power of V at the lowest velocities†. One may reasonably expect that the laws for α- and β-particles will be similar at these velocities too and in fact at all velocities, provided the effect of capture in changing the charge of the α-particle is allowed for. The law of variation of range with velocity depends essentially on the first factor in $S_1{}^2$. So too does the absolute amount of ionization per unit length of track. This first factor, V-factor, is therefore certainly in default at α-particle velocities. The logarithmic dependence of (3·41) on the ϵ's is probably a point in favour of this formula. But this results from the form of the second, V_2-factor, alone.

In view of these considerations it seems reasonable to suppose tentatively that in all circumstances

$$S_1{}^2(V_2, V) = \frac{Z^2\sigma f(V)}{(\epsilon_0 + \frac{1}{2}m_2 V_2{}^2)^2}. \qquad \text{......(3·5)}$$

In (3·5) $f(V)$ is some function of the velocity of the moving nucleus which is to be supposed practically independent of V_2, ϵ_0, Z and σ. Further, $f(V) = 0$ unless V is so great that the transfer of energy can exceed $(\epsilon_0 + \frac{1}{2}m_2 V_2{}^2)$; we shall assume that the maximum transfer is correctly given by the classical theory.

By (2·63) and (2·52) the associated formula for $S_2{}^1$ is

$$S_2{}^1(V_2, V_1) = \frac{\varpi h^3 Z^2}{4\pi m_2{}^2} \cdot \frac{\dfrac{2\epsilon_0(m_0+m_1)}{m_0 m_1} + V_1{}^2 + \dfrac{m_2(m_0+m_1)}{m_0 m_1} V_2{}^2}{V_1{}^2 V_2{}^2}$$
$$\times \frac{f\left\{\left(\dfrac{2\epsilon_0(m_0+m_1)}{m_0 m_1} + V_1{}^2 + \dfrac{m_2(m_0+m_1)}{m_0 m_1} V_2{}^2\right)^{\frac{1}{2}}\right\}}{(\epsilon_0 + \frac{1}{2}m_2 V_2{}^2)^2}. \qquad \text{...(3·6)}$$

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* Bohr, *Phil. Mag.*, 24, 10 (1913); 30, 581 (1915).

† Blackett, *Proc. Roy. Soc.*, A, 103, 68 (1923). The V^3-law appears to hold *when the logarithmic factors are ignored*. This is not the place to enter into details, but when the logarithmic factors are included the factor V^2 in the denominator of (3·41) must not actually be replaced by V to give an adequate representation of the observed velocity variation, but by a factor more like $V^{\frac{3}{2}}$. The logarithmic factors in fact themselves vary effectively in these ranges more or less like $V^{\frac{1}{2}}$.

This form of S_1^2 would be the more convincing if $f(V)$ were found to be independent of the atom from which the electron is to be extracted—if in fact $f(V)$ were a universal function of the relative velocity of the interacting bodies. It is too much to expect that any such universality really obtains. It would imply that the stopping powers of all atoms for α-particles so far as they are due to ionization should stand in the ratio of the corresponding

$$\Sigma_s \sigma_s \log (2m_2 V^2/\epsilon_s).$$

A study of actual stopping powers shows that this is improbable, for it is hardly possible to attribute the whole difference to the other sources of energy loss. A more satisfactory comparison would be between the average ionizations produced per atom at velocity V. But as yet the data hardly suffice for an exhaustive test. Even if $f(V)$ turns out to be a function of the atom or even of the group of electrons in the atom, formula (3·5) still seems, on the whole evidence, to be the most reasonable assumption that can be made.

In actual applications we require the numerical values of $f(V)$. These are probably best obtained from data as to the total ionization per cm. track due to an α-particle moving at a fairly high velocity. At a velocity equivalent to 1000 volts the α-particle makes $2·2 \times 10^4$ and $4·6 \times 10^3$ pairs of ions per cm. in Air and H_2 respectively[*]. We may take the ionization potential of H_2 to be 16 volts[†], and the potentials for N_2 and O_2 to be those specified in a paper already quoted[‡]. The mean potential ρ in $g(V_2)$ is 16 volts for H_2 and perhaps about 20 for Air. Making the calculations for 1000 volts ($V = 1·88 \times 10^9$ cm./sec.), we find

$$\begin{aligned} f(V) &= 1·30 \times 10^{-28},\ H_2 \\ &= 1·57 \times 10^{-28},\ \text{Air}. \end{aligned}$$

The value for Air is of course a mean for all its electronic groups. We shall adopt here the provisional value $f(V) = 1·4 \times 10^{-28}$, and for a range of velocities over which $f(V)$ varies like V^{-s} we shall take[§]

$$f(V) = 1·4 \times 10^{-28} \left(\frac{1·88 \times 10^9}{V}\right)^s. \qquad ......(3·61)$$

§ 4. *The capture and loss of electrons by swift α-particles.* In the former discussion of this phenomenon the effects of interactions with atomic cores were ignored, and the effects of electronic interactions were calculated on the basis of the classical ionization theory of J. J. Thomson. The result obtained was

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[*] Geiger, *Proc. Roy. Soc.*, A, 82, 486 (1909); Taylor, *Phil. Mag.*, 26, 402 (1913).

[†] Smyth, *Proc. Roy. Soc.*, A, 105, 116 (1924).

[‡] *Proc. Camb. Phil. Soc.*, *loc. cit.*

[§] The theoretical value of s is of course 2, and the theoretical value of the constant $1·0 \times 10^{-28}$.

$$\frac{[\mathrm{He}^{++}]\,\nu_2}{[\mathrm{He}^{+}]} = \frac{8\pi m_2 \sigma}{\varpi h^3}\left(\frac{m_2}{2}\right)^{\frac{1}{2}} \frac{W-\epsilon_0}{W\epsilon_0}\, W^{\frac{1}{2}}\,(W+\epsilon_0)^2. \quad \ldots(4{\cdot}11)$$

The notation has been adapted to that of the present paper, and $W\,(=\frac{1}{2}m_2 V^2)$ is the energy of an electron moving with the velocity V of the a-particle. Equation (4·11) was obtained using for $f(V)$ its classical value (or some constant multiple thereof)

$$f(V) = A/V^2. \quad \ldots\ldots(4{\cdot}12)$$

It is however easy enough to retain the general form of $f(V)$ throughout the calculations. If we do this we obtain again (4·11) with an extra factor F on the right-hand side, where

$$F = \frac{V^2 f(V)}{(2V^2 + 2\epsilon_0/m_2)\, f\,\{(2V^2 + 2\epsilon_0/m_2)^{\frac{1}{2}}\}}. \quad \ldots(4{\cdot}13)$$

If $f(V)$ varies as V^{-s} over the range of values concerned, then

$$F = \left\{\frac{V^2}{2V^2 + 2\epsilon_0/m_2}\right\}^{1-\frac{1}{2}s} = (\tfrac{1}{2})^{1-\frac{1}{2}s} \quad \ldots\ldots(4{\cdot}14)$$

approximately. We know that the value of s is between 1 and 2. Thus the only change in (4·11) due to retaining the general form of $f(V)$ is to introduce a somewhat trivial numerical factor, at most $1/\sqrt{2}$. *The form of $f(V)$, provided it can be approximately represented by a power law, is without effect on the equilibrium ratio* $[\mathrm{He}^{++}]/[\mathrm{He}^{+}]$ *in the beam.* The equilibrium ratio is controlled entirely by the V_2-factor in $S_1{}^2$, and Rutherford's V^5-law is a direct consequence of (or verification of) the form of the V_2-factor alone.

The new analysis of § 2 of this paper can be used to calculate the equilibrium ratio in the beam on the assumption that the only effective interactions are those in which atomic cores take part *provided that an atomic core interacts with* He^{++} *and* He^{+} *as if it were a point charge Ze.* We have seen that

$$\int_0^{(V_2)_{\max.}} S_1{}^2\,(V_2,\, V)\, m_2 V_2\, dV_2$$

is the total effective target area for ionization associated with each He^{+}. Using (3·5), we find that the total number of ionizations per second in the beam is

$$[\mathrm{He}^{+}]\,\nu_1 V \int_0^{(V_2)_{\max.}} S_1{}^2\,(V_2,\, V)\, m_2 V_2\, dV_2,$$

or

$$[\mathrm{He}^{+}]\,\nu_1 V f(V)\, Z^2 \sigma \left[\frac{1}{\epsilon_0} - \frac{1}{2m_2 V^2}\right]. \quad \ldots\ldots(4{\cdot}21)$$

In the same way the effective "target" for recombination is

$S_2^1(V, V)$ for each [He^{++}] in the beam. Therefore, by (3·6), the total number of recombinations per second in the beam is

$$[\mathrm{He}^{++}]\,\nu_1\nu_2 V^2 S_2^1(V, V)$$

or

$$[\mathrm{He}^{++}]\,\nu_1\nu_2 V^2 \frac{\varpi h^3 Z^2}{4\pi m_2^2} \frac{X^2 f(X)}{V^4(\epsilon_0 + \tfrac{1}{2}m_2 V^2)^2}, \quad \ldots\ldots(4{\cdot}22)$$

where

$$X^2 = \frac{m_1 m_2 + m_2 m_0 + m_0 m_1}{m_0 m_1} V^2 + \frac{2\epsilon_0(m_0 + m_1)}{m_0 m_1}. \quad (4{\cdot}221)$$

In equilibrium the numbers given by (4·21) and (4·22) must be equal. Since we have reason to believe that effective interactions only occur inside atoms or molecules, ν_2 is the intra-atomic (intra-molecular) density of free—that is lightly bound—electrons and ν_1 the corresponding density of atomic cores. We observe that in (4·221) $X = V$ to sufficient accuracy.

The equilibrium ratio in the beam is therefore determined by the equation

$$\frac{[\mathrm{He}^{++}]\,\nu_2}{[\mathrm{He}^{+}]} = \frac{4\pi\sigma m_2^2}{\varpi h^3} \frac{2m_2 V^2 - \epsilon_0}{2m_2 V^2 \epsilon_0} V(\epsilon_0 + \tfrac{1}{2}m_2 V^2)^2 \quad \ldots(4{\cdot}23)$$

$$= \frac{8\pi\sigma m_2}{\varpi h^3}\left(\frac{m_2}{2}\right)^{\frac{1}{2}} \frac{W - \tfrac{1}{4}\epsilon_0}{W\epsilon_0} W^{\frac{1}{2}}(W + \epsilon_0)^2. \quad \ldots(4{\cdot}24)$$

This differs from the completed form of (4·11) for electron-electron interactions in two ways only, both trivial. One of these is the replacement of the factor $W - \epsilon_0$ by $W - \tfrac{1}{4}\epsilon_0$. The other is the disappearance of the numerical factor F, which is here always 1 whatever be the value of s.

Since equations (4·11) and (4·24) are essentially the same, the equilibrium ratio will also agree with (4·11) and (4·24) when simultaneous independent contributions by both processes of interaction are included. So far therefore as these contributions adequately represent the effects of interactions with *atoms*, the conclusions of the former paper remain unaltered. It is necessary however at this stage to enter on a more exact discussion of the nature of the intra-atomic field from the points of view of loss and capture separately, before we can apply our results to these separate effects, and reach final conclusions as to the equilibrium ratio.

§ 5. *The nature of the atomic field in the process of loss.* In the process of loss He$^{+} \rightarrow$ He^{++}, an electron, originally (it is presumed) in the 1-quantum orbit of ionization potential $\epsilon_0 = 54$ volts, is removed by interaction with the field of an atom (or molecule), which may be thought of as passing the α-particle at velocity V. We are not here primarily concerned with effects on the passing atom, and it is clear that the best approximation we can make is to regard the field of force of the atom as made up of (1) the general central field due to the nucleus together with the electrons in their normal quantum orbits, (2) independent contributions by certain of the individual electrons. The general central field will be capable of extracting the electron from He⁺ if the centre of the moving atom passes near enough to it, without any specific interaction with any particular electron in the atom. We shall find that the critical value of p for such extraction is considerably less than the greatest radii of the electronic orbits in the moving atom. In addition therefore to the above interactions causing loss, loss can be caused by close interaction with those electrons which lie for the whole or greater part of the time outside this critical distance. These interactions will plainly take place almost independently of the general field. They can be calculated separately and added to the total for the general field of the atom. Electrons inside the critical distance will give no private contributions, for whenever the electron in He⁺ reaches such a region, it will be extracted by the general field. This analysis of the atomic field for loss replaces the cruder conception of a number of free electrons and a core field (point charge) which we have tacitly used hitherto. To estimate the effect of this general field, we can only use the classical theory of ionization. We proceed to make this estimate.

The general field can probably be roughly represented by a central force $Z(r)/r^2$, where $Z(r)$ is the effective nuclear charge as a function of r. The exact value of $Z(r)$ is uncertain, but it is not unreasonable to suppose that it will be similar to that which determines the X-ray and optical levels so far as these can be approximated to by central orbits. The numerical value of this field can be derived from a study of these levels with fair exactness*. The field so determined is of course the field experienced by an electron of the atom itself, and due allowance will have been made automatically for the phase relations of the electrons in their orbits. The field experienced by a swift electron from outside will naturally not be identical, but *on the average* there seems no reason to anticipate wide divergences, except that at greater distances $Z(r)$ will approach zero instead of unity. This however will prove to be trivial.

When $Z(r)$ has the constant value Z, the exact classical relation (3·211) between p and q can be written in the form

$$p^2 = \frac{2Z^2e^4}{m_2V^2}\left(\frac{1}{q} - \frac{1}{2m_2V^2}\right). \qquad \text{......(5·11)}$$

The critical value p_l of p is that value for which q first reaches ϵ_0 the ionization energy of He⁺. When V is fairly large so that $2m_2V^2$ is large compared to ϵ_0, we have approximately

$$p_l = Ze^2/(\tfrac{1}{2}m_2V^2\epsilon_0)^{\frac{1}{2}}, \qquad \text{......(5·12)}$$

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* See D. R. Hartree, *Camb. Phil. Soc. Proc.*, 21, 625 (1923) for a discussion and numerical values.

which is sufficiently accurate in this connection for all values of V covered by Rutherford's experiments. This approximation is exactly that to which we are led by calculating the impulse on the electron, assumed not to move, as the atomic field moves past in a straight line at constant velocity V. This method of calculation will therefore be accurate enough to determine p_l in all cases, and can be applied at once even when $Z(r)$ is not constant. If I is the impulse acting on the electron, it is easily shown that

$$I = e^2 \int_{-\infty}^{+\infty} \frac{Z(r)}{r^2} \cos\theta \, dt$$

$$= \frac{2e^2}{pV} \int_0^{\frac{1}{2}\pi} Z(p \sec\theta) \cos\theta \, d\theta. \quad \text{......(5·13)}$$

This integral can be computed if $Z(r)$ is known as a function of r. If the impulse is to transfer energy ϵ_0 to the electron (at rest), then $I = (2m_2\epsilon_0)^{\frac{1}{2}}$, and we have

$$\frac{p_l(\frac{1}{2}m_2V^2\epsilon_0)^{\frac{1}{2}}}{e^2} = \int_0^{\frac{1}{2}\pi} Z(p_l \sec\theta) \cos\theta \, d\theta \quad \text{......(5·14)}$$

$$= Z_l, \quad \text{......(5·15)}$$

say. Thus determined, $Z_l e$ is that point charge which will remove the electron from the normal orbit in He$^+$ exactly as often on the average as the general field of the atom. We may call $Z_l e/r^2$ the mean atomic field for loss.

Curves of $Z(r)$ are available* for Na and K. I have also by the kindness of Mr Hartree had the use of similar curves for Cu and Rb not yet published. The curves we really require are of course those for the atoms used in Rutherford's experiments. Thus only Cu is strictly relevant, and we must make the best we can of the general numerical tendencies. The following rough values have been obtained from (5·14) by numerical integration.

Mean atomic fields for loss, Z_l.

Velocity volts	Na	K	Rb	Cu	"Air"
1000	5·5	7·6	8·7	7·9	3·8
500	4·8	—	—	6·2	3·4
250	4·1	5·5	6·0	4·8	2·9

The velocity is that at which the atomic field and He$^+$ are supposed to pass by one another. The general nature of the changes in Z

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* Hartree, *loc. cit.*

from atom to atom of one type, and with varying velocity, is clearly shown. The effective field of the atom H will obviously be 1 at all these velocities. In the absence of precise information one can interpolate between Na and H to estimate very roughly the mean field for each atom in Air, given in the last column. The true values are probably of this order but may well be either less or greater.

To the effect of the general field we have to add the effects of electrons which lie outside the critical distance from the nucleus. It is easy to see that these effects contribute only a comparatively small part, perhaps 12–20 % of the total losses. For to a very rough approximation the mean effective field of the atom will be equal to the actual field at the critical distance, since by far the greatest contribution to (5·14) is made by that part of the integral near $\theta = 0$, where $Z(p_l \sec \theta) = Z(p_l)$. Further, $Z(p_l)$ will be roughly equal to the number of electrons which on the average lie at distances from the nucleus greater than p_l. We again emphasize that these approximations are very rough, but they suffice to justify us in taking the effective number of external electrons to be about Z_l. The contributions to the total losses by the general field and the external electrons will therefore be roughly in the ratio $Z_l : 1$.

We can now write down a formula for the number of losses per He⁺ per second, n_l, or for the mean free path for loss, λ_l. These quantities are connected by the equation $\lambda_l = V/n_l$, and it will suffice to give λ_l. We find

$$\lambda_l = \frac{1}{\nu_1 f(V)\, Z_l^2 \left[\dfrac{1}{\epsilon_0} - \dfrac{1}{2m_2 V^2}\right] + \nu_1' f'(V) \left[\dfrac{1}{\epsilon_0} - \dfrac{1}{\frac{1}{2} m_2 V^2}\right]}. \quad (5·21)$$

In this equation ν_1' is the effective electron density and so $\nu_1' = Z_l \nu_1$. Also it is probable that $f(V) = f'(V)$. These functions are theoretically identical, and there is no empirical evidence to distinguish between them. Since the first term is the more important, we can reduce (5·21) with sufficient accuracy to

$$\lambda_l = \frac{1}{\nu_1 Z_l (Z_l + 1) f(V) \left[\dfrac{1}{\epsilon_0} - \dfrac{1}{2m_2 V^2}\right]}, \quad \text{......}(5·22)$$

where $f(V)$ is given by (3·61). For Air at normal pressure and temperature this reduces to

$$\lambda_l = \frac{2·10 \times 10^{-4}}{Z_l (Z_l + 1)} \frac{W \epsilon_0}{W - \frac{1}{4}\epsilon_0} \left(\frac{W}{1000}\right)^{\frac{1}{2}s}; \; (\textit{volts}) \quad ...(5·23)$$

W and ϵ_0 are as usual the equivalent velocity of He⁺ and its ionization potential expressed in volts. The exact figures depend of course on s. If we take $s = \frac{3}{2}$, we find the following values:

Velocity V in terms of V_0	Velocity W in volts	λ_l observed mm.	λ_l calculated mm.
0·94	920	0·011	0·0061
0·76	600	0·0078	0·0050
0·47	230	0·0050	0·0035

The calculated values are now about 1·6 times too small, but such agreement is quite satisfactory*.

§ 6. *The nature of the atomic field in the process of capture.* We have seen in the last section how to determine at least roughly the value of Z proper to equation (4·21), which we called Z_l. Here we must determine the similar value Z_c proper to (4·22). In the process of capture we are definitely concerned with the extraction of an electron out of the passing atom, and here the original conception of an atmosphere of "free" electrons, any one of which may be captured, remains essential. What we have further to do is to estimate what is the effective part of the general field of the atom for the purpose of interaction with that one of the "free" electrons which is being or has been loosened by interaction with the He^{++}. The general field of the atom can still be thought of as roughly $Z(r)\, e/r^2$, but the critical region is different. The process of capture as here idealized is one in which an electron and a general atomic field pass a stationary He^{++} with equal velocities V and so interact that the electron remains behind in a quantized orbit. Consider this process in detail with the time scale reversed. It is then one of loss in which the electron has to be ejected from its orbit in He^+ with the velocity V of the moving field. We can therefore determine Z_c by determining what will be the mean point charge of the general atomic field for this type of loss, for the effective mean value must clearly be the same for the same process reversed in time.

The critical conditions now occur when $q = \frac{1}{2} m_2 V^2$, so that the last term in (5·11) is no longer strictly negligible. Instead of (5·12) we have

$$p^2 = \frac{Z^2 e^4}{(\frac{1}{2} m_2 V^2)^2} (1 - \tfrac{1}{4}),$$

or

$$p = \sqrt{\tfrac{3}{4}}\, \frac{Ze^2}{\frac{1}{2} m_2 V^2}. \qquad \text{......(6·11)}$$

It is probably sufficiently accurate in determining Z_c to include $\sqrt{\frac{3}{4}}$ as a correcting factor and then determine the transfer of energy by calculating the impulse as before. We therefore have the equations

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* It would actually be somewhat better with the classical value of $f(V)$, about 1·3 times too large. But we have no justification for using the classical value.

$$\sqrt{\frac{4}{3}}\,\frac{p_c\,(\tfrac{1}{2}m_2V^2)}{e^2} = \int_0^{\frac{1}{2}\pi} Z\,(p_c \sec\theta)\cos\theta\, d\theta \quad \text{......(6·12)}$$

$$= Z_c. \quad \text{......(6·13)}$$

To these interactions of the free electrons with the general field we must add their interactions with the other electrons in the atom. In this case it seems proper to suppose that *all* the other electrons make separate contributions of approximately the normal amount. The values of Z_c calculated from Hartree's curves are given below:

Mean atomic fields for capture, Z_c.

Velocity volts	Na	K	Rb	Cu
1000	8·5	12·9	18·8	17·0
500	7·4	—	—	13·0
250	6·0	8·3	9·7	9·0

Since the values of Z in (4·21) and (4·22) are different, equation (4·24) will contain a correcting factor. If we ignore the differences between the formulae for electrons and mean atomic fields as in the last section, we can include all interactions in the one formula, and the complete correcting factor to (4·24) is

$$\frac{Z_l\,(Z_l+1)}{Z_c^2+(Z-1)},$$

where Z is the atomic number. The values of this ratio are as follows:

$$\text{Ratio}\ \frac{Z_c^2+(Z-1)}{Z_l\,(Z_l+1)}.$$

Velocity volts	Na	K	Rb	Cu
1000	2·3	2·8	4·6	4·5
500	2·3	—	—	4·4
250	2·2	2·5	3·1	3·9

The large change of ratio for Rb can be traced back to the extra size of the Rb atom. In the absence of similar information for the atoms used in Rutherford's experiments we conclude that this correcting factor will be roughly independent of velocity, and have a value between 2·5 and 5. It is hardly possible to carry out similar calculations for velocities of 100 volts owing to uncertainties in the atomic field near its boundary. The effect of this correcting factor is that the value of ν_2 in the atom—that is the "free" electron density called ν_e in the previous paper—which must be postulated to explain the observed ratio is reduced to about 10^{24}. This is a not unreasonable value. It is in fact possible that actual densities are greater than this, so that there may be some margin of allowance for a possible reducing effect of the parallelism of velocities on the frequencies of capture mentioned in the previous paper.

It does not seem profitable to attempt to push the analysis of these results any further at this stage. We become involved with too many uncertain factors.

§ 7. *Summary.* Formulae are given for the frequency relations between the general 2- and 3-body collision processes of dissociation and recombination in a perfect gas. For ionization and its reverse the formulae take a simple form.

The results are applied to a more thorough investigation of the theory of the capture and loss of electrons by swift α-particles in which the effects of the general field of the atom are included. The final results are substantially unaltered. The theory can account satisfactorily for Rutherford's observations if the "free" electron density in atoms is about 10^{24} electrons per c.c.—a reasonable value.

It is shown that Rutherford's V^5-law for the equilibrium ratio $[He^{++}]/[He^{+}]$ is a necessary consequence simply of the classical formula for the distribution in velocity of the δ-particles ejected by α- or β-particles of given velocity.

A detailed discussion is given of the nature of the general field inside an atom from the point of view of its effectiveness in causing loss or capture.

It is a pleasure to acknowledge the advantages I have derived from discussions of these subjects with Messrs E. A. Milne and P. M. S. Blackett.

[*Addendum*, June 28, 1924. In my former paper on this subject, *Phil. Mag.*, 47, 257 (1924), I referred hurriedly to a paper by Becker, *Zeit. für. Phys.*, 18, 325, on the same subject then just to hand. This reference and my own paper are on certain points inaccurate, as Dr Becker has since been good enough to point out to me. The main point is that the necessary and sufficient condition for the preservation of the equilibrium laws is the condition there called (A) as given by Dr Becker, and not $(A) + (B)$, for it can be shown that (A) implies (B). There is little doubt that (C) is still the correct condition, but it is of course not deducible from the chosen hypothesis for the process there discussed. I wish to take this opportunity of thanking Dr Becker for his communication and of calling attention again to his paper.]

Integrals of Systems of Ordinary Differential Equations. By Dr T. M. CHERRY, Trinity College.

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Let
$$\frac{dx_r}{dt} = X_r \quad (r = 1, \dots n) \qquad \dots\dots(1)$$
be a system of differential equations in which $X_1, \dots X_n$ are analytic functions independent of t, expansible in convergent series of powers of $x_1, \dots x_n$. Suppose further that not all the functions $X_1, \dots X_n$ vanish when $x_1 = \dots = x_n = 0$. It will be shown in §§ 1–3 that these equations possess $(n - 1)$ independent integrals independent of t, expansible in convergent series of powers of $x_1, \dots x_n$; i.e. functions
$$\phi_r(x_1, \dots x_n) \quad (r = 1, \dots n - 1)$$
such that
$$\frac{d\phi_r}{dt}, = \frac{\partial \phi_r}{\partial x_1} X_1 + \dots + \frac{\partial \phi_r}{\partial x_n} X_n, = 0$$
identically. In § 4 it is shown how the series for $\phi_1, \dots \phi_{n-1}$ may be most directly constructed, and in § 5 is briefly considered the corresponding problem when the origin is a singular point of the equations, i.e. when the expansions of $X_1, \dots X_n$ all begin with terms of the first or higher degrees.

§ 1. *Formal solution of the equations* (1). We shall first show that the equations (1) may be formally solved by power series in $t - c_1, c_2, \dots c_n$ for each coordinate x_r, where $c_1, \dots c_n$ are arbitrary constants, and in § 2 we shall show that these series are absolutely and uniformly convergent provided $|t + c_1|, |c_2|, \dots |c_n|$ are sufficiently small. This result is of course similar to the well-known result that the equations (1) yield $x_1, \dots x_n$ as analytic functions of t and of the initial values $x_1^0, \dots x_n^0$*, but differs in that the n independent arbitrary constants are so chosen that one occurs additively with t.

Suppose that when $x_1 = \dots = x_n = 0$ the functions $X_1, \dots X_n$ take respectively the values $a_1, \dots a_n$, of which by hypothesis at least one is not zero, say a_1. We may replace x_r by a new coordinate x_r' such that
$$x_r' = x_r - \frac{a_r}{a_1} x_1 \quad (r = 2, \dots n)$$
and shall then have
$$\frac{dx_r'}{dt} = X_r - \frac{a_r}{a_1} X_1, = X_r' \text{ say},$$

---

* See, for example, Painlevé, *Leçons sur la théorie analytique des équations différentielles*, p. 394.

where $X_r' = 0$ at the origin; we have, moreover,

$$\frac{\partial(x_1, x_2', \dots x_n')}{\partial(x_1, x_2, \dots x_n)} = 1.$$

We may therefore always suppose that $X_2 = \dots = X_n = 0$ at the origin, while $X_1 \neq 0$, and shall write the equations

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \lambda + a_1^{(1)}x_1 + \dots + a_n^{(1)}x_n + \tfrac{1}{2}a_{11}^{(1)}x_1^2 + a_{12}^{(1)}x_1x_2 + \dots, \\ \frac{dx_r}{dt} &= \qquad a_1^{(r)}x_1 + \dots + a_n^{(r)}x_n + \tfrac{1}{2}a_{11}^{(r)}x_1^2 + a_{12}^{(r)}x_1x_2 + \dots \\ &\qquad\qquad\qquad\qquad (r = 2, \dots n). \end{aligned}\right\} \quad (2)$$

These equations we proceed to solve by successive approximation.

Substitute in (2)

$$x_r = x_r^{(1)} + x_r^{(2)} + \dots \quad (r = 1, \dots n) \qquad \dots\dots(3)$$

and equate separately the terms of successive orders, as follows:

$$\left.\begin{aligned} \frac{dx_1^{(1)}}{dt} &= \lambda, \\ \frac{dx_r^{(1)}}{dt} &= 0 \quad (r = 2, \dots n), \end{aligned}\right\} \qquad \dots\dots(4)$$

$$\frac{dx_r^{(2)}}{dt} = a_1^{(r)}x_1^{(1)} + \dots + a_n^{(r)}x_n^{(1)} \quad (r = 1, \dots n), \qquad \dots\dots(5)$$

$$\frac{dx_r^{(3)}}{dt} = a_1^{(r)}x_1^{(2)} + \dots + a_n^{(r)}x_n^{(2)} + \tfrac{1}{2}a_{11}^{(r)}x_1^{(1)2} + a_{12}^{(r)}x_1^{(1)}x_2^{(1)} + \dots, \qquad \dots\dots(6)$$

$$\begin{aligned} \frac{dx_r^{(4)}}{dt} &= a_1^{(r)}x_1^{(3)} + \dots + a_n^{(r)}x_n^{(3)} + a_{11}^{(r)}x_1^{(1)}x_1^{(2)} \\ &\quad + a_{12}^{(r)}(x_1^{(2)}x_2^{(1)} + x_1^{(1)}x_2^{(2)}) + \dots + a_{111}^{(r)}x_1^{(1)3} + \dots, \end{aligned}$$

....

Solving (4) we obtain

$$\left.\begin{aligned} x_1^{(1)} &= \lambda(t + c_1) \\ x_r^{(1)} &= c_r \quad (r = 2, \dots n) \end{aligned}\right\}, \qquad \dots\dots(7)$$

where $c_1, \dots c_n$ are arbitrary constants. From (5) we then have

$$\frac{dx_r^{(2)}}{dt} = a_1^{(r)}\lambda(t + c_1) + a_2^{(r)}c_2 + \dots + a_n^{(r)}c_n,$$

$$x_r^{(2)} = \tfrac{1}{2}a_1^{(r)}\lambda(t + c_1)^2 + a_2^{(r)}c_2(t + c_1) + \dots + a_n^{(r)}c_n(t + c_1), \quad \dots(8)$$

where the constant of integration has been so adjusted as a function of $c_1, \dots c_n$ that $x_r^{(2)}$ is a homogeneous polynomial of second degree in $t + c_1, c_2, \dots c_n$, vanishing with $t + c_1$. The equation (5) is then

$$\begin{aligned} \frac{dx_r^{(3)}}{dt} &= \sum_{s=1}^{n} a_s^{(r)}\{\tfrac{1}{2}a_1^{(s)}\lambda(t + c_1)^2 + (a_2^{(s)}c_2 + \dots + a_n^{(s)}c_n)(t + c_1)\} \\ &\quad + \tfrac{1}{2}a_{11}^{(r)}\lambda^2(t + c_1)^2 + a_{12}^{(r)}\lambda c_2(t + c_1) + \tfrac{1}{2}a_{22}^{(r)}c_2^2(t + c_1) + \dots, \end{aligned}$$

giving on integration

$$x_r^{(3)} = \sum_{s=1}^{n} a_s^{(r)} \{\tfrac{1}{6}a_1^{(s)}\lambda\,(t+c_1)^3 + \tfrac{1}{2}\,(a_2^{(s)}c_2 + \ldots + a_n^{(s)}c_n)\,(t+c_1)^2\}$$
$$+ \tfrac{1}{6}a_{11}^{(r)}\lambda^2\,(t+c_1)^3 + \tfrac{1}{2}a_{12}^{(r)}\lambda c_2\,(t+c_1)^2 + \tfrac{1}{2}a_{22}^{(r)}c_2^2\,(t+c_1) + \ldots.$$

The equations for $x_r^{(4)}$, $x_r^{(5)}$, ... are solved similarly, and in each case the constant of integration is so adjusted as a function of $c_1, \ldots c_n$ that $x_r^{(s)}$ is a homogeneous polynomial of degree s in $t+c_1, c_2, \ldots c_n$, vanishing with $t+c_1$.

We may thus construct for $x_1, \ldots x_n$ power series in $t+c_1$, $c_2, \ldots c_n$ which formally satisfy the equations (2).

§ 2. *Convergence of the series* (3). Let $Y_1, \ldots Y_n$ be power series in $y_1, \ldots y_n$ which are dominant for $X_1, \ldots X_n$ respectively, and take as comparison equations

$$\left.\begin{aligned} \frac{dy_1}{dt} &= Y_1 = \mu + b_1^{(1)}y_1 + \ldots + b_n^{(1)}y_n + \tfrac{1}{2}b_{11}^{(1)}y_1^2 + b_{12}^{(1)}y_1y_2 + \ldots, \\ \frac{dy_r}{dt} &= Y_r = \qquad b_1^{(r)}y_1 + \ldots + b_n^{(r)}y_n + \tfrac{1}{2}b_{11}^{(r)}y_1^2 + b_{12}^{(r)}y_1y_2 + \ldots \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r = 2, \ldots n) \end{aligned}\right\}$$

Put
$$y_r = y_r^{(1)} + y_r^{(2)} + \ldots \quad (r = 1, \ldots n) \qquad \ldots\ldots(9)$$

and determine $y_r^{(1)}$, $y_r^{(2)}$ ... in succession as in § 1. We obtain

$$\left.\begin{aligned} y_1^{(1)} &= \mu\,(t+d_1) \\ y_r^{(1)} &= d_r \quad (r = 2, \ldots n) \end{aligned}\right\}, \qquad \ldots\ldots(10)$$

$$y_r^{(2)} = \tfrac{1}{2}b_1^{(r)}\mu\,(t+d_1)^2 + b_2^{(r)}d_2\,(t+d_1) + \ldots + b_n^{(r)}d_n\,(t+d_1), \qquad \ldots\ldots(11)$$

$$y_r^{(3)} = \sum_{s=1}^{n} b_s^{(r)} \{\tfrac{1}{6}b_1^{(s)}\mu\,(t+d_1)^3 + \tfrac{1}{2}\,(b_2^{(s)}d_2 + \ldots + b_n^{(s)}d_n)\,(t+d_1)^2\}$$
$$+ \tfrac{1}{6}b_{11}^{(r)}\mu^2\,(t+d_1)^3 + \tfrac{1}{2}b_{12}^{(r)}\mu d_2\,(t+d_1)^2 + \tfrac{1}{2}b_{22}^{(r)}d_2^2\,(t+d_1) + \ldots.$$

Since the series $Y_1, \ldots Y_n$ are dominant respectively for $X_1, \ldots X_n$, each coefficient in the former series is positive and greater than the absolute value of the corresponding coefficient of the latter series. Thus supposing the constants $c_1, \ldots c_n, d_1, \ldots d_n$ so chosen that $t+d_1, d_2, \ldots d_n$ are positive and

$$|\,t+c_1\,| < t+d_1, \quad |\,c_2\,| < d_2, \ldots \quad |\,c_n\,| < d_n, \qquad \ldots\ldots(12)$$

we see from a comparison of (7), (10) that

$$|\,x_1^{(1)}\,| < y_1^{(1)}, \quad |\,x_2^{(1)}\,| < y_2^{(1)}, \ldots \quad |\,x_n^{(1)}\,| < y_n^{(1)};$$

from (8) and (11) that

$$|\,x_1^{(2)}\,| < y_1^{(2)}, \ldots \quad |\,x_n^{(2)}\,| < y_n^{(2)};$$

and in general $y_r^{(s)}$ is the sum of a number of positive terms of which each is greater than the modulus of the corresponding term in $x_r^{(s)}$, so that

$$| x_1^{(s)} | < y_1^{(s)}, \dots | x_n^{(s)} | < y_n^{(s)}.$$

It follows that if $t + d_1, d_2, \dots d_n$ be so chosen that the series (9) are convergent, the series (3) will be absolutely and uniformly convergent for values of their arguments satisfying the inequalities (12).

To obtain series with the required dominating property, it is well known that we may take for each of $Y_1 - \mu, Y_2, \dots Y_n$ the expansion of

$$\frac{M (y_1 + y_2 + \dots + y_n)}{1 - a (y_1 + y_2 + \dots + y_n)},$$

provided M and a are sufficiently large positive constants, say

$$M > M_0, \; a > a_0, \qquad \dots\dots(13)$$

and of course we must also choose μ such that $\mu > | \lambda |$. Putting

$$y_1 + y_2 + \dots + y_n = z$$

the comparison equations may then be written

$$\frac{d (y_1 - y_2)}{dt} = \frac{d (y_1 - y_3)}{dt} = \dots = \frac{d (y_1 - y_n)}{dt} = \mu, \quad \dots(14)$$

$$\begin{aligned} \frac{dz}{dt} &= \mu + \frac{nMz}{1 - az} \\ &= \frac{\mu + z (nM - a\mu)}{1 - az}. \qquad \dots\dots(15) \end{aligned}$$

Since μ is positive it is obviously possible to choose M, a so that

$$nM - a\mu = 0,$$

and so that at the same time the inequalities (13) are satisfied. The equation (15) is then

$$\frac{dz}{dt} = \frac{\mu}{1 - az},$$

giving on integration

$$z - \tfrac{1}{2} az^2 = \mu t + \text{const.} \qquad \dots\dots(16)$$

From (15) we obtain

$$y_1 - y_r = \mu t + \text{const.} \quad (r = 2, \dots n), \qquad \dots\dots(17)$$

which with (16) constitute the general solution of the equations (14), (15).

The solution which we have previously found by the series (9) is such that when $t + d_1 = 0$,

$$y_1 = 0, \; y_2 = d_2, \dots y_n = d_n.$$

From (16), (17) this solution must therefore be equivalent to

$$z - \tfrac{1}{2} az^2 = \mu (t + d_1) + d - \tfrac{1}{2} ad^2, \qquad \dots\dots(18)$$

$$y_1 - y_r = \mu (t + d_1) - d_r \quad (r = 2, \dots n), \qquad \dots\dots(19)$$

where we have written d for $d_2 + d_3 + \dots + d_n$. From (18) we have

$$z = \frac{1}{\alpha}\left\{1 - \sqrt{1 - 2\alpha\mu(t + d_1) - 2\alpha d + \alpha^2 d^2}\right\}$$

for the value of z which takes the value d when $t + d_1 = 0$; writing this

$$\alpha z = 1 - (1 - \alpha d)\left\{1 - \frac{2\alpha\mu(t + d_1)}{(1 - \alpha d)^2}\right\}^{\frac{1}{2}}$$

$$= \alpha d + \frac{\alpha\mu(t + d_1)}{1 - \alpha d} + \frac{1}{2}\cdot\frac{\alpha^2\mu^2(t + d_1)^2}{(1 - \alpha d)^3} + \frac{1}{2}\cdot\frac{\alpha^3\mu^3(t + d_1)^3}{(1 - \alpha d)^5} + \dots,$$

we see that z may be expanded in a series of powers of $t + d_1$, $d_2, \dots d_n$, absolutely convergent for sufficiently small values of $|t + d_1|$, $|d_2|, \dots |d_n|$. Since

$$z = y_1 + y_2 + \dots + y_n$$

we have finally from (19)

$$y_1 = \mu(t + d_1) + \frac{\mu\alpha(t + d_1)(d_2 + \dots + d_n)}{n\{1 - \alpha(d_2 + \dots + d_n)\}} + \frac{1}{2}\cdot\frac{\alpha\mu^2(t + d_1)^2}{n\{1 - \alpha(d_2 + \dots + d_n)\}^3} + \dots,$$

$$y_r = d_r + \frac{\mu\alpha(t + d_1)(d_2 + \dots + d_n)}{n\{1 - \alpha(d_2 + \dots + d_n)\}} + \frac{1}{2}\cdot\frac{\alpha\mu^2(t + d_1)^2}{n\{1 - \alpha(d_2 + \dots + d_n)\}^3} + \dots,$$

$$(r = 2, \dots n)$$

which when expanded in powers of $t + d_1, d_2, \dots d_n$ must be identical with the series (10). These series are therefore convergent, and hence the series (3) are absolutely and uniformly convergent when

$$|t + c_1|, \ |c_2|, \dots |c_n|$$

are sufficiently small.

§ 3. *Existence of integrals.* The series (3) expressing the general solution of the equations (2) vanish for $t + c_1 = c_2 = \dots = c_n = 0$, and are such that for these values of their arguments the Jacobian

$$\frac{\partial(x_1, x_2, \dots x_n)}{\partial(c_1, c_2, \dots c_n)}$$

has the value λ. The equations (3) may therefore be solved, giving $t + c_1, c_2, \dots c_n$ as power series in $x_1, x_2, \dots x_n$, convergent when these quantities are sufficiently small in absolute value. The series obtained are

$$\lambda(t + c_1) = x_1 - \frac{x_1}{\lambda}\{\tfrac{1}{2}a_1^{(1)}x_1 + a_2^{(1)}x_2 + \dots + a_n^{(1)}x_n\} + \dots \quad \dots\dots(20)$$

$$= \phi_1(x_1, \dots x_n), \text{ say;}$$

$$c_r = x_r - \frac{x_1}{\lambda}\{\tfrac{1}{2}a_1^{(r)}x_1 + a_2^{(r)}x_2 + \dots + a_n^{(r)}x_n\} + \dots$$

$$(r = 2, \dots n) \dots\dots(21)$$

$$= \phi_r(x_1, \dots x_n), \text{ say,}$$

in which it is readily seen (and may be formally proved by induction) that every term of second or higher degree contains x_1 as a factor.

Since $c_2, \dots c_n$ are arbitrary constants, we see from (21) that $\phi_2, \dots \phi_n$ are $(n-1)$ independent integrals of the equations (2), not involving t. In fact, writing the general solution (3) as

$$x_r = f_r\,(t + c_1, c_2, \dots c_n) \quad (r = 1, \dots n)$$

and writing $\left(\frac{\partial f_s}{\partial c_r}\right)$ for the cofactor of $\frac{\partial f_s}{\partial c_r}$ in the Jacobian

$$\frac{\partial\,(f_1, \dots f_n)}{\partial\,(c_1, \dots c_n)}, = J \text{ say},$$

we have from the reciprocal relationship between $f_1, \dots f_n$ and $\phi_1, \dots \phi_n$

$$\frac{\partial \phi_r}{\partial x_s} = \frac{1}{J} \cdot \left(\frac{\partial f_s}{\partial c_r}\right);$$

so that

$$\begin{aligned} \frac{d\phi_r}{dt} &= \frac{\partial \phi_r}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \phi_r}{\partial x_2}\frac{dx_2}{dt} + \dots \\ &= \frac{1}{J}\left\{\left(\frac{\partial f_1}{\partial c_r}\right)\frac{\partial f_1}{\partial c_1} + \left(\frac{\partial f_2}{\partial c_r}\right)\frac{\partial f_2}{\partial c_1} + \dots\right\} \\ &= 0, \end{aligned}$$

since the second factor is compounded of the elements of the first row of J with the cofactors of the rth row. Thus the equations (1), from which (2) are derived by linear transformation, possess $(n-1)$ independent integrals independent of t, expansible in convergent series of powers of $x_1, \dots x_n$. It is hardly necessary to add that they cannot possess any integrals not involving t independent of these.

§ 4. *Direct construction of the integrals.* Having in the preceding sections established the existence, form, and convergence of the series for the integrals, we shall now give a method for their direct construction. We shall write the equations (2) in the form

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \lambda + X_1^{(1)} + X_1^{(2)} + \dots, \\ \frac{dx_r}{dt} &= \phantom{\lambda +{}} X_r^{(1)} + X_r^{(2)} + \dots, \quad (r = 2, \dots n) \end{aligned}\right\} \quad \dots\dots(22)$$

where $X_r^{(s)}$ denotes the aggregate of the terms of degree s in X_r. If

$$\phi = \text{const.}$$

is an integral, where $\phi = \phi_1 + \phi_2 + \dots,$ ......(23)

we have the identity

$$\frac{\partial \phi}{\partial x_1} X_1 + \frac{\partial \phi}{\partial x_2} X_2 + \dots + \frac{\partial \phi}{\partial x_n} X_n = 0,$$

whence, equating to zero separately the terms of successive degrees,

$$\lambda \frac{\partial \phi_1}{\partial x_1} = 0, \qquad \text{......(24)}$$

$$\lambda \frac{\partial \phi_2}{\partial x_1} + X_1^{(1)} \frac{\partial \phi_1}{\partial x_1} + X_2^{(1)} \frac{\partial \phi_1}{\partial x_2} + \dots + X_n^{(1)} \frac{\partial \phi_1}{\partial x_n} = 0, \quad \text{...(25)}$$

$$\lambda \frac{\partial \phi_3}{\partial x_1} + X_1^{(1)} \frac{\partial \phi_2}{\partial x_1} + \dots + X_1^{(2)} \frac{\partial \phi_1}{\partial x_1} + \dots = 0, \quad \text{..... (26)}$$

....

The equation (24) possesses $(n-1)$ independent solutions linear in $x_1, \dots x_n$, which we may take to be

$$\phi_1 = x_2,\ \phi_1 = x_3,\ \dots \phi_1 = x_n.$$

Taking the first of these, for example, (25) becomes

$$\lambda \frac{\partial \phi_2}{\partial x_1} + (a_1^{(2)} x_1 + \dots + a_n^{(2)} x_n) = 0,$$

giving on integration

$$\phi_2 = -\frac{1}{\lambda} (\tfrac{1}{2} a_1^{(2)} x_1^2 + a_2^{(2)} x_1 x_2 + \dots + a_n^{(2)} x_1 x_n);$$

the complementary function, consisting of an arbitrary function of $x_2, \dots x_n$, is omitted in order that ϕ_2 may be a polynomial of the second degree with x_1 as a factor, as in the series (21) found in § 3. From equation (26) we then find for ϕ_3 a third degree polynomial divisible by x_1, and so on for $\phi_4, \dots$.

The $(n-1)$ series which are thus constructed are known from § 3 to be convergent when $|x_1|, \dots |x_n|$ are sufficiently small.

§ 5. *On the existence of integrals developable about singular points of the equations* (1). In the preceding sections we have supposed that not all of $X_1, \dots X_n$ vanish at the origin. We shall now suppose that at this point

$$X_1 = X_2 = \dots = X_n = 0,$$

i.e. that $X_1, \dots X_n$ are developable in convergent series of powers of $x_1, \dots x_n$, the lowest terms being of the first degree or higher; the origin is thus a singular point of the equations (1). If the first degree terms in X_r are $a_1^{(r)} x_1 + \dots + a_n^{(r)} x_n$ the roots $\alpha_1, \dots \alpha_n$ of the equation

$$\begin{vmatrix} a_1^{(1)} - \alpha, & a_2^{(1)}, & \dots & a_n^{(1)} \\ a_1^{(2)}, & a_2^{(2)} - \alpha, & \dots & a_n^{(2)} \\ \dots & \dots & \dots & \dots \\ a_1^{(n)}, & a_2^{(n)}, & \dots & a_n^{(n)} - \alpha \end{vmatrix} = 0$$

are called the *exponents* of the singular point. It will be sufficient here to consider the general case in which no two exponents are equal; we may then take the equations in a transformed form in which the linear terms in X_r reduce to $a_r x_r$. As in the last section we shall write $X_r^{(s)}$ for the aggregate of the terms of degree s in X_r.

Putting $$\phi = \phi_1 + \phi_2 + \dots$$
we find as in § 4 that $\phi = \text{const.}$ is an integral if

$$a_1 x_1 \frac{\partial \phi_1}{\partial x_1} + a_2 x_2 \frac{\partial \phi_1}{\partial x_2} + \dots + a_n x_n \frac{\partial \phi_1}{\partial x_n} = 0, \qquad \dots\dots(27)$$

$$\sum_{r=1}^{n} a_r x_r \frac{\partial \phi_2}{\partial x_r} + \sum_{r=1}^{n} X_r^{(2)} \frac{\partial \phi_1}{\partial x_r} = 0, \qquad \dots\dots(28)$$

$$\sum_{r=1}^{n} a_r x_r \frac{\partial \phi_3}{\partial x_r} + \sum_{r=1}^{n} X_r^{(2)} \frac{\partial \phi_2}{\partial x_r} + \sum_{r=1}^{n} X_r^{(3)} \frac{\partial \phi_1}{\partial x_r} = 0, \dots\dots(29)$$

....

If $$\phi_1 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$
the equation (27) requires for its identical satisfaction

$$c_1 a_1 = c_2 a_2 = \dots = c_n a_n = 0;$$

thus we may determine as many independent linear solutions of (27) as there are zero exponents, and since in general no exponent vanishes no integral ϕ can have a development starting with first degree terms. Pursuing this, the general case, equation (28) reduces to

$$\sum_{r=1}^{n} a_r x_r \frac{\partial \phi_2}{\partial x_r} = 0;$$

and putting $$\phi_2 = \tfrac{1}{2} c_{11} x_1^2 + c_{12} x_1 x_2 + \dots$$
this requires for its identical satisfaction

$$a_1 c_{11} = a_2 c_{22} = \dots = (a_1 + a_2)\, c_{12} = (a_1 + a_3)\, c_{13} = \dots = 0;$$

thus all the coefficients in ϕ_2 must vanish unless there is a pair of the exponents whose sum is zero. In general this is not the case, and we proceed to equation (29), which cannot be satisfied by a third degree polynomial unless we can find from amongst the exponents a group which satisfies a relation of one of the forms

$$2a_1 + a_2 = 0, \quad a_1 + a_2 + a_3 = 0.$$

Proceeding similarly we see that the equations (1) possess no integral developable about a singular point unless its exponents $a_1, \dots a_n$ satisfy a relation of the form

$$A_1 a_1 + A_2 a_2 + \dots + A_n a_n = 0, \qquad \dots\dots(30)$$

where $A_1, \dots A_n$ are positive integers or zero, but are not all zero. In general there exists no such relation.

Even if there is a relation of the form (30) it does not follow that the series for ϕ can be constructed, for if

$$A_1 + A_2 + \dots + A_n = m,$$

the equations for $\phi_{2m}, \phi_{3m}, \ldots$ will in general contain terms (which will be called *critical*) involving a power of

$$x_1^{A_1} x_2^{A_2} \ldots x_n^{A_n},$$

and it is readily seen that in virtue of (30) such an equation as

$$\alpha_1 x_1 \frac{\partial \phi_{rm}}{\partial x_1} + \ldots + \alpha_n x_n \frac{\partial \phi_{rm}}{\partial x_n} + B x_1^{rA_1} x_2^{rA_2} \ldots x_n^{rA_n} = 0$$

cannot be satisfied by a polynomial in $x_1, \ldots x_n$ unless $B = 0$. Thus it is impossible to construct a power series for ϕ unless the coefficients of all critical terms in the equations (28), (29), ... vanish, and this will not be the case unless the coefficients in the series $X_1, \ldots X_n$ are connected by an infinite number of relations. Thus, *unless there is some restriction upon the form of* $X_1, \ldots X_n$, *the equations* (1) *possess no integrals developable about their singular points.*

When the equations (1) are of Hamiltonian form we know that the characteristic exponents can be arranged in pairs, the sum of those in any pair being zero. If the system is of order $2m$ we may thus, in virtue of what has been seen above, find the second degree terms for m independent integrals. In a paper which appears below* it is shown, for a particular case which may legitimately be regarded as the most general, that the coefficients of all the critical terms vanish, and therefore that we may construct m power series which are, formally, integrals developable about the singular point of the equations.

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* "On Integrals developable about a Singular Point of a Hamiltonian System of Differential Equations," p. 325, below.

A System of Linear Equations with an Infinity of Unknowns.

By E. C. Titchmarsh. (Communicated by Mr G. H. Hardy, Trinity College.)

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1. I have collected in the present note some theorems regarding the solution of a certain system of linear equations with an infinity of unknowns. The general form of the equations is

$$a_1 x_n + a_2 x_{n+1} + a_3 x_{n+2} + \dots = c_n \quad (n = 1, 2, 3, \dots), \quad \dots(1\cdot1)$$

the numbers $a_1, a_2, \dots c_1, c_2, \dots$ being given. Equations of this type are of course well known; but in studying them it is generally assumed that the series depend for convergence on the convergence-exponent of the sequences involved, e.g. that Σa_n^2 and Σx_n^2 are convergent[*]. No assumptions of this kind are made here, and in fact the series need not be absolutely convergent. On the other hand rather special assumptions are made with regard to the monotonic character of the sequences a_n and c_n.

A system of equations of this type, with $a_n = 1/(2n+1)$ and $c_n = 0$, was encountered recently by Prof. Hardy and myself in the study of a certain integral equation[†]. Similar systems occur in the theory of power series[‡].

In Theorems I and II it is proved that under certain conditions the equations have a unique solution, and in Theorems III and IV a similar result is obtained with different hypotheses. Neither set of results includes the other, and it seems difficult to construct a general theory which would include both.

2. *Theorem I. Let $a_1, a_2, \dots$ be a non-increasing sequence of positive numbers, and suppose that*

$$a_1 x_n + a_2 x_{n+1} + \dots = 0 \quad (n = 1, 2, \dots). \quad \dots\dots(2\cdot1)$$

Then if $\lim a_n > 0$, *the equations have no solution other than the obvious null-solution. If* $\lim a_n = 0$, *they have no solution, other than the null-solution, for which* $x_n \to 0$.

Since the series $\Sigma a_n x_n$ is convergent, we have $a_n x_n \to 0$. Suppose that $a_n \to a$. Then if $a > 0$ it follows that $x_n \to 0$. Hence in either case we have to prove that if $x_n \to 0$, then $x_n = 0$ for all values of n.

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[*] For the ordinary theory of linear equations with an infinity of unknowns see F. Riesz, *Systèmes d'équations linéaires à une infinité d'inconnues* (Paris, 1913).

[†] G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 23 (1924), 1-26.

[‡] See H. von Koch, *Proceedings of the fifth International Congress of Mathematicians* (Cambridge, 1913).

Let $$\mu_n = \operatorname{Max}(|x_n|, |x_{n+1}|, \ldots).$$
Then $\mu_n \to 0$ since $x_n \to 0$. Now for all values of n
$$a_1 x_n + a_2 x_{n+1} + a_3 x_{n+2} + \ldots = 0$$
and $$a_1 x_{n+1} + a_2 x_{n+2} + a_3 x_{n+3} + \ldots = 0.$$
Subtracting, we obtain
$$a_1 x_n = (a_1 - a_2)\, x_{n+1} + (a_2 - a_3)\, x_{n+2} + \ldots.$$

Hence
$$\begin{aligned} a_1 |x_n| &\leqq (a_1 - a_2)\, |x_{n+1}| + (a_2 - a_3)\, |x_{n+2}| + \ldots \\ &\leqq (a_1 - a_2)\, \mu_{n+1} + (a_2 - a_3)\, \mu_{n+2} + \ldots \\ &= (a_1 - a)\, \mu_{n+1}. \end{aligned}$$
Hence $|x_n| \leqq \mu_{n+1}$, and consequently $\mu_n = \mu_{n+1}$. But since $\mu_n \to 0$ we may choose n so large that $\mu_{n+1} < \epsilon$. Then it follows that $\mu_n < \epsilon$, and so, repeating the argument n times, that $\mu_1 < \epsilon$. Hence $\mu_1 = 0$, i.e. $x_1 = x_2 = \ldots = 0$. This proves the theorem.

It may be remarked that if $a = 0$ there may be solutions for which x_n does not tend to zero. Suppose for example that $a_{2n} = a_{2n-1}$ for all values of n. Then the equations are satisfied by putting $x_n = (-1)^n$. Here x_n is bounded, so that the assumption $x_n \to 0$ was not unnecessarily stringent.

3. *Theorem II. Let* $a_1, a_2, \ldots$ *and* $c_1, c_2, \ldots$ *be non-increasing sequences of positive numbers such that*
$$(a_1 + a_2 + \ldots + a_n)\, c_{n+1} \to 0. \qquad \ldots\ldots(3{\cdot}1)$$
Then if $\lim a_n > 0$, *the equations* (1·1) *have just one solution; while if* $\lim a_n = 0$, *they have just one solution for which* $x_n \to 0$.

We define $\lambda_1, \lambda_2, \ldots$ in succession by the equations
$$\begin{aligned} &a_2 - \lambda_1 a_1 = 0, \\ &a_3 - \lambda_1 a_2 - \lambda_2 a_1 = 0, \\ &a_4 - \lambda_1 a_3 - \lambda_2 a_2 - \lambda_3 a_1 = 0, \text{ etc.} \end{aligned}$$
Then if we multiply equations (1·1), beginning at the nth, by 1, $-\lambda_1$, $-\lambda_2$, ... and add, we obtain formally
$$a_1 x_n = c_n - \lambda_1 c_{n+1} - \lambda_2 c_{n+2} - \ldots. \qquad \ldots\ldots(3{\cdot}2)$$

We proceed to verify that this is a solution such that $x_n \to 0$. It will then follow from Theorem I that it is the only such solution.

Let $$\theta_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$
Then the equation
$$a_n - \lambda_1 a_{n-1} - \ldots - \lambda_{n-1} a_1 = 0$$
may be written
$$a_n - \theta_1 a_{n-1} - (\theta_2 - \theta_1)\, a_{n-2} - \ldots - (\theta_{n-1} - \theta_{n-2})\, a_1 = 0$$
or $$a_n + \theta_1 (a_{n-2} - a_{n-1}) + \ldots + \theta_{n-2} (a_1 - a_2) = \theta_{n-1} a_1.$$

Hence if $\theta_m \geqq 0$ for $m = 1, 2, \ldots n-2$, then $\theta_{n-1} \geqq 0$ also. But $\theta_1 = \lambda_1 = a_2/a_1 \geqq 0$. Hence $\theta_n \geqq 0$ for all values of n.

Again the same equation may be written

$$(a_{n-1} - a_n) + (1-\theta_1)(a_{n-2} - a_{n-1}) + \ldots \\ + (1-\theta_{n-2})(a_1 - a_2) = (1-\theta_{n-1})\, a_1.$$

Hence if $1-\theta_m \geqq 0$ for $m = 1, 2, \ldots n-2$, then $1-\theta_{n-1} \geqq 0$ also. But $1-\theta_1 = 1-\lambda_1 = (a_1 - a_2)/a_1 \geqq 0$. Hence $1-\theta_n \geqq 0$ for all values of n. Thus

$$0 \leqq \theta_n \leqq 1 \qquad \ldots\ldots(3{\cdot}3)$$

for all values of n.

Also (3·1) implies that $c_n \to 0$. From this and (3·3) it follows that the series (3·2) are convergent, so that a value is actually obtained for x_n for all values of n. Also

$$\begin{aligned} a_1 x_n &= c_n - \theta_1 c_{n+1} - (\theta_2 - \theta_1)\, c_{n+2} - \ldots \\ &= c_n - \theta_1 (c_{n+1} - c_{n+2}) - \theta_2 (c_{n+2} - c_{n+3}) - \ldots \\ &= (c_n - c_{n+1}) + (1-\theta_1)(c_{n+1} - c_{n+2}) + \ldots \end{aligned}$$

so that

$$c_n - c_{n+1} \leqq a_1 x_n \leqq c_n.$$

Hence $x_n \to 0$. It remains to prove that equations (1·1) are actually satisfied. It is sufficient to consider the first equation. Taking the first n terms of the series, we have from (3·2)

$$a_1 (a_1 x_1 + a_2 x_2 + \ldots + a_n x_n) = a_1 (c_1 - \lambda_1 c_2 - \lambda_2 c_3 - \ldots) \\ + a_2 (c_2 - \lambda_1 c_3 - \ldots) + \ldots + a_n (c_n - \lambda_1 c_{n+1} - \ldots). \quad \ldots\ldots(3{\cdot}4)$$

The coefficients of $c_2, c_3, \ldots c_n$ are zero, so that

$$a_1 (a_1 x_1 + \ldots + a_n x_n) = a_1 (c_1 - \lambda_n c_{n+1} - \lambda_{n+1} c_{n+2} - \ldots) \\ + a_2 (-\lambda_{n-1} c_{n+1} - \lambda_n c_{n+2} - \ldots) + \ldots + a_n (-\lambda_1 c_{n+1} - \ldots). \\ \ldots\ldots(3{\cdot}5)$$

But it follows from (3·3) that

$$| \lambda_m c_{n+1} + \lambda_{m+1} c_{n+2} + \ldots | \leqq 2 c_{n+1}.$$

Hence

$$a_1 \,| a_1 x_1 + \ldots + a_n x_n - c_1 | \leqq 2 (a_1 + a_2 + \ldots + a_n)\, c_{n+1},$$

which tends to zero by hypothesis. This proves the theorem.

4. *Theorem III. Let* $a_1, a_2, \ldots$ *be a non-increasing sequence of positive numbers with limit zero, and such that*

$$a_n a_{n+2} \geqq a_{n+1}^2 \qquad \ldots\ldots(4{\cdot}1)$$

for all values of n. *Then the equations* (2·1) *have no solution, other than the null-solution, for which*

$$s_n = x_1 + x_2 + \ldots + x_n$$

is bounded.

The first n of the equations to be satisfied may be written

$$\begin{aligned} a_1x_1 + a_2x_2 + \dots + a_nx_n &= R_1, \\ a_1x_2 + \dots + a_{n-1}x_n &= R_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \\ a_1x_n &= R_n, \end{aligned}$$

where

$$\begin{aligned} -R_1 &= a_{n+1}x_{n+1} + a_{n+2}x_{n+2} + \dots \\ &= a_{n+1}(s_{n+1} - s_n) + a_{n+2}(s_{n+2} - s_{n+1}) + \dots \\ &= -a_{n+1}s_n + (a_{n+1} - a_{n+2})s_{n+1} + \dots . \end{aligned}$$

Hence if $|s_n| \leqq M$, it follows that $|R_1| \leqq 2Ma_{n+1}$. We may prove in the same way that

$$|R_m| \leqq 2Ma_{n-m+2} \quad (m = 1, 2, \dots n).$$

Now, defining $\lambda_1, \lambda_2, \dots$ as before, we have

$$a_1x_1 = R_1 - \lambda_1R_2 - \lambda_2R_3 - \dots - \lambda_{n-1}R_n.$$

Hence

$$a_1|x_1| \leqq 2M\{a_{n+1} + |\lambda_1|a_n + \dots + |\lambda_{n-1}|a_2\}. \quad \dots(4\cdot2)$$

Again we have the two equations

$$\begin{aligned} a_1\lambda_n &= a_{n+1} - \lambda_1a_n - \dots - \lambda_{n-1}a_2, \\ 0 &= a_n - \lambda_1a_{n-1} - \dots - \lambda_{n-1}a_1. \end{aligned}$$

Eliminating the terms independent of any λ, we have

$$\frac{a_1\lambda_n}{a_{n+1}} = \lambda_1\left(\frac{a_{n-1}}{a_n} - \frac{a_n}{a_{n+1}}\right) + \dots + \lambda_{n-1}\left(\frac{a_1}{a_n} - \frac{a_2}{a_{n+1}}\right).$$

The terms in the brackets are all positive, since by (4·1) a_n/a_{n-1} is a non-increasing sequence. Hence if $\lambda_1, \lambda_2, \dots \lambda_{n-1}$ are all positive, it follows that λ_n is also positive. Thus, λ_1 being positive, we have $\lambda_n \geqq 0$ for all values of n. Combining this result with (3·3) it is seen that $\Sigma\lambda_n$ is a convergent series of positive terms, its sum lying between 0 and 1. Hence we can find m and n so that

$$\lambda_{m+1} + \lambda_{m+2} + \dots + \lambda_{n-1} < \epsilon$$

and also

$$a_{n-m+1} < \epsilon.$$

Then by (4·2)

$$\begin{aligned} a_1|x_1| &\leqq 2M\{(a_{n+1} + \lambda_1a_n + \dots + \lambda_ma_{n-m+1}) \\ &\qquad\qquad + (\lambda_{m+1}a_{n-m} + \dots + \lambda_{n-1}a_2)\} \\ &\leqq 2M\{(1 + \lambda_1 + \dots + \lambda_m)a_{n-m+1} + (\lambda_{m+1} + \dots + \lambda_{n-1})a_2\} \\ &\leqq 2M(2 + a_2)\epsilon. \end{aligned}$$

Since the right-hand side may be made as small as we please by choice of ϵ, it follows that $x_1 = 0$. Similarly $x_n = 0$ for all values of n.

In this theorem we have assumed more about a_n than in Theorem I, but less (in one way) about x_n. Notice that the conditions of Theorem I are insufficient to secure that $\lambda_n \geqq 0$. For example, $\lambda_2 = (a_1 a_3 - a_2^2)/a_1^2$, which is negative if a_3 is small enough. And in fact the theorem ceases to be true if we omit the condition (4·1). This is shown by the example already mentioned, where $a_{2n-1} = a_{2n}$, $x_n = (-1)^n$. Here (4·1) is not satisfied, and $x_n \nrightarrow 0$, though s_n is bounded.

5. *Theorem IV. Suppose that the numbers a_n satisfy the conditions of Theorem III, and that*

$$c_1 + c_2 + \dots + c_n$$

is bounded. Then the equations (1·1) *have just one solution for which $s_n = x_1 + \dots + x_n$ is bounded.*

As in the proof of Theorem II we have only to verify that the formulae (3·2) give a solution of the type stated. The series in (3·2) are in this case absolutely convergent, since c_n is bounded and $\Sigma\lambda_n$ is absolutely convergent. Also if

$$| c_1 + c_2 + \dots + c_n | \leqq K, \qquad \text{......(5·1)}$$

then
$$a_1 | s_n | \leqq 2K (1 + \lambda_1 + \lambda_2 + \dots),$$

so that s_n is bounded. As in the proof of Theorem II it is now sufficient to verify that

$$a_1 x_1 + \dots + a_n x_n - a_1 c_1 \to 0.$$

We obtain equation (3·5) as before. Now rearrange the right-hand side as a series of λ's, and use (5·1). This gives

$$\begin{aligned} a_1 | a_1 x_1 + a_2 x_2 + \dots + a_n x_n - c_1 | & \\ \leqq 2K (a_n \lambda_1 + a_{n-1}\lambda_2 + \dots + a_2 \lambda_{n-1} + a_1 \lambda_n + a_1 \lambda_{n+1} + \dots) & \\ = 2K a_{n+1} + 2K a_1 (\lambda_{n+1} + \lambda_{n+2} + \dots), & \end{aligned}$$

which tends to zero. This proves the theorem.

On Poincaré's Theorem of "The non-existence of Uniform Integrals of Dynamical Equations." By Dr T. M. CHERRY, Trinity College.

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In Chapter v of his *Les Méthodes Nouvelles de la Mécanique Céleste* Poincaré proves a theorem which he calls *The non-existence of Uniform Integrals.* This theorem is as follows:

Let
$$\frac{dx_r}{dt} = \frac{\partial F}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial F}{\partial x_r} \quad (r = 1, \dots n) \qquad \text{......(1)}$$
be a system of equations in which the characteristic function F is expanded in a series
$$F \equiv F_0 + \mu F_1 + \mu^2 F_2 + \dots \qquad \text{......(2)}$$
convergent for sufficiently small values of $|\mu|$, for all real values of $y_1, \dots y_n$, and for values of $x_1, \dots x_n$ within certain finite real intervals; while $F_0, F_1, \dots$ are analytic functions of $x_1, \dots x_n$, $y_1, \dots y_n$, of which F_0 is independent of $y_1, \dots y_n$ and $F_1, F_2, \dots$ are periodic with respect to these variables with period 2π. *Then the equations possess in general besides F no other "uniform" integral* Φ *of the same form, viz. an integral*
$$\Phi \equiv \Phi_0 + \mu\Phi_1 + \mu^2\Phi_2 + \dots = \text{const.} \qquad \text{......(3)}$$
in which $\Phi_0, \Phi_1, \dots$ *are analytic functions of* $x_1, \dots x_n$, $y_1, \dots y_n$ *periodic with respect to* $y_1, \dots y_n$, *and the series is convergent for sufficiently small values of* $|\mu|$, *for all real values of* $y_1, \dots y_n$, *and for values of* $x_1, \dots x_n$ *within certain finite real intervals**.

In this paper I shall show that this theorem no longer holds if we relax the restriction that the integral is to be developable in powers of μ.

§ 1. *Proof of Poincaré's Theorem.* The following treatment differs somewhat from Poincaré's, and appears to bring out more clearly the true significance of the Theorem.

Let us seek to determine integrals of the equations (1) which are power series in μ, with coefficients which are analytic functions of $x_1, \dots x_n, y_1, \dots y_n$, periodic with period 2π with respect to $y_1, \dots y_n$. Writing (3) for such an integral, the condition to be satisfied is
$$(F, \Phi) = 0, \qquad \text{......(4)}$$
the symbol on the left denoting the "Poisson bracket"
$$\sum_{r=1}^{n} \left(\frac{\partial F}{\partial x_r}\frac{\partial \Phi}{\partial y_r} - \frac{\partial F}{\partial y_r}\frac{\partial \Phi}{\partial x_r}\right).$$

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* These intervals are of course to be such that within them the derivatives of F are finite and continuous.

Substituting for F, Φ their expansions (2), (3) and equating separately to zero the coefficient of each power of μ we have the equations

$$(F_0, \Phi_0) = 0, \qquad \text{......(5)}$$

$$(F_0, \Phi_1) + (F_1, \Phi_0) = 0, \qquad \text{......(6)}$$

$$(F_0, \Phi_2) + (F_1, \Phi_1) + (F_2, \Phi_0) = 0, \qquad \text{......(7)}$$

....

from which we are to determine Φ_0, Φ_1, ... in succession.

Since Φ_0 is to be periodic with respect to $y_1, \ldots y_n$ we substitute in (5) an expression of the form

$$\Phi_0 = \Sigma A_{m_1, \ldots m_n} e^{i(m_1 y_1 + \ldots + m_n y_n)}, \qquad \text{......(8)}$$

where the A's are undetermined functions of $x_1, \ldots x_n$, and the summation is over all integral values (positive, negative and zero) of $m_1, \ldots m_n$. Since F_0 is independent of $y_1, \ldots y_n$ (5) becomes

$$\Sigma A_{m_1, \ldots m_n} \left(m_1 \frac{\partial F_0}{\partial x_1} + \ldots + m_n \frac{\partial F_0}{\partial x_n} \right) e^{i(m_1 y_1 + \ldots + m_n y_n)} = 0. \quad \text{......(9)}$$

In general the derivatives of F_0 are connected by no identical relation of the form

$$m_1 \frac{\partial F_0}{\partial x_1} + \ldots + m_n \frac{\partial F_0}{\partial x_n} = 0, \qquad \text{......(10)}$$

where $m_1, \ldots m_n$ are integers not all zero; assuming this to be the case, the identical satisfaction of (9) requires that the series (8) should reduce to the term for which $m_1 = \ldots = m_n = 0$, and this may be an arbitrary function of $x_1, \ldots x_n$. We may determine $(n-1)$ such functions independent of F_0, and suppose for Φ_0 some such definite choice made.

Since Φ_0 as well as F_0 is independent of $y_1, \ldots y_n$, the equation (6) is

$$\sum_{r=1}^{n} \left(\frac{\partial F_0}{\partial x_r} \frac{\partial \Phi_1}{\partial y_r} - \frac{\partial \Phi_0}{\partial x_r} \frac{\partial F_1}{\partial y_r} \right) = 0. \qquad \text{......(11)}$$

Here F_1 is by hypothesis expansible in the form

$$F_1 = \Sigma B_{m_1, \ldots m_n} e^{i(m_1 y_1 + \ldots + m_n y_n)},$$

and we are searching for a function Φ_1 of the same form, say

$$\Phi_1 = \Sigma C_{m_1, \ldots m_n} e^{i(m_1 y_1 + \ldots + m_n y_n)}. \qquad \text{......(12)}$$

Equation (11) thus becomes

$$\Sigma \left\{ C_{m_1, \ldots m_n} \left(m_1 \frac{\partial F_0}{\partial x_1} + \ldots + m_n \frac{\partial F_0}{\partial x_n} \right) \right.$$
$$\left. - B_{m_1, \ldots m_n} \left(m_1 \frac{\partial \Phi_0}{\partial x_1} + \ldots + m_n \frac{\partial \Phi_0}{\partial x_n} \right) \right\} e^{i(m_1 y_1 + \ldots + m_n y_n)} = 0,$$

an identity in which the coefficient of each exponential must vanish separately; this gives

$$C_{m_1,\dots m_n} = B_{m_1,\dots m_n} \frac{m_1 \dfrac{\partial \Phi_0}{\partial x_1} + \dots + m_n \dfrac{\partial \Phi_0}{\partial x_n}}{m_1 \dfrac{\partial F_0}{\partial x_1} + \dots + m_n \dfrac{\partial F_0}{\partial x_n}}, \quad \dots\dots(13)$$

and the series (12) is determined except for the term independent of $y_1, \dots y_n$, which remains arbitrary.

The equation (7) may now be solved similarly for Φ_2, and so in succession the terms of the series (3) may be constructed.

Under the assumed non-existence of identities of the form (10) no denominator in the series for $\Phi_1, \Phi_2, \dots$ will vanish identically. In general also no coefficient $B_{m_1,\dots m_n}$ in the series for F_1 will contain as factor the corresponding divisor

$$m_1 \frac{\partial F_0}{\partial x_1} + \dots + m_n \frac{\partial F_0}{\partial x_n},$$

and since Φ_0 is not a function of F_0 the numerator of (13) will not contain the denominator as a factor. Thus in general a term of the series (12) will become infinite if $x_1, \dots x_n$ be given such values as to satisfy any relation of the form (10) for which not all the integers $m_1, \dots m_n$ are zero. In any case in which the derivatives of F_0 do not reduce to mere constants, such sets of values of $x_1, \dots x_n$ are everywhere dense in that domain of real values of these variables for which the derivatives of F_0 are finite and continuous*.

We have therefore constructed a series which is formally an integral of the equations (1), but as regards $x_1, \dots x_n$ it has no real finite domain of convergence in common with that for which the right-hand sides of the equations (1) are real, finite and continuous. The conditions initially laid down for Φ are therefore not satisfied.

§ 2. *Example of the process of* § 1. In the following example the multiple-Fourier series for F_1 reduces to a single term, so that the conditions for the applicability of Poincaré's Theorem are not satisfied; it will be found in fact that the series obtained for Φ has a finite domain of convergence. The example is interesting because this series can be summed in finite terms.

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* This statement requires some justification. If P be any point (at which the derivatives of F_0 are finite and continuous) of a space for which $x_1, \dots x_n$ are Cartesian coordinates, we may determine a locus, L_0 say, of the family (10), in which we regard $m_1, \dots m_n$ as *continuously variable* parameters, to pass through P. Allowing $m_1, \dots m_n$ to vary continuously from their values for L_0, the locus represented by (10) will undergo a continuous displacement in the neighbourhood of P. In any such variation, however small, $m_1, \dots m_n$ simultaneously pass through rational values; thus, since $m_1, \dots m_n$ enter (10) only through their ratios, we can find a locus of the family for which $m_1, \dots m_n$ have integral values, to pass arbitrarily near P.

Let
$$F_0 = \lambda_1 x_1 + \lambda_2 x_2 - \nu_1 x_1^2 - \nu_2 x_2^2,$$
$$F_1 = x_1^{a_1} x_2^{a_2} e^{i(m_1 y_1 + m_2 y_2)},$$
where $\lambda_1, \lambda_2, \nu_1, \nu_2, a_1, a_2$ are any constants, and m_1, m_2 are integers. For Φ_0 we may take any function of x_1, x_2 independent of F_0, and shall take
$$\Phi_0 = \lambda_1 x_1 - \nu_1 x_1^2.$$
The equation (6) is then
$$(\lambda_1 - 2\nu_1 x_1)\frac{\partial \Phi_1}{\partial y_1} + (\lambda_2 - 2\nu_2 x_2)\frac{\partial \Phi_1}{\partial y_2} = i(\lambda_1 - 2\nu_1 x_1) m_1 x_1^{a_1} x_2^{a_2} e^{i(m_1 y_1 + m_2 y_2)},$$
which is satisfied by
$$\Phi_1 = \frac{m_1(\lambda_1 - 2\nu_1 x_1) x_1^{a_1} x_2^{a_2} e^{i(m_1 y_1 + m_2 y_2)}}{m_1(\lambda_1 - 2\nu_1 x_1) + m_2(\lambda_2 - 2\nu_2 x_2)}.$$
Since Φ_1 is of the form ψF_1, where ψ is a function only of x_1, x_2, the equation (7) reduces to
$$(\lambda_1 - 2\nu_1 x_1)\frac{\partial \Phi_2}{\partial y_1} + (\lambda_2 - 2\nu_2 x_2)\frac{\partial \Phi_2}{\partial y_2} - F_1\left(\frac{\partial \psi}{\partial x_1}\frac{\partial F_1}{\partial y_1} + \frac{\partial \psi}{\partial x_2}\frac{\partial F_1}{\partial y_2}\right) = 0,$$
i.e.
$$(\lambda_1 - 2\nu_1 x_1)\frac{\partial \Phi_2}{\partial y_1} + (\lambda_2 - 2\nu_2 x_2)\frac{\partial \Phi_2}{\partial y_2}$$
$$- \frac{2im_1 m_2\{m_2\nu_2(\lambda_1 - 2\nu_1 x_1) - m_1\nu_1(\lambda_2 - 2\nu_2 x_2)\} F_1^2}{\{m_1(\lambda_1 - 2\nu_1 x_1) + m_2(\lambda_2 - 2\nu_2 x_2)\}^2} = 0,$$
giving
$$\Phi_2 = \frac{AF_1^2}{D^3},$$
where we have written for brevity
$$A = m_1 m_2\{m_2\nu_2(\lambda_1 - 2\nu_1 x_1) - m_1\nu_1(\lambda_2 - 2\nu_2 x_2)\},$$
$$D = m_1(\lambda_1 - 2\nu_1 x_1) + m_2(\lambda_2 - 2\nu_2 x_2).$$
The equation for Φ_3 is now
$$(\lambda_1 - 2\nu_1 x_1)\frac{\partial \Phi_3}{\partial y_1} + (\lambda_2 - 2\nu_2 x_2)\frac{\partial \Phi_3}{\partial y_2} + (F_1, \Phi_2) = 0,$$
which reduces in a similar manner to that for Φ_2 and gives
$$\Phi_3 = \frac{2A(\nu_1 m_1^2 + \nu_2 m_2^2) F_1^3}{D^5}.$$
We now find in succession
$$\Phi_4 = \frac{5A(\nu_1 m_1^2 + \nu_2 m_2^2)^2 F_1^4}{D^7} = 2 \,.\, \frac{5}{4} \,.\, \frac{(\nu_1 m_1^2 + \nu_2 m_2^2) F_1}{D^2} \,.\, \Phi_3,$$
$$\Phi_5 = \frac{14A(\nu_1 m_1^2 + \nu_2 m_2^2)^3 F_1^5}{D^9} = 2 \,.\, \frac{7}{5} \,.\, \frac{(\nu_1 m_1^2 + \nu_2 m_2^2) F_1}{D^2} \, \Phi_4,$$
and it may be verified by induction that
$$\Phi_{n+1} = 2 \,.\, \frac{(2n-1)}{n+1} \,.\, \frac{(\nu_1 m_1^2 + \nu_2 m_2^2) F_1}{D^2} \,.\, \Phi_n.$$

The terms $\mu^2\Phi_2 + \mu^3\Phi_3 + \dots$ are thus to a constant factor the later terms of a binomial expansion, and we find that the series for Φ is the expansion in powers of μ of

$$\Phi = \lambda_1 x_1 - \nu_1 x_1^2 + \mu \frac{\nu_1 m_1^2}{\nu_1 m_1^2 + \nu_2 m_2^2} F_1$$

$$- \frac{AD}{2\,(\nu_1 m_1^2 + \nu_2 m_2^2)^2} \left(\sqrt{1 - \frac{4\mu\,(\nu_1 m_1^2 + \nu_2 m_2^2)\,F_1}{D^2}} - 1\right).$$

It is easy to verify directly that $(F, \Phi) = 0$. We observe that the series for Φ is convergent for no values of x_1, x_2 which make $D = 0$ unless $\mu = 0$, and, moreover, that such values are not in general singularities of Φ, since the part of Φ involving D is

$$\sqrt{D^2 - 4\mu\,(\nu_1 m_1^2 + \nu_2 m_2^2)\,F_1} - D.$$

§ 3. *Integrals not expansible in powers of* μ. We have shown in § 1 that unless the expansion of F satisfies certain special conditions, there exists no integral Φ which is a function of $x_1, \dots x_n, y_1, \dots y_n$, μ, periodic with respect to $y_1, \dots y_n$ with period 2π, and analytic throughout a domain D for which $|\,\mu\,|$ is sufficiently small, $x_1, \dots x_n$ lying within fixed finite real ranges, and $y_1, \dots y_n$ taking all real values. The question now arises whether this theorem remains true if we no longer require that Φ is to be developable in powers of μ: in other words, can we conclude the non-existence of integrals analytic throughout a domain D' for which μ has a fixed value, or lies in a certain real range not containing the point $\mu = 0$, while the x's and ϕ's lie in ranges of the same nature as before?

It seems to have been generally assumed that the answer to this question is in the affirmative. Thus Poincaré states that the development in powers of μ is an unessential restriction to his theorem, though his remarks in support of this statement* are hardly convincing. Levi-Cività†, and following him Whittaker‡, state that for the Problem of Three Bodies the theorem disproves the existence of integrals uniform with respect to the Keplerian variables. Fermi§ holds that the theorem, if extended so as to include the non-existence of Invariant Relations expansible in powers of μ, establishes for dynamical systems in general the property of "quasi-continuity of path." While various writers on the Quantum Theory make the theorem an important link in the argument that the quantum conditions as at present understood do not apply to dynamical systems in general, and that such systems cannot give sharp spectral lines‖.

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* *Méthodes Nouvelles*, § 87.

† *Acta Math.*, xxx (1905), 305.

‡ *Analytical Dynamics* (2nd ed.), p. 385.

§ *Phys. Zeit.*, xxiv (1923), 261.

‖ See, for example, Born and Pauli, *Zeit. für Phys.*, x, 137, § 3.

I shall now proceed to show that such conclusions are not justified, by establishing, for a large class of equations to which Poincaré's Theorem applies, the existence of integrals independent of F which fulfil all Poincaré's conditions except that they are not expansible in powers of μ. For this purpose I utilize a result established in this number of the *Proceedings**, viz. that if in the equations

$$\frac{dx_r}{dt} = X_r \quad (r = 1, \dots m), \qquad \text{......(14)}$$

the quantities $X_1, \dots X_m$ are convergent power series in $x_1, \dots x_m$ which do not all vanish when $x_1 = \dots = x_m = 0$, there exist $(m-1)$ independent convergent power series

$$\Phi_r(x_1, \dots x_m) \quad (r = 1, \dots m-1) \qquad \text{......(15)}$$

such that

$$\Phi_r = \text{const.}$$

is an integral of the equations. Supposing that (14) are of Hamiltonian form, the conjugate sets of variables being $\xi_1, \dots \xi_n$ and $\eta_1, \dots \eta_n$, the first degree terms in the expansion of the characteristic function F in powers of these quantities must therefore not be all zero. We shall suppose that F is expanded in powers of a parameter μ

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots,$$

where $F_0, F_1, \dots$ are separately expansible in powers of $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$, and that it becomes of Poincaré's form under the contact transformation

$$\xi_r = \sqrt{2x_r} \cos y_r, \quad \eta_r = \sqrt{2x_r} \sin y_r \quad (r = 1, \dots n). \qquad \text{......(16)}$$

Since F_0 is to be a function of $x_1, \dots x_n$ only it must before transformation be a power series in $\xi_1^2 + \eta_1^2, \dots \xi_n^2 + \eta_n^2$. Moreover, for Poincaré's theorem to be applicable the derivatives of F_0 are not to reduce to constants, and are to be connected by no identity of the form (10), and no coefficient $B_{m_1, \dots m_n}$ in F_1 is to contain as factor the corresponding divisor

$$m_1 \frac{\partial F_0}{\partial x_1} + \dots + m_n \frac{\partial F_0}{\partial x_n}.$$

Of these conditions the first two will be satisfied in general if F_0 is not merely a linear function of $\xi_1^2 + \eta_1^2, \dots \xi_n^2 + \eta_n^2$, e.g. if

$$F_0 = \lambda_1 (\xi_1^2 + \eta_1^2) + \dots + \lambda_n (\xi_n^2 + \eta_n^2) + \nu_1 (\xi_1^2 + \eta_1^2)^2 + \dots + \nu_n (\xi_n^2 + \eta_n^2)^2,$$

where the λ's and ν's are constants, while the third is obviously satisfied in general. Thus the equations

$$\frac{d\xi_r}{dt} = \frac{\partial F}{\partial \eta_r}, \quad \frac{d\eta_r}{dt} = -\frac{\partial F}{\partial \xi_r} \quad (r = 1, \dots n) \qquad \text{......(17)}$$

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* "Integrals of Systems of Ordinary Differential Equations," p. 273, above.

possess in general no integral apart from F periodic in $y_1, \dots y_n$ and expansible in a series of powers of μ convergent for sufficiently small values of $|\mu|$ and for finite real ranges of $x_1, \dots x_n$; but they have been seen to possess always $(2n-1)$ integrals (of which $(2n-2)$ must be independent of F) expansible in series of powers of $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$ convergent when the moduli of these arguments are sufficiently small. On applying the transformation (16) the equations (17) take Poincaré's form (1), while the integrals become periodic with respect to $y_1, \dots y_n$, and are given by series convergent for all real values of $y_1, \dots y_n$ and for sufficiently small positive values of $x_1, \dots x_n$.

Thus all systems of equations of Poincaré's form ((1) above) *will possess* $(2n-2)$ *independent integrals, independent of* F, *which fulfil all Poincaré's conditions except that they are not expansible in powers of* μ, *provided the characteristic function* F *transforms under* (16) *to a convergent series of powers of* $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$ *in which the linear terms are not identically zero.* This condition is sufficient for the existence of integrals as stated; I do not enquire here into the necessary conditions, as my object is merely to show that Poincaré's theorem of non-existence is untrue if we relax the restriction relating to development in powers of μ.

It is easy to verify that the series giving the integrals additional to F are not expansible in powers of μ. This is because the expansion of F_0 starts with quadratic terms in $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$, so that the linear terms in F must be divisible by μ, and on transforming the equations linearly into the form (2) on p. 274 of the paper cited above the quantity λ will contain μ as a factor. But it is readily seen that the process of § 3 of that paper gives a series in which certain terms of degree r contain the factor λ^{-r} uncompensated by any positive powers of λ. Hence the integrals constructed are not expansible in powers of μ, but have all the points for which $\mu = 0$ as essential singularities.

§ 4. *Concluding remarks.* The result of the last section is that Poincaré's theorem of non-existence of integrals applies only to integrals expansible in powers of μ. This suggests that if the complete solution of the equations (1) be exhibited in the form

$$x_r, y_r = \phi\,(t + c_1, c_2, \dots c_{2n}, \mu),$$

where $c_1, \dots c_{2n}$ are constants of integration, the functions ϕ would be found to possess some sort of singularity for $\mu = 0$. This throws a striking light upon practically the whole of Poincaré's conclusions respecting the complication of such problems as that of Three Bodies—conclusions drawn for the most part from the divergence of certain expansions in powers of μ or $\sqrt{\mu}$, and resting indirectly upon his theorem of non-existence of "uniform" integrals; for it suggests that the divergence of these series is due to the expansion of the functions they represent in an unnatural manner about a singular point, and not to any great complication in the functions themselves.

It is not difficult to show that the series (15) giving the integrals of the equations (17) are convergently expansible about any value of μ for which the derivatives of F do not all vanish when

$$\xi_1 = \dots = \xi_n = \eta_1 = \dots = \eta_n = 0; \qquad \dots\dots(18)$$

the singling out of the value $\mu = 0$ is therefore correlated with the fact that the origin (18) is then a singular point of the equations*. We are thus brought back to the fact, which has been noted in § 5 of the paper already quoted†, that the problem of integrals developable about a singular point is of far greater complication than that of integrals developable about an ordinary point.

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* If the equations be taken in Poincaré's form (1), the special property of the value $\mu = 0$ is that then the equations simplify in such a far-reaching manner that their solution becomes trivial; we thus reach the paradoxical conclusion that expansions in powers of μ are unsuitable for the precise reason which originally led to their adoption! In fact, as has been remarked by Poincaré, we get into unnecessary difficulties by taking too simple a first approximation: it is this same idea which underlies Gyldén's contributions to Celestial Mechanics.

† See p. 281, above.

*On the zeros of the Riemann zeta-function**. By Mr J. E. LITTLEWOOD, Trinity College, Cayley Lecturer.

[*Received* 16 June 1924.]

1. Let $N(T)$ denote, as usual, the number of zeros of $\zeta(s)$ whose imaginary part γ satisfies $0 < \gamma < T$, and $N(\sigma, T)$ the number of these for which, in addition, the real part is greater than σ. In this definition we suppose, in the first place, that no zero actually lies on the line $t = T$: if the line contains zeros we define

$$N(T) = \lim_{\epsilon \to 0} \tfrac{1}{2}\{N(T+\epsilon) + N(T-\epsilon)\},$$

$$N(\sigma, T) = \lim_{\epsilon \to 0} \tfrac{1}{2}\{N(\sigma, T+\epsilon) + N(\sigma, T-\epsilon)\}\dagger.$$

Zeros on a boundary $t =$ *constant* are, in fact, only to count half the full value‡.

We define a function $f(s) = \log \zeta(s)$ as follows. In the first place when $\sigma > 1$,

$$f(s) = \sum_{m,p} \frac{1}{m} p^{-ms},$$

the familiar Dirichlet series. Next, if no zero of $\zeta(s)$ has ordinate t, we define $f(\sigma + it)$ to be the value obtained from $f(2 + it)$ by continuous variation of the argument along the straight line from $2 + it$ to $\sigma + it$. Finally, for a t which is the ordinate of a zero we define

$$f(\sigma + it) = \lim_{\epsilon \to 0} \tfrac{1}{2}\{f(\sigma + it + i\epsilon) + f(\sigma + it - i\epsilon)\}.$$

It is known§ that, as $T \to \infty$,

$$(1.1) \qquad N(T) = M(T) + S(T) + \tfrac{7}{8} + O(T^{-1})$$
$$= M(T) + S(T) + O(1),$$

where

$$(1.2) \qquad M(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi},$$

$$(1.3) \qquad S(T) = \frac{1}{\pi} \mathfrak{I} f(\tfrac{1}{2} + iT).$$

It is known further that

$$(1.4) \qquad \mathfrak{I} f(s) = O(\log t) \quad (\sigma \geqslant \tfrac{1}{2}),$$

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* Most of the results of the paper were announced, without proof, in the *Records of Proceedings of the London Mathematical Society*, February 10, 1921.

† These equations are, of course, then valid for *all* T.

‡ We adhere strictly, however, to the condition $\beta > \sigma$ for the real part of a zero counted in $N(\sigma, T)$. A halving convention here has something against it and nothing in its favour.

§ R. Backlund, *Acta Mathematica*, 41 (1918), 345–375. With our conventions the results are evidently true for all T if true for "*wurzelfrei*" T.

and in particular

$$(1.5) \qquad S(T) = O(\log t).$$

The present paper is devoted to the study of the functions $N(\sigma, T)$ and $S(T)$: it falls naturally into two parts. In Theorem 2 we prove that

$$\int_\sigma^1 N(\sigma, T)\, d\sigma \text{ and } \frac{1}{2\pi}\int_0^T \log |\zeta(\sigma + it)|\, dt$$

on the one hand, and in Theorem 3 that

$$\int_0^T \mathfrak{I} f(\sigma + it)\, dt \text{ and } \int_\sigma^\infty \log |\zeta(\sigma + iT)|\, d\sigma$$

on the other, are nearly equal when $\sigma \geqslant \frac{1}{2}$. From the first result we deduce upper bounds for $N(\sigma, T)$ when σ is near the value $\frac{1}{2}$; and from the second it follows eventually that

$$\int_0^T S(t)\, dt = O(\log T),$$

an upper bound which may be improved to $o(\log T)$ if we assume the hypothesis of Lindelöf (*a fortiori* if we assume Riemann's hypothesis). These are the principal results of the first part. In the second we assume the truth of the Riemann hypothesis, and prove, among other things, that

$$S(T) = O\left(\frac{\log T}{\log\log T}\right).$$

Throughout the paper A's and a's denote positive absolute constants, not always the same from one occurrence to another*. On the other hand, an A or a affected with a suffix is the same wherever it occurs. O's are absolute unless the contrary is stated†.

When we are concerned with approximations valid when $T \to \infty$ we need only consider values of T greater than any convenient constant. We shall suppose in such cases, without always making an explicit statement, that T satisfies any inequality $T > A$ which is required by the run of the argument.

2. We begin by collecting some results whose proof would interrupt the development of later arguments.

Lemma 1.

$$\int_2^T |\zeta(\sigma + it)|^2\, dt < AVT \quad (T \geqslant 2, \tfrac{1}{2} \leqslant \sigma \leqslant 2),$$

where
$$V = \text{Min}\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right).$$

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* Thus we write, e.g. $(\sin\theta + 2)^a < (A + A)^a < A$.

† $f(x) = O\{\phi(x)\}$ means $|f(x)/\phi(x)| < K \; (x > x_0)$: an O involves two constants, K and x_0. Our convention is that both are A's.

The lemma is closely allied to known results[*], but it does not actually follow from any published theorem, and a proof must be given *de novo*. We have[†], for $\frac{1}{2} \leqslant \sigma \leqslant 2$, $t \geqslant 2$,

$$|\zeta(s)| = \left|\sum_{n<t} n^{-s} + O(1)\right| < \left|\sum_{n<t}\right| + A,$$

$$(2.1) \qquad \int_2^T |\zeta(\sigma+it)|^2\,dt \leqslant 2\int_2^T \left|\sum_{n<t} n^{-s}\right|^2 dt + 2A.$$

Now

$$\int_2^T \left|\sum_{n<t} n^{-s}\right|^2 dt = \int_2^T \sum_{m,n<t} (mn)^{-\sigma}\left(\frac{m}{n}\right)^{it} dt = \sum_{m,n<T}\int_{T_1}^T (mn)^{-\sigma}\left(\frac{m}{n}\right)^{it} dt,$$

where $T_1 = \text{Max}\,(m, n, 2)$,

$$\leqslant \sum_{m=n}\int_{T_1}^T + \sum_{m \neq n}\left|\int_{T_1}^T\right|$$

$$(2.2) \qquad \leqslant \sum_{m<T} Tm^{-2\sigma} + 2\sum_{m,n<T}{}' (mn)^{-\sigma} \Big/ \left|\log\frac{n}{m}\right| = S_1 + S_2$$

say, where the dash denotes that terms for which $m = n$ are to be omitted in Σ'. Now

$$(2.3) \qquad S_1 \leqslant T\,\text{Min}\left(\sum_{m<T} m^{-1}, \sum_1^\infty m^{-2\sigma}\right)$$

$$< AT\,\text{Min}\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right) = AVT.$$

Also if $n = m + r$ we have

$$n^{-\sigma}\Big/\left|\log\frac{n}{m}\right| < \begin{cases} Am^{-\sigma}\Big/\dfrac{|r|}{m} & (|r| < \frac{1}{2}m) \\ An^{-\sigma} & (|r| \geqslant \frac{1}{2}m). \end{cases}$$

Hence

$$S_2 < \sum_{m<T} m^{-\sigma}\left(m^{1-\sigma}A\sum_{r<\frac{1}{2}m}\frac{1}{r} + A\sum_{n<T} n^{-\sigma}\right)$$

$$< A\sum_{m<T} m^{1-2\sigma}\log(m+1) + A\left(\sum_{m<T} m^{-\sigma}\right)^2$$

$$< A\int_1^T u^{1-2\sigma}\log(u+1)\,du + A\log T + A\left(\sum_{m<T} m^{-\frac{1}{2}}\right)^2.$$

If $\sigma > \frac{3}{4}$,

$$S_2 < A\int_1^T u^{-\frac{1}{2}}Au^{\frac{1}{4}}\,du + AT + AT < AT < AVT,$$

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[*] See G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 21 (1921), 39–74.

[†] *Loc. cit.*, 53.

since $V \geqslant \frac{1}{\sigma - \frac{1}{2}} \geqslant \frac{1}{2 - \frac{1}{2}} = A$ when T is large. If $\sigma \leqslant \frac{3}{4}$

$$S_2 < A \left[\frac{u^{2-2\sigma} \log (u+1)}{2 - 2\sigma}\right]_1^T - \int_1^T \frac{u^{2-2\sigma}}{2-2\sigma} \frac{du}{u+1} + AT$$

$$< \frac{AT^{2-2\sigma} \log (T+1)}{2 - \frac{3}{2}} + AT < AT \log T \,.\, T^{-2(\sigma - \frac{1}{2})} + AT.$$

The first term does not exceed $AT \log T$ on the one hand, and is equal to

$$\frac{T}{2(\sigma - \frac{1}{2})} xe^{-x} \leqslant \frac{T}{2(\sigma - \frac{1}{2})} \quad (x = (2\sigma - 1) \log T)$$

on the other. Thus

$$S_2 < AVT + AT < AVT$$

also in this case, and so for $\frac{1}{2} \leqslant \sigma \leqslant 2$. From this and (2.3) the lemma follows.

3. *Lemma* 2. *Suppose that* z_1 *and* z_2 *lie within* $|z| \leqslant 4$, *and that* $z_2 - z_1$ *is real, positive, and not greater than* η. *Then*

$$\left| \mathfrak{R} \int_{z_1}^{z_2} \log z \, dz \right| < A\eta \log \left(\frac{1}{\eta} + 2\right),$$

the integral being taken along a straight line.

This is, of course, nearly trivial. Suppose first that $|z_1| > 2\eta$. Then $|z| > \eta$ for any point of the path of integration, and so

$$|\mathfrak{R} \log z| \leqslant \left|\log \frac{1}{\eta}\right| + \log 4,$$

$$\left|\mathfrak{R} \int_{z_1}^{z_2} \log z \, dz\right| \leqslant \eta \left(\left|\log \frac{1}{\eta}\right| + \log 4\right) < A\eta \log \left(\frac{1}{\eta} + 2\right).$$

Suppose now $|z_1| \leqslant 2\eta$. Then $|z_2| \leqslant 3\eta$. Also, $\log z$ being interpreted to have its principal value, we have, writing

$$r_1 = |z_1|, \quad r_2 = |z_2|,$$

(3.1)

$$\left|\int_{z_1}^{z_2} \log z \, dz\right| \leqslant \left|\int_0^{z_1}\right| + \left|\int_0^{z_2}\right| \leqslant |z_1 \log z_1| + |z_1| + |z_2 \log z_2| + |z_2|$$

$$\leqslant r_1 |\log r_1| + (\pi + 1) r_1 + r_2 |\log r_2| + (\pi + 1) r_2.$$

Since $r |\log r|$ increases to a maximum at $r = e^{-1}$, then decreases to a minimum at $r = 1$, and then increases, we have

$$r_1 |\log r_1| \leqslant \begin{cases} 3\eta \log \dfrac{1}{3\eta} & (3\eta < e^{-1}) \\ A & (3\eta > e^{-1}), \end{cases}$$

and in either case

$$r_1 \mid \log r_1 \mid + Ar_1 < A\eta \log \left(\frac{1}{\eta} + 2\right).$$

Since a similar result holds for r_2 (3.1) leads to the desired conclusion also when $\mid z_1 \mid \leqslant 2\eta$.

Lemma 3. *Let C_1, C_2, C_3 be three simple closed curves, C_3 containing C_2, and C_2 containing C_1. Suppose that $f(z)$ is regular and satisfies $|f| \leqslant M_3$ in the interior of C_3, and satisfies $|f| \leqslant M_1$ on C_1. Then within and upon C_2*

$$|f| \leqslant M_1^{\alpha} M_3^{\beta},$$

where α, β are positive numbers satisfying $\alpha + \beta = 1$, and depending only on the geometrical configuration of the set of curves (not on its absolute position or orientation).

This generalization by conformal representation of Hadamard's "three circles" theorem is a known result*.

4. Suppose $\phi(s)$ is meromorphic in and upon the boundary of a rectangle bounded by the lines $t = 0$, $t = T$; $\sigma = \alpha$, $\sigma = \beta$ $(\beta > \alpha)$; and regular and never zero on the side, l, of the rectangle for which $\sigma = \beta$. The function $\log \phi(s)$ is regular in a neighbourhood containing l, and we define

$$F(s) = \log \phi(s)$$

for s of l. For other s of the rectangle we define $F(s)$ to be the value obtained from $\log \phi(\beta + it)$ by continuous variation along $t = constant$ from $\beta + it$ to $\sigma + it$. This provided the path does not cross a zero or pole of $\phi(s)$; if it does we define

$$F(s) = \tfrac{1}{2} \lim \{F(\sigma + it + i\epsilon) + F(\sigma + it - i\epsilon)\}.$$

Let now $\nu(\sigma', T)$ denote the excess of the number of zeros over the number of poles in the part of the rectangle for which $\sigma > \sigma'$, zeros or poles on $t = 0$ or $t = T$ counting one-half only. Then we have

Theorem 1. $$\int F(s)\, ds = -2\pi i \int_{\alpha}^{\beta} \nu(\sigma, T)\, d\sigma,$$

the integral on the left being taken round the rectangle in the positive direction.

This formula has obvious affinities with Jensen's well-known theorem. Various proofs are possible; the one given here depends on the inversion of the order of a double integration. We may suppose $t = 0$ and $t = T$ to be free of zeros and poles of $\phi(s)$; it is easily verified that our halving conventions then ensure the truth of the theorem in the general case. We have

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* H. Bohr, E. Landau, and J. E. Littlewood, *Bull. Acad. Belgique*, 15 (1913), 1144–1175, 1166.

$$\int F(s)\,ds = \int_{\alpha}^{\beta} F(\sigma)\,d\sigma - \int_{\alpha}^{\beta} F(\sigma+iT)\,d\sigma + \int_{0}^{T}\{F(\beta+it) - F(\alpha+it)\}\,idt$$

$$= \int_{\alpha}^{\beta} F(\sigma)\,d\sigma - \int_{\alpha}^{\beta} F(\sigma+iT)\,d\sigma + \int_{0}^{T} idt \int_{\alpha}^{\beta} \frac{\phi'}{\phi}(\sigma+it)\,d\sigma^{*}$$

$$= \int_{\alpha}^{\beta} F(\sigma)\,d\sigma - \int_{\alpha}^{\beta} F(\sigma+iT)\,d\sigma + \int_{\alpha}^{\beta} d\sigma \int_{\sigma}^{\sigma+iT} \frac{\phi'}{\phi}(s)\,ds.$$

Now by the theorem of residues the (inner) integral last written is equal to

$$\int_{\sigma}^{\beta} \frac{\phi'}{\phi}(\sigma)\,d\sigma + \int_{0}^{T} \frac{\phi'}{\phi}(\beta+it)\,idt - \int_{\sigma}^{\beta} \frac{\phi'}{\phi}(\sigma+iT)\,d\sigma - 2\pi i\nu(\sigma, T).$$

Hence

$$\int F(s)\,ds = \int_{\alpha}^{\beta}\{F(\sigma) - F(\sigma+iT) + F(\beta) - F(\sigma) + F(\beta+iT)$$

$$- F(\beta) - F(\beta+iT) + F(\sigma+iT) - 2\pi i\nu(\sigma, T)\}\,d\sigma$$

$$= -2\pi i \int_{\alpha}^{\beta} \nu(\sigma, T)\,d\sigma.$$

5. *Theorem* 2.

(5.1)

$$2\pi \int_{\sigma}^{1} N(\sigma, T)\,d\sigma = \int_{0}^{T} \log|\zeta(\sigma+it)|\,dt + c(\sigma) + \mathfrak{I}\int_{\sigma}^{\infty} f(\sigma+iT)\,d\sigma$$

(5.2) $$= \int_{0}^{T} \log|\zeta(\sigma+it)|\,dt + O(\log T) \quad (-1 \leqslant \sigma \leqslant 1),$$

where $c(\sigma)$ *is a constant depending only on* σ.

Theorem 3.

(5.3)

$$\int_{0}^{T} \mathfrak{I} f(\sigma+it)\,dt = \int_{\sigma}^{\infty} \log|\zeta(\sigma+iT)|\,d\sigma + c'(\sigma) \quad (-1 \leqslant \sigma \leqslant 2),$$

where $c'(\sigma)$ *is a constant depending only on* σ. *In particular*

(5.4) $$\int_{0}^{T} S(t)\,dt = \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \log|\zeta(\sigma+iT)|\,d\sigma + c_1$$

(5.5) $$= \frac{1}{\pi}\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma+iT)|\,d\sigma + O(1).$$

---

[*] It is easily verified that $\int d\sigma \int |\phi'/\phi|\,dt$ exists: in fact near a singularity $\sigma_0 + it_0$ of ϕ'/ϕ we have

$$\int |\phi'/\phi|\,dt = O(\log|\sigma - \sigma_0|)$$

and the integral of this with respect to σ converges. We may invert the order of integration and ignore lines on which the inner integrals do not exist.

Theorems 2 and 3 are simple consequences of Theorem 1 and known results. We take in Theorem 1 $\alpha = \sigma$, β large and positive, $\phi(s) = \zeta(s)$. Then (on account of the pole at $s = 1$)

$$\nu(\sigma, T) = \begin{cases} N(\sigma, T) - \frac{1}{2} & (\sigma < 1) \\ 0 & (\sigma \geqslant 1). \end{cases}$$

Theorem 1 gives

$$-2\pi i \int_\sigma^1 \nu(\sigma, T)\, d\sigma = -2\pi i \int_\sigma^\beta \nu(\sigma, T)\, d\sigma$$

$$= \int_\sigma^\beta f(\sigma)\, d\sigma - \int_\sigma^\beta f(\sigma + iT)\, d\sigma - \int_0^T f(\sigma + it)\, idt + \int_0^T f(\beta + it)\, idt.$$

If in this we make $\beta \to \infty$ the last term disappears, and the first two integrals tend to integrals $\int_\sigma^\infty$. Taking real and imaginary parts we obtain (5.1) and (5.3), and observe further that c_σ, c_σ' are of the form $O(1)$ for $\sigma \geqslant -1$. (5.2) now follows from (5.1) in virtue of (1.4)*, (5.4) is the special case $\sigma = \frac{1}{2}$ of (5.3), and (5.5) follows at once.

6. We prove next the following results.

Theorem 4.

(6.1)

$$\int_\sigma^1 N(\sigma, T)\, dt = O(T) \operatorname{Min}\left(\log\log T, \log \frac{1}{\sigma - \frac{1}{2}}\right) \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1).$$

In particular

(6.2) $$\int_{\frac{1}{2}}^1 N(\sigma, T)\, d\sigma = O(T \log\log T).$$

Theorem 5.

(6.3) $$N(\sigma, T) = O\left(T \frac{1}{\sigma - \frac{1}{2}} \log \frac{1}{\sigma - \frac{1}{2}}\right) \quad (\tfrac{1}{2} < \sigma \leqslant 1).$$

From (5.1)

$$2\pi \int_\sigma^1 N(\sigma, T)\, d\sigma = \int_2^T \log |\zeta(\sigma + it)|\, dt + O(\log T)$$

$$= \tfrac{1}{2} \int_2^T \log |\zeta(\sigma + it)|^2\, dt + O(\log T)$$

$$\leqslant \tfrac{1}{2}(T - 2) \log \left\{\int_2^T |\zeta(\sigma + it)|^2\, dt \middle/ (T - 2)\right\} + O(\log T)$$

$$< AT \log \left\{\operatorname{Min}\left[O(\log T),\ O\left(\log \frac{1}{\sigma - \frac{1}{2}}\right)\right]\right\} + O(\log T),$$

---

* Since $f = O(2^{-\sigma})$ $(\sigma \geqslant 2)$, so that $\int_2^\infty f\, d\sigma = O(1)$.

by Lemma 1; and this proves (6.1). We now observe that if $\sigma_1 = \frac{1}{2} + \frac{1}{2}(\sigma - \frac{1}{2})$,

$$N(\sigma, T) \leqslant \frac{1}{\sigma_1 - \sigma} \int_{\sigma_1}^{\sigma} N(\sigma, T)\, d\sigma \leqslant \frac{2}{\sigma - \frac{1}{2}} \int_{\sigma_1}^{1} N(\sigma, T)\, d\sigma$$

$$< AT(\sigma - \tfrac{1}{2})^{-1} \log(\sigma_1 - \tfrac{1}{2})^{-1} < AT(\sigma - \tfrac{1}{2})^{-1} \log(\sigma - \tfrac{1}{2})^{-1},$$

which is (6.3).

Theorem 6. *If $\phi(T)$ is positive, and increases to ∞ with T, then all but an infinitesimal proportion of the zeros of $\zeta(s)$ in the upper half plane lie in the region*

$$(6.4) \qquad |\sigma - \tfrac{1}{2}| < \phi(t) \frac{\log \log t}{\log t}, \quad t > e^2.$$

That is to say: the number of zeros outside the region and with imaginary part between 0 and T is of the form $o(T \log T)$ (the total number between these limits being approximately $\frac{1}{2\pi} T \log T$). It is clearly enough to prove that for large T the number of zeros in the region

$$(6.5) \qquad \sigma \geqslant \tfrac{1}{2} + \phi(t) \frac{\log \log t}{\log t}, \quad \sqrt{T} < t < T,$$

is $o(T \log T)$. The curved boundary of (6.5) lies to the right of $\sigma = \sigma_1$, where

$$\sigma_1 - \tfrac{1}{2} = \phi(\sqrt{T}) \frac{\log \log T}{\log T}.$$

But evidently $N(\sigma_1, T) = o(T \log T)$ in virtue of Theorem 5[*], and the required result is true *a fortiori*.

It should be observed that Theorems 4, 5 and 6 are novel only so far as they concern a *small* neighbourhood of the line $\sigma = \frac{1}{2}$: for *fixed* $\sigma > \frac{1}{2}$ Carlson's well-known theorem[†] gives results much more precise than ours.

7. We turn now to the study of the function $S(T)$, or rather of its integral $\int_0^T S(t)\, dt$. We prove two theorems.

Theorem 7[‡]. $\qquad \int_0^T S(t)\, dt = O(\log T).$

---

[*] If $\sigma_1 > 1$ the result is trivial. If not, then $\sigma_1 - \frac{1}{2} \leqslant \frac{1}{2}$, and

$$\log \frac{1}{\sigma_1 - \frac{1}{2}} = O(\log \log T).$$

[†] F. Carlson, *Arkiv för Mat., Ast. och Fys.*, 15 (1920), No. 20. The details of the proof can be lightened by Theorem 2, which enables us to reduce the double integration to a single one. Carlson's theorem lies much deeper than ours for fixed σ, but it appears to throw no light on the present problem.

[‡] H. Cramér, *Math. Zeitschr.*, 4 (1919), 104–130, 122–130, proves, by an entirely different method, that $\int_0^T S(t)\, dt = O(T^{\epsilon})$.

Theorem 8. *It follows from Lindelöf's hypothesis that*

$$\int_0^T S(t)\,dt = o(\log T).$$

We begin with the proof of the more difficult Theorem 8. Lindelöf's hypothesis is that

$$|\zeta(\tfrac{1}{2}+it)| < t^{o(1)},$$

and this is known* to involve the proposition that the number of zeros of $\zeta(s)$ in any region, D,

$$\sigma \geqslant \tfrac{1}{2}+\delta, \quad |t-T| \leqslant 1,$$

is of the form $o(\log T)$†. We suppose $\delta < \frac{1}{10}$. Let C_3 be the (boundary of the) rectangle with corners at $\frac{1}{2}+2\delta+i\{T \pm (1-\delta)\}$, $2 \pm i\{T \pm (1-\delta)\}$; C_2 that with corners at

$$\tfrac{1}{2}+3\delta+i\{T\pm(1-2\delta)\},\ \ 2-\delta+i\{T\pm(1-2\delta)\};$$

C_1 that with corners at $\frac{3}{2}+i\{T\pm(1-3\delta)\}$, $\frac{5}{3}+i\{T\pm(1-3\delta)\}$. We denote by Σ_0 a summation over the zeros ρ of $\zeta(s)$ for which $|\gamma - T| \leqslant 1$, Σ_1 a summation over those zeros lying in D (the terms of Σ_1 are therefore contained in Σ_0). It is known that Σ_0 has $O(\log T)$ terms‡. Let now

$$\psi(s) = \psi(s, T, \delta) = \frac{\zeta'(s)}{\zeta(s)} - \Sigma_1 \frac{1}{s-\rho},$$

so that $\psi(s)$ is regular in C_3.

8. *Lemma* 3. *bis* $\quad \psi(s) = O\left(\dfrac{\log T}{\delta}\right)$

(*uniformly*) *for s in* C_3.

We have, for s in C_3,

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log t) + \Sigma\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)\S,$$

$$(8.1)\quad \psi(s) = O(\log t) + (\Sigma - \Sigma_1)\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + \Sigma_1 \frac{1}{\rho}$$

$$= O(\log T) + (\Sigma - \Sigma_1)\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

If Σ' denotes a summation over ρ for which $|\gamma - t| \geqslant 1$ we have||

$$(8.2)\qquad \Sigma'\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) = O(\log t) = O(\log T).$$

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* R. Backlund, *Öfversigt Finska Vetensk. Soc.*, 61 (1918–19), No. 9.

† o's are not uniform in δ. O's (by our current convention) are absolute, and so uniform in δ. We must therefore bear in mind that, e.g., a $o(\log T)$ is not necessarily of the form $O(\log T)$; for example δ^{-1} is of the first form but not of the second.

‡ E. Landau, *Handbuch*, p. 337. We refer to this treatise as *H.B.*

§ *H.B.*, pp. 316–318.

|| *H.B.*, p. 339.

Since the number of ρ within a vertical distance $O(1)$ of T is $O(\log T)$ we may write

$$\Sigma - \Sigma_1 = \Sigma' + \Sigma_2 - \Sigma_3,$$

where Σ_2 and Σ_3 are exclusive and contain $O(\log T)$ terms, and where further $|\mathfrak{I}(s-\rho)| \geqslant 1$ in Σ_3, and in Σ_2

$$|s-\rho| \geqslant \textit{distance between } C_2 \textit{ and boundary of } D \geqslant \delta.$$

Hence

$$(\Sigma - \Sigma_1)\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \Sigma'\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \Sigma_2 O\left(\frac{1}{\delta}\right) - \Sigma_3 O(1)$$
$$= O\left(\frac{\log T}{\delta}\right)$$

by (8.2). The lemma now follows from (8.1).

9. *Lemma* 4.

(9.1) $\psi(s) = o(\log T) \quad (\tfrac{1}{2} + 3\delta \leqslant \sigma \leqslant \tfrac{5}{3},\ |t - T| \leqslant \tfrac{1}{2}).$

Consider $\psi(s)$ on C_1. Here $\sigma \geqslant \tfrac{3}{2}$ and so

$$\frac{\zeta'(s)}{\zeta(s)} = O(1).$$

Since $1/(s-\rho) = O(1)$ in Σ_1 and Σ_1 has $o(\log T)$ terms, we have $\psi(s) = o(\log T)$. Let now

$$M_1 = \underset{C_1}{\text{Max}}\,|\psi|, \quad M_3 = \underset{C_3}{\text{Max}}\,|\psi|.$$

C_1 is interior to C_2 and C_2 to C_3 (since $\delta < \tfrac{1}{10}$), and Lemma 3 is applicable to $\psi(s)$, the indices α, β depending only on δ. Thus in and upon C_2

$$|\psi(s)| \leqslant M_1^{\alpha} M_3^{\beta} < \{o(\log T)\}^{\alpha}\left\{O\left(\frac{\log T}{\delta}\right)\right\}^{\beta} = o(\log T).$$

But C_2 includes the region with which the lemma is concerned.

10. We suppose now that neither t nor T is the ordinate of a ρ. Then we may integrate (9.1) between s and $\tfrac{5}{3} + it$, and obtain (all logarithms having their principal values)

$$f(s) - f(\tfrac{5}{3} + it) = \Sigma_1\{\log(s-\rho) - \log(\tfrac{5}{3} + it - \rho)\} + \int_{\frac{5}{3}+it}^{s} \psi(s)\,ds,$$

(10.1) $f(s) = O(1) + \Sigma_1 \log(s-\rho) - \Sigma_1 O(1) + o(\log T)$

$= o(\log T) + \Sigma_1 \log(s-\rho) \quad (\tfrac{1}{2} + 3\delta \leqslant \sigma \leqslant \tfrac{5}{3},\ |t - T| \leqslant \tfrac{1}{2}),$

since Σ_1 has $o(\log T)$ terms. Integrating again between the same limits (s always satisfying the conditions of (10.1))

$$\int_{\sigma}^{\frac{5}{3}} f(\sigma + it)\,d\sigma = o(\log T) + \Sigma_1 \int_{\sigma}^{\frac{5}{3}} \log(s-\rho)\,d\sigma.$$

It is clear (and follows strictly from Lemma 2 with $s - \rho = z, \eta = 2$) that the integral last written is $O(1)$. Therefore

$$\int_\sigma^2 f(\sigma+it)\,d\sigma = O(1) + \int_\sigma^{\frac{5}{3}} f(\sigma+it)\,d\sigma = o(\log T) + \Sigma_1 O(1) = o(\log T). \tag{10.2}$$

From this follows

Lemma 5. *There exists a* σ_0 *satisfying* $\tfrac{1}{2} + 3\delta \leqslant \sigma_0 \leqslant \tfrac{1}{2} + 4\delta$ *and such that*

$$\log|\zeta(s_0)| = o(\log T) \quad (s_0 = \sigma_0 + iT), \tag{10.3}$$

$$\int_{\sigma_0}^2 \log|\zeta(\sigma+iT)|\,d\sigma = o(\log T), \tag{10.4}$$

$$\Sigma_1 \log|s_0 - \rho| = o(\log T). \tag{10.5}$$

In fact (10.4) is true, by (10.2), for any $\sigma_0 \geqslant \tfrac{1}{2} + 3\delta$. Also, by the same equation,

$$\begin{aligned}\left|\frac{1}{\delta}\int_{\frac12+3\delta}^{\frac12+4\delta} \log|\zeta(\sigma+iT)|\,d\sigma\right| &\leqslant \frac{1}{\delta}\left|\int_{\frac12+3\delta}^{\frac12+4\delta} f(s)\,d\sigma\right| \\ &\leqslant \frac{1}{\delta}\left|\int_{\frac12+3\delta}^{2} f(s)\,d\sigma\right| + \frac{1}{\delta}\left|\int_{\frac12+4\delta}^{2} f(s)\,d\sigma\right| \\ &< o(\log T) + o(\log T) = o(\log T).\end{aligned}$$

Since the average of $\log|\zeta|$ over $\sigma = \tfrac{1}{2} + 3\delta$ to $\tfrac{1}{2} + 4\delta$ has a modulus $o(\log T)$ (and $\log|\zeta|$ is continuous), the value of $\log|\zeta|$ itself must be $o(\log T)$ for some $\sigma = \sigma_0$. Finally (10.5) follows from (10.1) and (10.3).

11. For $s = \sigma + iT$ we have (since $\Sigma = \Sigma_0 + \Sigma'$ when $t = T$)

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} - \Sigma_0 \frac{1}{s-\rho} \tag{11.1} \\ = O(\log T) + \Sigma'\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \Sigma_0 \frac{1}{\rho} = O(\log T) \quad (\sigma \geqslant \tfrac{1}{2}).\end{aligned}$$

Still supposing only $\sigma \geqslant \tfrac{1}{2}$, integrating between s and s_0, and taking real parts, we obtain

$$\begin{aligned}\log|\zeta(s)| - \log|\zeta(s_0)| &= \mathfrak{R}\,\Sigma_0 \int_{s_0}^{s} \frac{ds}{s-\rho} + O(\log T) \\ &= \Sigma_0 \log(s-\rho) - \Sigma_0 \log(s_0 - \rho) + O(\log T),\end{aligned}$$

and so

$$\begin{aligned}\log|\zeta(s)| &= O(\log T) + o(\log T) \tag{11.2} \\ &\quad + \Sigma_0 \log|s-\rho| - (\Sigma_0 - \Sigma_1)\log|s_0 - \rho|,\end{aligned}$$

since $\log|\zeta(s_0)|$ and $\Sigma_1 \log|s_0 - \rho|$ are $o(\log T)$.

Now for a term of $\Sigma_0 - \Sigma_1$ we have $\mathfrak{R}\rho \leqslant \frac{1}{2} + \delta$, and so

$$\mathfrak{R}\,(s_0 - \rho) \geqslant 2\delta,$$

and

$$\log|s_0 - \rho| = O\left(\log\frac{1}{\delta}\right).$$

Hence, integrating (11.2) from $\sigma = \frac{1}{2}$ to $\sigma = \sigma_0$ (a distance not exceeding 4δ), we have

$$(11.3)\quad \int_{\frac{1}{2}}^{\sigma_0} \log|\zeta(s)|\,d\sigma = O(\delta\log T) + o(\log T)$$

$$+ \Sigma_0\,\mathfrak{R}\int_{\frac{1}{2}}^{\sigma_0} \log(s-\rho)\,d\sigma + (\Sigma_0 - \Sigma_1)\,O\left(\delta\log\frac{1}{\delta}\right).$$

Now in Σ_0 $\quad \mathfrak{R}\displaystyle\int_{\frac{1}{2}}^{\sigma_0} \log(s-\rho)\,d\sigma = \mathfrak{R}\int_{z_1}^{z_2} \log z\,dz,$

where z_1, z_2 satisfy the conditions of Lemma 2, with $\eta = 4\delta$. Hence

$$(11.4)\qquad \int_{\frac{1}{2}}^{\sigma_0} \log|s-\rho|\,d\sigma = O\left(\delta\log\frac{1}{\delta}\right).$$

Since Σ_0 and $\Sigma_0 - \Sigma_1$ have $O(\log T)$ terms we have finally, from (11.3) and (11.4),

$$(11.5)\quad \int_{\frac{1}{2}}^{\sigma_0} \log|\zeta(s)|\,d\sigma = O\left(\delta\log\frac{1}{\delta}\log T\right) + o(\log T).$$

Combining (11.5) and (10.4), we obtain

$$\int_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma = o(\log T) + O\left(\delta\log\frac{1}{\delta}\log T\right),$$

and so, since the left side is independent of δ, and $\delta\log 1/\delta \to 0$ with δ,

$$(11.6)\qquad \int_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma = o(\log T).$$

It follows from Theorem 3 that $\displaystyle\int_0^T S(t)\,dt = o(\log T)$, when, as we have supposed, T is not the ordinate of a ρ. But the result, if true for such T, is true for all, and the proof of Theorem 8 is completed.

12. The proof of Theorem 7 follows similar, but much simpler lines. Let

$$\chi(s) = \chi(s, T) = \frac{\zeta'(s)}{\zeta(s)} - \Sigma_0\frac{1}{s-\rho}.$$

We suppose $t = T$, or $s = \sigma + iT$, throughout. Then (by (8.2))

$$\chi(s) = O(\log T).$$

Integration gives

$$\begin{aligned} f(s) - \Sigma_0\log(s-\rho) &= f(2+iT) - \Sigma_0\log(2+iT-\rho) + O(\log T) \\ &= O(1) + \Sigma_0\,O(1) + O(\log T) = O(\log T). \end{aligned}$$

Integrating again and taking real parts we obtain

$$\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma+iT)|\,d\sigma = \mathfrak{R}\int_{\frac{1}{2}}^{2} f(s)\,d\sigma = O(\log T) + \Sigma_0\,\mathfrak{R}\int_{\frac{1}{2}}^{2} \log(s-\rho)\,d\sigma$$

$$= O(\log T) + \Sigma_0\,\mathfrak{R}\,O(1) = O(\log T).$$

Theorem 7 follows from this and (5.5).

13. *Theorem* 9. *It follows from Lindelöf's hypothesis that*

$$S(T) = o(\log T).$$

This is a known theorem, due to Cramér*. It is, however, interesting to observe that it is an almost immediate consequence of Theorem 8. Suppose the result false. Then there exists a positive h such that either $S(T) > h \log T$ for arbitrarily large T, or else $S(T) < -h \log T$ for arbitrarily large T (both events, of course, can occur). We observe that

$$(13.1)\quad S(T+x) - S(T)$$

$$= \{N(T+x) - N(T)\} - \{M(T+x) - M(T)\} + O(1)$$

$$\geqslant -M(T+x) + M(T) - A$$

$$\geqslant -A_1 x \log T - A_2 \quad (T > 2,\ 0 \leqslant x \leqslant T),$$

and similarly

$$(13.2)\quad S(T-x) - S(T) \leqslant A_3 x \log T + A_4 \quad (T > 2,\ 0 \leqslant x \leqslant T).$$

If now $S(T) > h \log T$ for arbitrarily large T we have, writing $\xi = h/A_1$,

$$\int_0^{T+\xi} S(t)\,dt - \int_0^T S(t)\,dt = \int_0^{\xi} S(T+x)\,dx$$

$$= \xi S(T) + \int_0^{\xi} \{S(T+x) - S(T)\}\,dx$$

$$\geqslant \xi h \log T + \int_0^{\xi} (-A_1 x \log T - A_2)\,dx$$

$$= (\xi h - \tfrac{1}{2}A_1\xi^2)\log T + O(1) = \frac{h^2}{2A_1}\log T + O(1),$$

which is false, since the left side is $o(\log T)$. A similar proof disposes of the other case: we consider $\int_0^{\xi} S(T-x)\,dx$ and use (13.2).

14. We now enter upon the second part of the paper. We always assume the Riemann hypothesis, and do not always state it explicitly in enunciating our theorems and lemmas.

Let
$$S_0(T) = S(T),$$

$$S_{2n}(T) = \frac{(-1)^n}{\pi}\,\mathfrak{I}\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\int_{\sigma}^{\infty}\dots f(\sigma+iT)\,(d\sigma)^{2n} \quad (n \geqslant 1),$$

$$S_{2n-1}(T) = \frac{(-1)^{n-1}}{\pi}\,\mathfrak{R}\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\int_{\sigma}^{\infty}\dots f(\sigma+iT)\,(d\sigma)^{2n-1} \quad (n \geqslant 1).$$

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* *Math. Zeitschr.*, 2 (1918), 237–241.

Theorem 10.

$$(14.1) \qquad S_n(T) = \int_0^T S_{n-1}(t)\,dt + c_n \quad (n > 0),$$

where c_n is a constant depending only on n.

This is deducible from Theorems 2 and 3 by repeated integration, if we observe that $N(\sigma, T) = 0$ $(\sigma \geqslant \frac{1}{2})$. It is, however, unnecessary to use these theorems. In fact, since $\log \zeta$ is regular in $\sigma > \frac{1}{2}$ (except at $s = 1$) we have, for $\sigma > \frac{1}{2}$, and by continuity for $\sigma \geqslant \frac{1}{2}$,

$$i\int_0^T f(\sigma + it)\,dt + \int_\sigma^\infty f(\sigma + iT)\,d\sigma = \int_\sigma^\infty f(s)\,ds,$$

the path of the last integral lying in $t > 0$. Deforming this path into the real axis, and taking account of the discontinuity of $f(s)$ along the whole stretch $\frac{1}{2} \leqslant \sigma \leqslant 1$, $t = 0$, we have, supposing $T > 0$, $\sigma \geqslant \frac{1}{2}$,

(14.2)

$$i\int_0^T f(\sigma + it)\,dt + \int_\sigma^\infty f(\sigma + iT)\,d\sigma = \int_\sigma^\infty f(s)\,ds + iC_\sigma' = C_\sigma + iC_\sigma',$$

where C_σ, C_σ' depend only on σ, $C_\sigma' = 0$ for $\sigma > 1$, and $C_\sigma = O(2^{-\sigma})$ as $\sigma \to +\infty$. Taking real and imaginary parts in (14.2) we obtain

$$(14.3) \quad \int_0^T \mathfrak{I}f(\sigma + it)\,dt = \int_\sigma^\infty \log|\zeta(\sigma + iT)|\,d\sigma - C_\sigma',$$

$$(14.4) \quad \int_0^T \log|\zeta(\sigma + it)|\,dt = -\int_\sigma^\infty \mathfrak{I}f(\sigma + iT)\,d\sigma + C_\sigma.$$

If we write $\sigma = \frac{1}{2}$ in (14.3) we obtain (14.1) with $n = 1$. A second integration gives, by means of (14.4),

$$\begin{aligned}\int_0^T S_1(t)\,dt &= \int_{\frac{1}{2}}^\infty d\sigma \int_0^T \log|\zeta(\sigma + it)|\,dt \\ &= -\int_{\frac{1}{2}}^\infty d\sigma \int_\sigma^\infty \{\mathfrak{I}f(\sigma + iT)\,d\sigma + C_\sigma\}\,d\sigma \\ &= -\int_{\frac{1}{2}}^\infty d\sigma \int_\sigma^\infty \mathfrak{I}f(\sigma + iT)\,d\sigma + c_2.\end{aligned}$$

The process may be repeated indefinitely, (14.3) and (14.4) being used alternately.

15. *Theorem* 11. *On the Riemann hypothesis*

$$S(T) = O\left(\frac{\log T}{\log\log T}\right)\text{[*]},$$

and, more generally,

$$S_n(T) = O\left(\frac{\log T}{(\log\log T)^{n+1}}\right) \quad (n \geqslant 0).$$

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[*] Results intermediate between this and the $S(T) = O(\log T)$ which is true without the Riemann hypothesis have been given by Bohr, Cramér, and Landau.

The proof of this theorem is long and difficult, and depends on a singularly varied set of ideas. I begin with some preliminary lemmas.

Throughout the sequel $T > 4$. We shall adopt the abbreviations

$$l(t) = \log t, \quad \lambda(t) = \log\log t, \quad \mu(t) = 1/\lambda(t);$$

$$l = l(T) = \log T, \quad \lambda = \lambda(T) = \log\log T, \quad \mu = \mu(T) = 1/\lambda(T).$$

Thus we drop the argument T in l, λ, μ, but never the argument t.

Lemma 6 (i). *Let* $f(T)$ *be an increasing function,* $f(T) > 1$, *and* $f(T) = O(l^2)$. *Then for large* T *there exists a* t *between* T *and* $T + l - 2$ *for which* $f(t+2) < \tfrac{3}{2} f(t)$.

(ii) *Let* f_1, f_2 *be increasing functions of* T, *greater than unity, and of the form* $O(l^2)$. *Let* $T_1(T) = O(l)$, $T_2(T) = O(l)$. *Then for large* T *there exists a* t *between* T *and* $T + l^4$ *for which* (*simultaneously*)

$$f_1\{t + T_1(t)\} < \tfrac{3}{2} f_1(t), \tag{15.1}$$

$$f_2\{t + T_2(t)\} < \tfrac{3}{2} f_2(t). \tag{15.2}$$

If (i) is false we have, for some large T,

$$f(T + 2n) \geqslant \tfrac{3}{2} f(T + 2n - 2) \quad (0 < n \leqslant \tfrac{1}{2} l - 1).$$

Hence for some large T and $\nu = [\tfrac{1}{2} l] - 1$

$$f(2T) \geqslant f(T + 2\nu) \geqslant \tfrac{3}{2} f(T + 2\nu - 2) \geqslant \ldots \geqslant (\tfrac{3}{2})^\nu f(T) > (\tfrac{3}{2})^\nu > l^3(2T),$$

which is false.

Consider now (ii). Let $T_n = T_{n-1} + l^2(T_{n-1})$ $(n > 0)$, $T_0 = T$. Then $T_n - T_{n-1} < l^2(T_n)$, and if $n \leqslant l$

$$T_n - T < n l^2(T_n) < l^3(T_n),$$

from which it follows (for large T) that $T_n < 2T$, and so, for $n \leqslant N = [l]$,

$$T_n - T < l^4. \tag{15.3}$$

Now the pair of inequalities

$$\left.\begin{aligned} &f_1(T_n) < \tfrac{3}{2} f_1(T_{n-1}) && (15.4)\\ &f_2(T_n) < \tfrac{3}{2} f_2(T_{n-1}) && (15.5)\end{aligned}\right\} \quad (n > 0)$$

must hold simultaneously for some $n \leqslant N$. For if not, one of them, say the first, must be false for at least $\tfrac{1}{2} N$ values of n, from which it follows that

$$f_1(T_N) \geqslant (\tfrac{3}{2})^{\frac{1}{2} N} f_1(T) > e^{AN}$$

and so

$$f_1(T + l^4) \geqslant f_1(T_N) > l^3,$$

which is false. But for an n giving (15.4) and (15.5) we have, taking $t = T_{n-1}$,

$$f_1\{t + T_1(t)\} \leqslant f_1\{t + l^2(t)\} = f_1(T_n) < \tfrac{3}{2} f_1(t)$$

for large T. This is (15.1), and (15.2) follows similarly.

16. *Lemma* 7. *Suppose $R(T)$ is continuous in $T > 0$ except at isolated points, and*

$$R_1(T) = \int_0^T R(t)\,dt + b_1,\quad R_2(T) = \int_0^T R_1(t)\,dt + b_2,$$

where b_1, b_2 are constants. Suppose further that as $T \to \infty$

$$R(T) = O\{g(T)\,h(T)\},$$
$$R_2(T) = O\{g(T)\}^*,$$

where $g(T)$ is increasing and greater than unity, and $T^2 h(T)$ is increasing and greater than unity, for $T > T_0$. Then

$$R_1(T) = O\{g(T)\sqrt{h}(T)\}.$$

This is practically included in a known result†.

We require also another lemma, belonging to the same order of ideas, and embodying an argument of which we have had an example already in § 13.

Lemma 8. *Suppose $\psi(T) = O(l)$ is an increasing function greater than unity $(T > 4)$, and*

$$R_1(T) = \int_0^T R(t)\,dt + O(1).$$

Suppose further that

$$(16.1)\qquad R(T+x) - R(T) > -Kxl - K' \qquad (0 \leqslant x < \tfrac{1}{2}T),$$

where K and K' are constants, and that

$$(16.2)\qquad R_1(T) = O\{l\mu^2\psi(T)\}.$$

Then

$$(16.3)\qquad R(T) = O(l\mu)\sqrt{\psi}(2T).$$

We write, for brevity, $u = \sqrt{\psi}(2T)$. If (16.3) is false, then, for any positive h (however large), there are arbitrarily large T for which either

$$(16.4)\qquad R(T) > hl\mu u,$$

or else

$$(16.5)\qquad R(T) < -hl\mu u.$$

We suppose $h > K + 1$ (as we may), and choose $\xi = x = \mu u$, or

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* The O's here are naturally not absolute.

† See G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 11 (1913). If $R(T)$ is continuous the theorem is a case of Theorem 6 (a), with

$$r = 2,\ x = T,\ f(x) = R_2(T),\ \phi(x) = g(T),\ \psi(x) = T^2 g(T)\,h(T).$$

In the actual case we can modify $R(T)$ in a small neighbourhood of each discontinuity so as to make it continuous, altering $R_1(T)$, $R_2(T)$ by arbitrarily little.

$\xi = -x$, according as (16.4) or (16.5) holds. Then $x < \frac{1}{2}T$ for large T, and we have respectively in the two cases

$$\int_T^{T+\xi} R(t)\,dt = \int_0^x R(T+y)\,dy = xR(T) + \int_0^x \{R(T+y) - R(T)\}\,dy$$
$$\geqslant xR(T) - \int_0^x (Ky\log T + K')\,dy,$$

and

$$\int_T^{T+\xi} R(t)\,dt = -\int_0^x R(T-y)\,dy = -xR(T) + \int_0^x \{R(T) - R(T-y)\}\,dy$$
$$\geqslant -xR(T) - \int_0^x \{Ky\log(T-y) + K'\}\,dy.$$

In either case

$$(16.6)\quad \left|\int_T^{T+\xi} R(t)\,dt\right| \geqslant x\,|R(T)| - \int_0^x (Ky\log T + K')\,dy$$
$$\geqslant xhl\mu u - \tfrac{1}{2}Klx^2 - K'x = x(h - \tfrac{1}{2}K)\,l\mu u - K'x$$
$$\geqslant x(\tfrac{1}{2}hl\mu u - K') > x(\tfrac{1}{4}hl\mu u) = \tfrac{1}{4}hl\mu^2\psi(2T).$$

But the left side does not exceed

$$|R_1(T+\xi)| + |R_1(T)| + O(1)$$
$$= O\left(\frac{\log(\frac{3}{2}T)}{\lambda(\frac{1}{2}T)}\,\psi(\tfrac{3}{2}T)\right) + O\{l\mu\psi(T)\} = O(l\mu)\,\psi(2T),$$

and it is impossible for (16.6) to hold for arbitrarily large h.

17. Let

$$P(T) = \underset{4\leqslant t\leqslant T}{\mathrm{Max}}\,(\pi\,|S(t)|) + l\mu + 1, \qquad \bar{P}(T) = \underset{\substack{4\leqslant t\leqslant T\\ \sigma\geqslant\frac{1}{2}}}{\mathrm{Max}}\,(|\mathfrak{I}f(s)|) + l\mu + 1,$$

$$Q(T) = \underset{4\leqslant t\leqslant T}{\mathrm{Max}}\,(|f(\tfrac{1}{2}+\mu+iT)|) + l\mu + 1, \quad \bar{Q}(T) = \underset{\substack{4\leqslant t\leqslant T\\ \sigma\geqslant\frac{1}{2}+\mu}}{\mathrm{Max}}\,(|f(s)|) + l\mu + 1.$$

A word or two on the relations of these four functions may make them more easily distinguishable. Since $S(t)$ is substantially $\mathfrak{I}f(s)$ with $\sigma = \frac{1}{2}$ we have, substantially, $P(t) \leqslant \bar{P}(t)$; and similarly $Q(t) \leqslant \bar{Q}(t)$. The Q's differ from the corresponding P's (1) by the presence of f instead of $\mathfrak{I}f$; and (2) by the addition of μ to σ or the lower bound of σ (f, but not $\mathfrak{I}f$, has infinities when $\sigma = \frac{1}{2}$, and σ must be kept from too near an approach to $\frac{1}{2}$). $\lambda\mu + 1$ is added to all the maxima to secure that the four functions exceed unity and are of order at least $l\mu$. Finally we take maxima over $4 \leqslant t \leqslant T$ to make our functions increasing. It will appear eventually that all four functions are of the same order, viz. $O(l\mu)$.

Lemma 9. *We have*

$$\bar{P}(T) = O(l), \quad \bar{Q}(T) = O(l\lambda).$$

The first of these follows from (1.4). To prove the second we observe that on the circle with centre at $s_0 = \frac{3}{2} + it_0$ ($t_0 \geqslant 4$) and radius $r = 1$,

$$|\mathfrak{I} f(s)| < A \log t_0.$$

Since $|f(s_0)| < A$ we have by Carathéodory's theorem

$$|f(s)| < \{A + Al(t_0)\} \frac{2r}{r - \rho}$$

in and upon the concentric circle of radius $\rho < r$. Taking

$$\rho \leqslant r - \mu = 1 - \mu,$$

we have, in particular,

$$|f(\sigma + it_0)| < Al(t_0)\lambda \quad (\tfrac{1}{2} + \mu \leqslant \sigma \leqslant \tfrac{3}{2}).$$

We may clearly omit the restriction $\sigma \leqslant \frac{3}{2}$, and the second result of the lemma follows by taking the maximum in the range $4 \leqslant t_0 \leqslant T$.

18. *Lemma* 10. *We have*

$$\bar{P}(T) = O\{P(T + l)\},$$
$$\bar{Q}(T) = O\{Q(T + l)\}.$$

In proving this we use the method of "rotation of domains" introduced by Lindelöf*. Consider the functions

$$g(s) = e^{\pm i f(s)}$$

in the strip bounded by the vertical lines $\sigma = \frac{1}{2}$, $\sigma = \frac{3}{2}$, and the horizontal line $t = 4$. Let s_0 be a point of the strip for which $5 \leqslant t_0 \leqslant T$. Rotate the strip about the point s_0 through an angle $-\frac{1}{2}\pi$, and let $g_1(s)$ be the function that takes the value $g(s)$ at the point s_1 of the new strip corresponding to the point s of the original. Then s_0 is enclosed in a square, the length of whose side is unity, formed by the original vertical lines and their new positions. On the vertical boundaries of the original strip

$$|g(s)| \leqslant \exp |\mathfrak{I} f(s)|$$

does not exceed $\exp\{P(t)\}$ or $\exp A$ according as the boundary is the left or right hand one. In either case $|g(s)| < \exp\{P(t) + A\}$. For internal points of the strip we have $|g(s)| < \exp\{\bar{P}(t)\}$. It follows that for any point s of a vertical boundary of the square

$$|g(s)| < \exp\{P(T + 1) + A\}, \quad |g_1(s)| < \exp\{\bar{P}(T + 1)\},$$

and that for any point of a horizontal boundary of the square

$$|g(s)| < \exp\{\bar{P}(T + 1)\}, \quad |g_1(s)| < \exp\{P(T + 1) + A\},$$

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* E. Lindelöf, *Acta Soc. Scient. Fenn.*, 46 (1915).

and so

(18.1) $|h(s)| = |g(s)g_1(s)| < \exp\{P(T+1) + \bar{P}(T+1) + A\}$

on the whole boundary of the square. Since $h(s)$ is regular and bounded in the interior of the square, and continuous at the boundary except at a finite number of points, we conclude, by a well-known principle, that (18.1) is valid in the interior of the square, and in particular at $s = s_0$, where $g(s_0) = g_1(s_0)$. Thus

$$|g(s_0)|^2 < \exp\{P(T+1) + \bar{P}(T+1) + A\}.$$

Since one or other of the $\pm$ signs in the definition of g gives $|g| = \exp|\mathfrak{I}f|$ we conclude, dropping the suffix 0, that

$$2|\mathfrak{I}f(s)| \leqslant P(T+1) + \bar{P}(T+1) + A \quad (\tfrac{1}{2} < \sigma < \tfrac{3}{2},\ 5 \leqslant t \leqslant T).$$

In this we may further write $4 \leqslant t$ for $5 \leqslant t$, and suppress the condition $\sigma < \frac{3}{2}$. Next we may, by continuity, infer the inequality at $\sigma = \frac{1}{2}$ except when $\frac{1}{2} + it$ is a zero of $\zeta(s)$. Finally, since $\mathfrak{I}f(s) = \frac{1}{2}\{f(s+0i) + f(s-0i)\}$, we may infer the inequality also in this remaining case if we replace $T+1$ by $T+2$. Thus

$$2|\mathfrak{I}f(s)| \leqslant P(T+2) + \bar{P}(T+2) + A \quad (\sigma \geqslant \tfrac{1}{2},\ 4 \leqslant t \leqslant T).$$

Taking the maximum of this for the range of t we have now

$$2\bar{P}(T) \leqslant P(T+2) + \bar{P}(T+2) + A + 2l\mu + 2.$$

By Lemma 6 (i) there exists a $t = t(T)$ between T and $T + l - 2$ for which

$$\bar{P}(t+2) < \tfrac{3}{2}\bar{P}(T).$$

Then

$$2\bar{P}(t) \leqslant P(t+2) + \bar{P}(t+2) + Al\mu < P(t+2) + \tfrac{3}{2}\bar{P}(t) + Al\mu,$$

$$\tfrac{1}{2}\bar{P}(t) < P(t+2) + Al\mu,$$

$$\bar{P}(T) \leqslant \bar{P}(t) < 2P(t+2) + Al\mu \leqslant 2P(T+l) + Al\mu < AP(T+l),$$

which proves the first part.

The second part is similar and a little more direct. Arguing with the strip bounded by $\sigma = \frac{1}{2} + \mu$, $\sigma = \frac{3}{2}$, $t = 4$, and with f in place of g, we obtain

$$|f(s_0)|^2 \leqslant \{Q(T+1) + A\}\bar{Q}(T+1) < AQ(T+1)\bar{Q}(T+1).$$

This time there is continuity at the boundary; therefore, taking the square root and the maximum,

$$\bar{Q}(T) < A\{Q(T+1)\bar{Q}(T+1)\}^{\frac{1}{2}} + l\mu + 1 < A\{Q(T+1)\bar{Q}(T+1)\}^{\frac{1}{2}}.$$

By Lemma 6 (i), for some t between T and $T + l - 2$,

$$\bar{Q}(t+1) < \tfrac{3}{2}\bar{Q}(t),$$

and so

$$\bar{Q}(t) < A\{\bar{Q}(t)Q(t+1)\}^{\frac{1}{2}},$$

$$\bar{Q}(t) < AQ(t+1),$$

$$\bar{Q}(T) \leqslant \bar{Q}(t) < AQ(T + l - 2 + 1) < AQ(T+l)$$

the desired result.

19. *Lemma* 11.

(19.1) $f(s) = O\{\bar{Q}(T+1)\,e^{-a_1(\sigma-\frac{1}{2})\lambda}\} = O\{Q(T+2l)\,e^{-a_1(\sigma-\frac{1}{2})\lambda}\}$
(uniformly) for $\frac{1}{2}+\mu \leqslant \sigma \leqslant 2,\ t \leqslant T.$

The second part follows from Lemma 10 and the first part, which we therefore consider. We write $x = \sigma - \frac{1}{2}$. It is sufficient to prove (19.1) for $\mu \leqslant x \leqslant \frac{2}{3} - \mu$. Consider three circles with a common centre at $s_0 = \frac{3}{2} + \mu + it$ and radii $r_3 > r_2 > r_1$, going through the points $\frac{1}{2} + \mu + it$, $\frac{1}{2} + x + it$, $\frac{7}{8} + it$ respectively. Let M_1, M_2, M_3 be the upper bounds of $|f(s)|$ in the circles. Then by Hadamard's "three circles" theorem

$$M_2 \leqslant M_1^{\vartheta} M_3^{1-\vartheta},$$

where $\vartheta = \log\frac{r_3}{r_2}\Big/\log\frac{r_3}{r_1}$ (and $0 < \vartheta < 1$). Now for $\mu \leqslant x \leqslant 1$, and large T,

$$\vartheta = \log\{1-(x-\mu)\}^{-1}\Big/\log\frac{1}{\frac{1}{3}+\mu} \geqslant A(x-\mu)/A = A(x-\mu),$$

$$M_1 \leqslant f(\tfrac{7}{8}) = A,$$

$$M_3 \leqslant \operatorname*{Max}_{\substack{\sigma \geqslant \frac{1}{2}+\mu \\ 3 < t < T+1}} |f(s)| < \bar{Q}(T+1) + A < A\bar{Q}(T+1).$$

Hence

$$M_2 < A\bar{Q}(T+1)\{\bar{Q}(T+1)\}^{-\vartheta} < A\bar{Q}(T+1)(l\mu)^{-\vartheta}$$
$$< A\bar{Q}(T+1)(l^{\frac{1}{2}})^{-A(x-\mu)} < A\bar{Q}(T+1)\,e^{-A\lambda(x-\mu)} < A\bar{Q}(T+1)\,e^{-A\lambda x},$$

the desired result.

20. *Lemma* 12. $Q(T) = O\{P(T + l^5)\}$.

We have, from Lemmas 10 and 11,

(20.1) $\mathfrak{R} f(s) = O\{P(T+2l)\}$ $(x \geqslant 0,\ t \leqslant T)$,

(20.2) $f(s) = O\{Q(T+2l)\,e^{-a_1\lambda x}\}$ $(\mu \leqslant x \leqslant \frac{3}{2},\ t \leqslant T)$.

Let* $\xi = 2\,\dfrac{1+a_1}{a_1}\,\mu \log\left(\dfrac{Q(T+2l)}{P(T+2l)} + e\right) > 2\mu,$

so that $\xi = o(1)$ (by Lemma 9), and $Q(T+2l)\,e^{-\frac{1}{2}a_1\lambda\xi} < P(T+2l)$. Then, for $\xi \leqslant x \leqslant \frac{3}{2}$, $t \leqslant T$, we have from (20.2)

(20.3) $|f(s)| = O\{Q(T+2l)\,e^{-a_1\lambda x}\} = O\{Q(T+2l)\,e^{-\frac{1}{2}a_1\lambda(x+\xi)}\}$
$= O\{P(T+2l)\,e^{-\frac{1}{2}a_1\lambda x}\}.$

Consider now the range $0 < x \leqslant \xi$, and the circles with common centre at $s_0 = \frac{1}{2} + \xi + it$, and radii ξ and x. On the outer circle

(20.4) $\mathfrak{R} f(s) = O\{P[T + 1 + 2l(T+1)]\} = O\{P(T+3l)\}$

by (20.1). Also, by (20.3) with $x = \xi$,

(20.5) $f(s_0) = O\{P(T+2l)\}$.

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* This part of the argument is one of the main keys to the proof of Theorem 11.

By (20.4), (20.5) and Carathéodory's theorem we have, on the inner circle and in particular at $s = \frac{1}{2} + x + it$,

$$(20.6) \qquad f(s) = O\left\{\frac{\xi}{x} P(T+3l)\right\} \quad (0 < x \leqslant \xi).$$

Combining this with (20.3) we obtain

$$(20.7) \quad f(s) = O\{P(T+2l)\, e^{-a_1\lambda x}\} + O\left\{\frac{\xi}{x} P(T+3l)\right\} \quad (x > 0).$$

In this we take $x = \mu$, vary t and take the maximum. This gives

$$Q(T) = O\{P(T+2l)\} + O\{\xi\lambda\, P(T+3l)\} + O(l\mu)$$

$$= O\{\xi\lambda P(T+3l)\} = O\left\{P(T+3l)\left[\log\left(\frac{Q(T+2l)}{P(T+2l)} + e\right)\right]\right\},$$

since $\xi\lambda > 2$, $P(T+3l) \geqslant l\mu$. Hence

$$(20.8) \qquad U(T) = \frac{Q(T)}{P(T+3l)} < A \log\left[\frac{Q(T+2l)}{P(T+2l)} + e\right]$$

$$= A\left\{\log[U(T)+e] + \log\left[\frac{Q(T+2l)+eP(T+2l)}{Q(T)+eP(T+3l)}\right] + \log\frac{P(T+3l)}{P(T+2l)}\right\}$$

$$< A\log[U(T)+e] + A\log\frac{Q(T+2l)}{Q(T)} + A\log\frac{P(T+3l)}{P(T)}.$$

Now by Lemma 6 (ii) there exists a t between T and $T + l^4$ for which

$$\frac{Q[t+2l(t)]}{Q(t)} < A, \quad \frac{P[t+3l(t)]}{P(t)} < A.$$

Then from (20.8)

$$U(t) < A\log[U(t)+e] + A,$$

from which it follows that $U(t) < A$, or

$$Q(t) < AP[t+3l(t)].$$

Hence $$Q(T) \leqslant Q(t) = O\{P(T+l^5)\},$$
the desired result.

Lemma 13.

$$\mathfrak{I} f(s) = O\{P(T+l^6)\, e^{-a_1\lambda x}\} \quad (0 \leqslant x \leqslant \tfrac{3}{2},\ t \leqslant T).$$

The result is true for $x \leqslant \mu$ by (20.1), and for $\mu \leqslant x \leqslant \frac{3}{2}$ by (20.2) and Lemma 12.

21. *Lemma* 14. $P(T) = O(l\mu)$.

We have

$$S_2(T) = -\frac{1}{\pi}\int_0^\infty dx \int_0^\infty \mathfrak{I} f(\tfrac{1}{2}+x+iT)\,dx = -\frac{1}{\pi}\int_0^{\frac{3}{2}} dx \int_x^{\frac{3}{2}} \mathfrak{I} f\,dx + O(1)$$

$$= O(\mu^2)\, P(T+l^6) + O(1) = O(\mu^2)\, P(T+l^6)$$

by Lemma 13. Also $S(T) = O\{P(T+l^6)\}$.

The conditions of Lemma 7 are fulfilled if $R(T) = S(T)$, $R_1(T) = S_1(T)$, $R_2(T) = S_2(T)$, and we conclude that

$$S_1(T) = O(\mu)\, P(T + l^6). \tag{21.1}$$

Further, if R and R_1 have their meanings above, and if

$$\psi(T) = \operatorname*{Max}_{4 \leqslant t \leqslant T} \left(\frac{\lambda(t)}{l(t)} P[t + l^6(t)]\right),$$

$\psi(T)$ is greater than unity, increasing, and of the form $O(\lambda) = O(l)$. Also, for $0 \leqslant x \leqslant \frac{1}{2}T$,

$$\begin{aligned} &R(T+x) - R(T) \\ &= \{N(T+x) - N(T)\} - \{M(T+x) - M(T)\} + O(1) \\ &> -\{M(T+x) - M(T)\} - A > -Ax \log T - A. \end{aligned}$$

The conditions of Lemma 8 are therefore satisfied, and we have

$$|S(T)| = |R(T)| = O(l\mu)\sqrt{\psi}(2T).$$

Since the function on the right is increasing we may write t for T on the left and take the maximum for $t \leqslant T$, obtaining

$$P(T) = O(l\mu)\sqrt{\psi}(2T) + O(l\mu) = O(l\mu)\sqrt{\psi}(2T).$$

Hence again, for $t \leqslant T$,

$$P[t + l^6(t)] = O\{l(t)\,\mu(t)\}\sqrt{\psi}(3t),$$

$$\frac{\lambda(t)}{l(t)} P[t + l^6(t)] = O\{\sqrt{\psi}(3t)\} = O\{\sqrt{\psi}(3T)\},$$

and taking the maximum on the left,

$$\psi(T) < A\sqrt{\psi}(3T). \tag{21.2}$$

Now (21.2) and the evident inequality

$$\psi(T) < Al \tag{21.3}$$

together imply

$$\psi(T) < A. \tag{21.4}$$

For we must have

$$\psi(3t) < 3\psi(t) \tag{21.5}$$

for some arbitrarily large values of t; otherwise

$$\psi(4 \,.\, 3^n) \geqslant 3^n \psi(4)$$

and $\psi(T) > AT$ for some arbitrarily large T, contrary to (21.3). But now (21.5) and (21.2) involve

$$\psi(t) < A\sqrt{\psi}(t), \quad \psi(t) < A,$$

for arbitrarily large t, and so, since ψ is increasing, for all t. Hence, since

$$P(T) \leqslant P(T + l^6) \leqslant l\mu\psi(T),$$

we have finally $$P(T) = O(l\mu).$$

22. Theorem 11 now follows without difficulty. We have at once

$$S(T) = O\{P(T)\} = O(l\mu)$$

the first of the desired results. Next we have

$$\begin{aligned} S_{2n}(T) &= \frac{(-1)^n}{\pi} \int_0^\infty dx \int_x^\infty dx \ldots \int_x^\infty \mathfrak{I} f(\tfrac{1}{2} + x + iT)\, dx \\ &= \frac{(-1)^n}{\pi} \int_0^{\frac{3}{2}} dx \int_x^{\frac{3}{2}} dx \ldots \int_x^{\frac{3}{2}} \mathfrak{I} f dx + O(1) \\ &= O\{\mu^{2n} P(T + l^6)\} = O(l\mu \,.\, \mu^{2n}) \end{aligned}$$

by Lemma 13, the desired result for S_n when n is even. Finally the results for odd n follow at once by Lemma 7 if we take $R = S_{2n-2}$, $R_1 = S_{2n-1}$, $R_2 = S_{2n}$.

23. Our material enables us to prove also

Theorem 12.

$$\zeta(\tfrac{1}{2} + iT) = O\left\{\exp\left(\frac{A \log T}{\log\log T}\right)\right\}.$$

In fact

$$\log \zeta(\tfrac{1}{2} + \mu + iT) = O\{Q(T)\} = O\{P(T + l^6)\} = O(l\mu).$$

Also*, for $s = \sigma + iT$,

$$\mathfrak{R} \frac{\zeta'(s)}{\zeta(s)} = O(1) - \tfrac{1}{2} \log T + \sum_\rho \mathfrak{R}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \quad (\tfrac{1}{2} < \sigma \leqslant 1).$$

Since every term in $\sum\limits_\rho$ *is positive*

$$\mathfrak{R} \frac{\zeta'(s)}{\zeta(s)} > -\tfrac{1}{2} \log T - A,$$

$$\begin{aligned} \mathfrak{R} \log \zeta(\tfrac{1}{2} + iT) &= \mathfrak{R} \log \zeta(\tfrac{1}{2} + \mu + iT) - \int_{\frac{1}{2}}^{\frac{1}{2}+\mu} \mathfrak{R} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma \\ &\leqslant O(l\mu) + \mu(\tfrac{1}{2} \log T + A) = O(l\mu), \\ |\zeta(\tfrac{1}{2} + iT)| &\leqslant \exp\{O(l\mu)\}. \end{aligned}$$

24. We prove, in conclusion, another theorem about the orders of $S(T)$, $S_1(T)$, ... as functions of $\log T$. Let

$$\alpha_n = \overline{\lim_{T \to \infty}} \frac{\log |S_n(T)|}{\log\log T},$$

and $\alpha = \alpha_0$. That is, on the one hand

$$S(T) = O\{(\log T)^{\alpha + \epsilon}\},$$

and, on the other, this result becomes false if α is replaced by any smaller number; and similar relations hold for S_n and α_n. We may call α the "order" of $S(T)$. It is known that $\alpha > 0$; and it is in fact true that

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* *H.B.*, p. 316.

$$(24.1) \qquad \alpha \geqslant \tfrac{1}{2}, \ \alpha_n \geqslant \tfrac{1}{2};$$

but the proof of this would take us too far afield. We can prove, however, within our present circle of ideas:

Theorem 13. *The number α_n is a convex function of n, and*

$$1 \geqslant \alpha \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \dots.$$

It follows at once from Theorem 11 that $\alpha_n \leqslant 1$. Further, supposing $\alpha_n \geqslant 0$*, and taking $R(T) = S_n(T)$, $R_1 = S_{n+1}$, $R_2 = S_{n+2}$, we see that Lemma 7 is valid with

$$g(T) = (\log T)^{\alpha_{n+2}+\epsilon}, \ g(T)\,h(T) = (\log T)^{\alpha_n+\epsilon}.$$

It therefore gives (O's depending on n and ϵ)

$$S_{n+1}(T) = O\{(\log T)^{\frac{1}{2}(\alpha_n+\alpha_{n+2})+\epsilon}\},$$

$$\alpha_{n+1} \leqslant \tfrac{1}{2}(\alpha_n + \alpha_{n+2}),$$

so that α_n is a convex function of n. Since the α's are bounded above it follows from the convexity that α_n never increases, and this completes the proof.

The upper bounds for the $S_n(T)$ given in Theorem 11 decrease by a factor $1/\log\log T$ for each addition of unity to n. Theorem 13 and the result $\alpha_n > 0$ together show that this factor cannot be replaced (throughout) by any constant power of $1/\log T$, however small.

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* The modifications necessary if $\alpha_n < 0$ or $\alpha_{n+2} = -\infty$ can be supplied by the reader. Since actually $\alpha_n > 0$ it is not worth while to set out the details.

On loci which have two systems of generating spaces. By Mr H. W. Richmond and F. Bath, King's College.

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Introduction.

In hypergeometry loci defined by parametric equations

$$(x) = K_{rs}\alpha_r\beta_s \quad (r = 0, 1, 2, \ldots p;\ s = 0, 1, 2, \ldots q), \quad \ldots(1)$$

have been frequently discussed from various points of view*. Among other properties these loci possess two systems of generating spaces obtained by keeping the α's (or β's) fixed while the β's (or α's) vary, the simplest example being the generators of a quadric in S_3. Another method of obtaining loci having these properties is by taking the flat spaces through the sets of a linear series of sets of points on an elliptic curve and the flat spaces through the sets of the residual series (the curve being supposed *normal*. In this way we shall obtain a configuration of $\infty^p\, S_p$'s and $\infty^q\, S_q$'s the dual of which gives a locus included among those represented by (1).

From this point of view the only work that has come to our notice makes no mention of the dual configuration. Thus, Segre has discussed the scrolls obtained from $\infty^1 S_1$'s, the joins of pairs of points of constant parameter sum on an elliptic curve; Veneroni discusses the manifold generated by three pencils of S_4's in S_5 (the dual of (1) for $p = q = 2$) and obtains its two systems of $\infty^2 S_2$'s as trisecants of a normal elliptic sextic which is double on the manifold; Rosati discusses the same manifold but starts from the (1, 1) correspondence between the conics of a plane and S_4's of an S_5. Finally, Marletta investigates g_m^{m-1}'s on a normal elliptic curve of order $2m$ showing that there are four such g_m^{m-1}'s *self-residual*—again the case of $p = q\ (= m)$. By consideration of the dual configuration we are able to extend the use of the elliptic curve, proving thereby the existence of loci with similar properties for $p \neq q$.

We add a note (§ 7) giving a geometrical construction for this dual locus and indicating the extension to loci of triple or higher multiple generation.

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(A) Using the parametric definition (1):

(1) C. Segre, *Rend. Circ. Mat. Palermo*, 5 (1891), p. 192.
(2) F. Schur, *Math. Annalen*, 23 (1884), p. 437.
(3) E. Study, *Methoden...ternären Formen* (Leipzig, 1889), p. 119.

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[*] See C. Segre, *Encyk. d. Math. Wiss.*, 3, Chap. VII, §§ 9 and 10, pp. 825–830; § 28, p. 904; § 38, pp. 927–932; §§ 43 and 44, pp. 954–959.

(4) C. Segre, *Math. Annalen*, 40 (1892), p. 413.
(5) C. Segre, *Rend. Circ. Mat. Palermo*, 30 (1910), p. 93.
(6) A. Duschek, *Monatshefte f. Math. u. Physik* (1923), p. 63.
Also *Encyk.*, p. 958, above cited.

(B) In connection with the "Equivalence of ∞^λ series of bilinear forms":
(1) S. Kantor, *Sitzungsberichte Königl. Bayr. Akad. München*, 27 (1897), p. 367.
(2) S. Kantor, *Monatshefte f. Math. u. Physik*, 11 (1900), p. 193.

(C) In connection with manifolds represented by the vanishing of all the determinants of given order in a matrix (writing down the conditions for a "plane" to touch the locus given by the parametric equations (1), it will be clear that the dual of (1) is in general represented by the vanishing of all the determinants of a certain order in a matrix whose terms are linear functions of the coordinates).
See *Encyk.*, pp. 927–932, above cited.

(D) Special case $p = 1$:
(1) Kötter, *Preisschrift* (1887).
(2) S. Kantor, *Journ. f. Math.*, 118 (1897), p. 74.
(3) K. Zindler, *Journ. f. Math.*, 111 (1893), p. 303.
(4) U. Perazzo, *Mem. Torino Accad.* (2), 54 (1904), p. 149, also *Encyk.*, p. 43, above cited.

(E) Special case $p = q$:
(i) A (1), A (4) and (B).
(ii) With special reference to elliptic curves:
(1) C. Segre, *Math. Annalen*, 27 (1886), p. 296.
(2) C. Rosati, *Rend. Ist. Lombardo* (2), 35 (1902), p. 407.
(3) E. Veneroni, *Rend. Ist. Lombardo* (2), 38 (1905), p. 523.
(4) G. Marletta, *Atti Accad. Gioenia di Catania* (5), 1 (1908), also *Encyk.*, p. 904, above cited.
When $p = q = 2$ the planes of the conics of a Veronese surface give a particular case of the locus, see
(5) G. Veronese, *Mem. Accad. Lincei* (3), 19 (1884), p. 344.
Another particular case for $p = q = 2$ is discussed by Perazzo whose locus has nine double lines, a case arising from the projection of the normal manifold for $p = q = 2$ (a locus in S_8) when the vertex of projection, an S_2, has 3 points in common with the manifold.
(6) U. Perazzo, *Accad. Sci. Torino, Atti*, 36 (1901), p. 891.

1. In S_{p+q+1}, i.e. in a space of $p + q + 1$ dimensions, an S_p and an S_q in general have no common point. Exceptionally they have a point (or a line, or S_2, or S_3, ...) in common and lie wholly in a space of $p + q$ (or $p + q - 1$, or $p + q - 2$, or $p + q - 3$, ...) dimensions; they are then said to *intersect*.

"*For all values of* p *and* q *there exist in space of* $p + q + 1$ *dimensions systems of* ∞^q S_p*'s and* ∞^p S_q*'s such that any* S_p *and any* S_q *intersect. The two systems generate a 'surface,' i.e. the coordinates of their points satisfy one relation* $F(x \ldots) = 0$, *and one* S_p *and one* S_q *pass through any arbitrary point of* $F(x \ldots) = 0$."

2. That complementary reguli exist in ordinary space may be seen from the properties of the curve of intersection of two quadrics, a curve of order four and genus one. The coordinates of points on

the curve are elliptic functions of a parameter and the parameters of four coplanar points have a constant sum, which may be taken to be zero. Thus all the chords which join two points of the curve whose parameters have a sum c intersect all the chords which join two points whose parameters have a sum $-c$. A similar method applies to space of many dimensions. [It may be remarked that when the elliptic functions degenerate the curves they represent become rational curves, of unchanged order, having a double point (node, isolated point, or cusp); in our preliminary work these rational curves were used at first to establish the Existence-Theorem.]

3. Elliptic curves of order $n+1$ are normal curves in S_n and can be represented by the equations

$$x_0 : x_1 : x_2 : x_3 : x_4 : \ldots : x_n :: 1 : \wp : \wp' : \wp^2 : \wp\wp' : \wp^3 : \wp^2\wp' : \ldots : P_n,$$

these being elliptic functions of a parameter θ. The form of the last function P_n depends upon the parity of n; in fact

$$P_{2s-1} = \wp^s; \quad P_{2s} = \wp^{s-1}\wp'.$$

Suppose that $n = p + q + 1$, so that the order of the elliptic curve is $p + q + 2$. Take on the curve any $p + 1$ points whose parameters have the sum c; they determine an S_p, one of a system of $\infty^p S_p$'s, since p of the $p + 1$ points may be taken arbitrarily. Take also $q + 1$ points whose parameters have sum $-c$; they determine one of a system of $\infty^q S_q$'s. Further, since the $p + q + 2$ points have parameter-sum zero, these points, and the S_p and the S_q lie wholly in a space of $p + q$ dimensions, and the S_p and the S_q intersect. Hence

"*For all values of p and q, there exist in space of* $p + q + 1$ *dimensions systems of* $\infty^p S_p$*'s and* $\infty^q S_q$*'s, such that any* S_p *and any* S_q *intersect.*"

4. Duality gives to this result the more convenient form stated in § 1. In S_{p+q+1} the dual of an S_p is an S_q and *vice versa*. Also if an S_p and an S_q lie wholly in an S_{p+q} and therefore intersect, the same is true of their duals. Hence

"*For all values of p and q, there exist in space of* $p + q + 1$ *dimensions systems of* $\infty^p S_q$*'s and* $\infty^q S_p$*'s, such that any* S_q *and any* S_p *intersect.*"

But further, an S_q is defined by $p + 1$ linear equations in $x_0, x_1, x_2, \ldots x_n$, (where $n = p + q + 1$), and the coefficients in these equations depend upon $p + 1$ parameters θ which are arbitrary except for the fact that their sum vanishes. We have just enough equations to eliminate the parameters and obtain an equation $F(x_0, x_1, \ldots x_n) = 0$, representing a surface upon which all the points of the S_q's and the S_p's lie. A clearer statement of this follows.

5. By duality the simply infinite system of points which form the elliptic curve of § 3 is replaced by a system of planes (i.e. S_{p+q}'s), defined by the equations

$$u_0 : u_1 : u_2 : \dots : u_n :: 1 : \wp : \wp' : \wp^2 : \wp\wp' : \wp^3 : \dots : P_n,$$

or $$x_0 + x_1\wp + x_2\wp' + x_3\wp^2 + x_4\wp\wp' + \dots + x_nP_n = 0, \quad \dots(\text{i})$$

the coefficients of the x's being elliptic functions of a parameter θ. If $p + 1$ values of θ whose sum is c are taken, the corresponding planes intersect in an S_q, one of the system of $\infty^p S_q$'s under consideration. But if we regard the point (x) as known, the equation (1) determines $p + q + 2$ values of θ whose sum is zero. In order to obtain the equation of the locus $F(x_0, x_1, x_2, \dots x_n) = 0$ of § 4, we have to express the condition that $p + 1$ of the values of θ have sum c, and the remaining $q + 1$ consequently have sum $-c$. We will denote the values of $\wp, \wp', \wp^2, \dots P_n$ when θ is put equal to c by $\gamma, \gamma', \gamma^2, \dots C_n$; their values when θ is put $= -c$ are

$$\gamma, -\gamma', +\gamma^2, -\gamma\gamma', \dots (-1)^{n-1} C_n.$$

To express the condition that $p + 1$ of the roots of (i) should have sum c, introduce a factor $(\wp - \gamma)$ into the left-hand member of (i). Then

$$(\wp - \gamma)(x_0 + x_1\wp + x_2\wp' + x_3\wp^2 + \dots + x_nP_n) \quad \dots\dots(\text{ii})$$

vanishes when $\theta = +c$ or $-c$ or any of the $p + q + 2$ roots of (i). But if $p + 1$ of these have sum c, and we associate with them the value $-c$, we have $p + 2$ values of θ whose sum is zero and which are the roots of an expression

$$\alpha_0 + \alpha_1\wp + \alpha_2\wp' + \alpha_3\wp^2 + \dots + \alpha_{p+1}P_{p+1};$$

similarly the remaining $q + 1$ values of θ together with the value c give $q + 2$ values of θ whose sum is zero, roots of

$$\beta_0 + \beta_1\wp + \beta_2\wp' + \beta_3\wp^2 + \dots + \beta_{q+1}P_{q+1}.$$

Thus the required condition is that (ii) shall be identically equal to the product

$$(\alpha_0 + \alpha_1\wp + \alpha_2\wp' + \alpha_3\wp^2 + \dots + \alpha_{p+1}P_{p+1})$$
$$\times (\beta_0 + \beta_1\wp + \beta_2\wp' + \beta_3\wp^2 + \dots + \beta_{q+1}P_{q+1})$$

for certain values of the α's and β's. Now $\theta = c$ is a root of the second factor and $\theta = -c$ a root of the first; hence

$$\beta_0 + \beta_1\gamma + \beta_2\gamma' + \beta_3\gamma^2 + \dots + \beta_{q+1}C_{q+1} = 0,$$
$$\alpha_0 + \alpha_1\gamma - \alpha_2\gamma' + \alpha_3\gamma^2 - \dots + (-1)^p \alpha_{p+1}C_{p+1} = 0;$$

by help of these α_0 and β_0 are eliminated. On multiplying out and substituting for $\wp'^2$ and γ'^2 it is seen that the factor $\wp - \gamma$ can be removed; and, equating coefficients of like terms, we express $x_0, x_1, x_2 \dots$ in terms of the α's and β's, which we now regard as parameters. The statement of § 1 is thus proved. Although the equation $F(x_0, x_1, \ldots) = 0$ of § 4 has not been obtained explicitly, yet we have obtained a set of parametric equations by which the coordinates (x) of points on $F = 0$ are expressed in terms of parameters $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{p+1}, \beta_1, \beta_2, \ldots \beta_{q+1}$. If the coefficients α are kept constant and the β's vary we obtain an S_q lying on $F = 0$, for this is tantamount to keeping the $p + 1$ values of θ whose sum is c, and which determine an S_q, constant, and allowing the $q + 1$ other values of θ to vary. It will be noticed that the number of parameters α and β is effectively only $p + q$, since only (i) the p ratios of the α's to one another and (ii) the q ratios of the β's to one another are involved.

6. The actual expressions for the coordinates (x) need not be found. They are particular cases of the general form,

$$(x) = K_{rs}\alpha_r\beta_s \quad (r = 0, 1, 2, \ldots p;\ s = 0, 1, 2, \ldots q), \ \ldots(1)$$

since the constants K_{rs} which appear are all definite functions of γ and of the g_2, g_3 of the elliptic functions. It is clear, therefore, that the $(p + q)$-folds we have here generated by $\infty^q S_p$'s and $\infty^p S_q$'s are particular cases obtained by projection from the most general manifold admitting of definition by parameter equations of type (1), viz. the $(p + q)$-fold in S_{pq+p+q} given by

$$x_{rs} = \alpha_r\beta_s \ldots \quad (r = 0, 1, \ldots p;\ s = 0, 1, \ldots q). \ \ldots\ldots(2)$$

7. A geometrical construction for the locus represented by (2) may be easily obtained and extended.

Take $p + 1$ S_q's and set up in each S_q a homogeneous coordinate system. We can arrange that all the coordinates x_{rs} of a point in the ith S_q vanish, except those for which $r = i$. Supposing this done, take $p + 1$ points, one in each S_q, the non-vanishing coordinates in each case having the same value, i.e. take the $p + 1$ points

$$\left.\begin{cases} x_{is} = \beta_s & s = 0, 1, \ldots p \\ x_{rs} = 0 & s = 0, 1, \ldots p,\ r \neq i \end{cases}\right\} \quad i = 0, \ldots p.$$

These $p + 1$ points determine an S_p passing through them. Any point on the S_p has coordinates given by (2). Hence the system of $\infty^q S_p$'s so obtained generates the required locus.

This construction is the dual of that adopted by Veneroni in his discussion of the case when $p = q = 2$, and clearly holds for the $p + q$-folds obtainable in spaces whose dimensions are less than $pq + p + q$.

The extension to loci of triple or higher multiple generation will be amply illustrated by the simplest example.

Let us take the locus of triple generation obtained for $p = q = r = 1$ containing three ∞^2 systems of lines (S_1's), and existing in space of at most 7 dimensions. All such loci are obtainable from the one in S_7 which we proceed to construct.

$(x_0, \dots x_7)$ are coordinates of a point in S_7 in which we consider the quadrics

(1), $x_0x_3 = x_1x_2$ belonging to the S_3 whose equations are

$$x_4 = x_5 = x_6 = x_7 = 0,$$

and (2), $x_4x_7 = x_5x_6$ belonging to the S_3 whose equations are

$$x_0 = x_1 = x_2 = x_3 = 0.$$

Now points on (1) are given by

$$x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 :: \beta_0\gamma_0 : \beta_0\gamma_1 : \beta_1\gamma_0 : \beta_1\gamma_1 : 0 : 0 : 0 : 0$$

and points on (2) are given by

$$y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 :: 0 : 0 : 0 : 0 : \beta_0\gamma_0 : \beta_0\gamma_1 : \beta_1\gamma_0 : \beta_1\gamma_1.$$

So, any point on a line joining a point of (1) to a corresponding point of (2) determined by

$$x_0 : x_1 : x_2 : x_3 :: y_4 : y_5 : y_6 : y_7$$

has coordinates

$$x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7$$

$$:: \alpha_0\beta_0\gamma_0 : \alpha_0\beta_0\gamma_1 : \alpha_0\beta_1\gamma_0 : \alpha_0\beta_1\gamma_1 : \alpha_1\beta_0\gamma_0 : \alpha_1\beta_0\gamma_1 : \alpha_1\beta_1\gamma_0 : \alpha_1\beta_1\gamma_1.$$

In other words, the locus obtained by joining corresponding points on the two quadrics is the one sought and the joining lines form one of the ∞^2 systems of lines. The other two systems are obtained from the two systems of generators of the ∞ quadrics which can be taken to serve as bases for the construction.

More generally, when p, q, r have any integral values, the $(r + 1)$ base varieties whose corresponding points are joined to give the $\infty^{p+q} S_r$'s are loci containing $\infty^q S_p$'s and $\infty^p S_q$'s (i.e. loci of double generation of the type already considered).

On Integrals developable about a Singular Point of a Hamiltonian System of Differential Equations. By Dr. T. M. CHERRY, Trinity College.

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§ 1. *Introduction.*

Let $$\frac{dx_r}{dt} = \frac{\partial H}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial H}{\partial x_r} \quad (r = 1, \ldots n) \qquad \ldots\ldots(1)$$

be a system of differential equations of Hamiltonian form, the characteristic function H being independent of t and expansible in a convergent series of powers of $x_1, \ldots x_n, y_1, \ldots y_n$ in which the terms of lowest degree are

$$\lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \ldots + \lambda_n x_n y_n.$$

The origin is thus for the equations (1) a singular point, of which the exponents are $\pm \lambda_1, \pm \lambda_2, \ldots \pm \lambda_n$. We shall suppose further that the constants $\lambda_1, \ldots \lambda_n$ are connected by no relation of the form

$$A_1\lambda_1 + A_2\lambda_2 + \ldots + A_n\lambda_n = 0, \qquad \ldots\ldots(2)$$

where $A_1, \ldots A_n$ are integers (positive, negative or zero) not all zero; in particular no exponent is zero and no two are equal. Any Hamiltonian system possessing a singular point whose exponents satisfy these conditions may be reduced to the above form by a linear contact transformation*. Since, amongst the aggregate of sets of values of $\lambda_1, \ldots \lambda_n$, sets for which a relation of the form (2) holds form an aggregate of measure zero, the case to be treated may legitimately be regarded as the most general. The reason for the restriction will appear later (see below, p. 327).

It will be shown in § 3 that *there exist n independent power series in $x_1, \ldots x_n, y_1, \ldots y_n$ which formally satisfy the condition* ((3) below) *for integrals of the equations* (1). In § 4 a direct process is given for the construction of these series, and it is found possible to give an expression for the general term. As far as I am aware previous methods of attacking equations of such general form as (1) have given processes for constructing series, but have never been pushed so far as to obtain for these series explicit expression. In § 5 it is shown that there are no integrals developable in powers of $x_1, \ldots x_n, y_1, \ldots y_n$ independent of the n whose existence has been demonstrated†, and that these integrals are in general in involution. § 6 contains some remarks on the convergence of the series found in § 4, but leaves this important question undecided.

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* The coefficients in this transformation are not necessarily real.

† If the origin had been an ordinary point of the equations there would have been $2n - 1$ integrals developable about it. See p. 273, above, "Integrals of Systems of Ordinary Differential Equations."

§ 2. *Method of construction of integrals.*

The condition that a quantity Z, which is a function of $x_1, \dots x_n$, $y_1, \dots y_n$ but not of t, should be an integral of (1) is

$$(Z, H) = 0, \qquad \dots\dots(3)$$

where the expression on the left denotes the Poisson bracket

$$\sum_{r=1}^{n} \left(\frac{\partial Z}{\partial x_r}\frac{\partial H}{\partial y_r} - \frac{\partial Z}{\partial y_r}\frac{\partial H}{\partial x_r}\right).$$

Suppose that Z is a power series in $x_1, \dots x_n, y_1, \dots y_n$ and write

$$Z = Z^{(1)} + Z^{(2)} + Z^{(3)} + \dots, \qquad \dots\dots(4)$$

where $Z^{(r)}$ denotes the aggregate of the terms of degree r; write similarly

$$H = H_2 + H_3 + H_4 + \dots \qquad \dots\dots(5)$$

so that $$H_2 = \lambda_1 x_1 y_1 + \dots + \lambda_n x_n y_n.$$

Then in (3) the terms of each degree must vanish separately, so that substituting (4) and (5) we obtain

$$(Z^{(1)}, H_2) = 0, \qquad \dots\dots(6)$$

$$(Z^{(2)}, H_2) + (Z^{(1)}, H_3) = 0, \qquad \dots\dots(7)_1$$

$$(Z^{(3)}, H_2) + (Z^{(2)}, H_3) + (Z^{(1)}, H_4) = 0, \qquad \dots\dots(7)_2$$

....

a series of partial differential equations from which polynomials $Z^{(1)}, Z^{(2)}, \dots$ are to be determined in succession. Writing

$$Z^{(1)} = c_1 x_1 + \dots + c_n x_n + c_{n+1} y_1 + \dots + c_{2n} y_n,$$

the equation (6) will be satisfied provided

$$c_1\lambda_1 = \dots = c_n\lambda_n = c_{n+1}\lambda_1 = \dots = c_{2n}\lambda_n = 0,$$

and thus since no λ vanishes $Z^{(1)}$ must vanish identically. If now we put

$$Z^{(2)} = \Sigma A_{rs} x_r x_s + \Sigma B_{rs} x_r y_s + \Sigma C_{rs} y_r y_s,$$

$(7)_1$ will be satisfied provided

$$A_{rs}(\lambda_r + \lambda_s) = B_{rs}(\lambda_r - \lambda_s) = C_{rs}(-\lambda_r - \lambda_s) = 0.$$

Thus the coefficients of $x_1 y_1, \dots x_n y_n$ in $Z^{(2)}$ are arbitrary, and since no two of the λ's are equal every other coefficient must vanish. We may thus determine second degree terms for n, and only n, independent integrals; we shall call these integrals $Z_1, \dots Z_n$, and writing as in (4)

$$Z_k = Z_k^{(2)} + Z_k^{(3)} + \dots \quad (k = 1, \dots n) \qquad \dots\dots(8)$$

we may take $$Z_k^{(2)} = x_k y_k \quad (k = 1, \dots n).$$

Now suppose that from the equations $(7)_1$, $(7)_2$, ... we have determined $Z^{(2)}, \dots Z^{(m-1)}$ as polynomials of degrees 2, ... $(m-1)$ respectively; then the equation for $Z^{(m)}$, viz.

$$(Z^{(m)}, H_2) + (Z^{(m-1)}, H_3) + \dots + (Z^{(2)}, H_m) = 0,$$

is of the form

$$\lambda_1 \left(x_1 \frac{\partial Z^{(m)}}{\partial x_1} - y_1 \frac{\partial Z^{(m)}}{\partial y_1} \right) + \dots + \lambda_n \left(x_n \frac{\partial Z^{(m)}}{\partial x_n} - y_n \frac{\partial Z^{(m)}}{\partial y_n} \right) = F(x_1, \dots y_n),$$

where F is a known homogeneous polynomial of degree m. This equation is to be satisfied by a polynomial of degree m. Writing therefore

$$F = \Sigma A x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n},$$

$$Z^{(m)} = \Sigma B x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n},$$

where $r_1 + s_1 + \dots + r_n + s_n = m$ and the A's and B's are constant coefficients, we must have identically

$$\Sigma B x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n} \{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)\}$$
$$= \Sigma A x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n},$$

giving $$B = \frac{A}{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)}.$$

We may thus determine $Z^{(m)}$ as a polynomial *provided that each term in F for which the indices satisfy the relation*

$$\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n) = 0$$

has a vanishing coefficient. Such terms will be called *critical.* The terms for which

$$r_1 = s_1, \; r_2 = s_2, \dots r_n = s_n$$

are always critical, and the necessary and sufficient condition that no other terms should be critical is that the λ's should satisfy no relation of the form (2) with integral coefficients not all zero. We have already supposed this to be the case, and have thereby reduced to a minimum the difficulty arising from the presence of critical terms.

When the coefficient of a critical term in the equation for $Z^{(m)}$ vanishes, this term appears in $Z^{(m)}$ with an *arbitrary* coefficient.

I have not been able to find a direct general proof that the coefficients of the critical terms vanish, though the verification for the first few such terms is given in § 4. This fundamental point is established in § 3 in an indirect manner, by showing that there exist n independent power series which are integrals of the equations (1). The demonstration turns of course on the Hamiltonian form of the equations.

The process just outlined for constructing integrals is of course not new; for example Whittaker* in 1916 gave a process which, with a change of coordinates, is the same. Whittaker however makes no explicit mention of the difficulty arising from the possible presence of critical terms, the satisfactory treatment of which is vital to establishing the validity of the process.

§ 3. *Existence of n integrals developable in powers of* $x_1, \dots x_r, y_1, \dots y_n$.

(i) Let the equations (1) be solved by successive approximation by substituting

$$x_r = x_r^{(1)} + x_r^{(2)} + \dots,$$
$$y_r = y_r^{(1)} + y_r^{(2)} + \dots,$$

and equating separately the groups of terms on either side whose orders, as measured by the sum of the superscripts in their factors, are equal. We thus obtain the equations

$$\frac{dx_r^{(1)}}{dt} = \lambda_r x_r^{(1)}, \quad \frac{dy_r^{(1)}}{dt} = -\lambda_r y_r^{(1)}, \qquad \text{......(9)}$$

$$\frac{dx_r^{(2)}}{dt} - \lambda_r x_r^{(2)} = \text{homogeneous quadratic function of the } x_r^{(1)}, y_r^{(1)},$$

$$\frac{dy_r^{(2)}}{dt} + \lambda_r y_r^{(2)} = \text{homogeneous quadratic function of the } x_r^{(1)}, y_r^{(1)},$$

....

From (9) we have

$$x_r^{(1)} = \alpha_r e^{\lambda_r t}, \quad y_r^{(1)} = \beta_r e^{-\lambda_r t},$$

where α_r, β_r are arbitrary constants. The equation for $x_r^{(2)}$ then has the form

$$\frac{dx_r^{(2)}}{dt} - \lambda_r x_r^{(2)} = \Sigma A \alpha_k \alpha_s e^{(\lambda_k+\lambda_s)t} + \Sigma B \alpha_k \beta_s e^{(\lambda_k-\lambda_s)t} + \Sigma C \beta_k \beta_s e^{-(\lambda_k+\lambda_s)t}.$$

Now if $\lambda \neq \mu$ the Particular Integral of

$$\frac{dx}{dt} - \lambda x = Ae^{\mu t}$$

is
$$x = \frac{Ae^{\mu t}}{\mu - \lambda},$$

and the Particular Integral of

$$\frac{dx}{dt} - \lambda x = Ae^{\lambda t}$$

is
$$x = Ate^{\lambda t}.$$

Thus $x_r^{(2)}$, and similarly $y_r^{(2)}$, are homogeneous quadratic functions of $\alpha_1 e^{\lambda_1 t}, \dots \alpha_n e^{\lambda_n t}, \beta_1 e^{-\lambda_1 t}, \dots \beta_n e^{-\lambda_n t}$ whose coefficients are either constants or linear functions of t. It is easy to see similarly that $x_r^{(3)}$, $y_r^{(3)}$ are homogeneous cubic functions of these same quantities with coefficients which are polynomials in t of at most second degree; and similarly for $x_r^{(4)}, \ldots$. Thus *the equations* (1) *may be formally solved by power series in*

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* "On the Adelphic Integral of the Differential Equations of Dynamics" (*Proc. Roy. Soc. Edin.*, November, 1916).

$$\alpha_1 e^{\lambda_1 t}, \ldots \alpha_n e^{\lambda_n t}, \ \beta_1 e^{-\lambda_1 t}, \ldots \beta_n e^{-\lambda_n t}, t \ldots \qquad \ldots\ldots(10)$$

where $\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n$ *are arbitrary constants.* Suppose that

$$\left.\begin{aligned} x_r &= f_r(\alpha_1 e^{\lambda_1 t}, \ldots \alpha_n e^{\lambda_n t}, \ \beta_1 e^{-\lambda_1 t}, \ldots \beta_n e^{-\lambda_n t}, t) \\ y_r &= g_r(\alpha_1 e^{\lambda_1 t}, \ldots \alpha_n e^{\lambda_n t}, \ \beta_1 e^{-\lambda_1 t}, \ldots \beta_n e^{-\lambda_n t}, t) \end{aligned}\right\} (r = 1, \ldots n). \quad (11)$$

We shall show that *if* (11) *be solved so as to obtain* $\alpha_1 e^{\lambda_1 t}, \ldots \beta_n e^{-\lambda_n t}$ *as power series in* $x_1, \ldots x_n, y_1, \ldots y_n, t$, *say*

$$\left.\begin{aligned} \alpha_r e^{\lambda_r t} &= \phi_r(x_1, \ldots x_n, y_1, \ldots y_n, t) \\ \beta_r e^{-\lambda_r t} &= \psi_r(x_1, \ldots x_n, y_1, \ldots y_n, t) \end{aligned}\right\} (r = 1, \ldots n), \ \ldots(12)$$

then the products $\phi_1\psi_1, \phi_2\psi_2, \ldots \phi_n\psi_n$ *are independent of* t. Since

$$\phi_r\psi_r = \alpha_r\beta_r$$

and α_r, β_r are arbitrary constants, $\phi_r\psi_r$ must be an integral of equations (1); as just stated we shall show that this integral does not involve t, while from its mode of formation it is a power series in $x_1, \ldots x_n, y_1, \ldots y_n$.

(ii) If we substitute in (11) the values of $\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n$ given by (12) we obtain $2n$ identities in $x_1, \ldots x_n, y_1, \ldots y_n, t$; differentiating these partially with respect to t we obtain

$$\left.\begin{aligned} 0 &= \frac{\partial x_r}{\partial t} + \sum_{s=1}^{n}\left(\frac{\partial x_r}{\partial \alpha_s}\frac{\partial \alpha_s}{\partial t} + \frac{\partial x_r}{\partial \beta_s}\frac{\partial \beta_s}{\partial t}\right) \\ 0 &= \frac{\partial y_r}{\partial t} + \sum_{s=1}^{n}\left(\frac{\partial y_r}{\partial \alpha_s}\frac{\partial \alpha_s}{\partial t} + \frac{\partial y_r}{\partial \beta_s}\frac{\partial \beta_s}{\partial t}\right) \end{aligned}\right\} (r = 1, \ldots n), \ \ldots(13)$$

where, indicating the arguments which remain constant during differentiation by subscripts,

$$\frac{\partial x_r}{\partial t} \text{ denotes } \left(\frac{\partial x_r}{\partial t}\right)_{\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n},$$

$$\frac{\partial x_r}{\partial \alpha_1} \text{ denotes } \left(\frac{\partial x_r}{\partial \alpha_1}\right)_{t, \alpha_2, \ldots \alpha_n, \beta_1, \ldots \beta_n},$$

$$\frac{\partial \alpha_s}{\partial t} \text{ denotes } \left(\frac{\partial \alpha_s}{\partial t}\right)_{x_1, \ldots x_n, y_1, \ldots y_n},$$

and so on. We may thus put

$$\frac{\partial x_r}{\partial t} = \frac{\partial H}{\partial y_r}, \ \frac{\partial y_r}{\partial t} = -\frac{\partial H}{\partial x_r} \quad (r = 1, \ldots n). \qquad \ldots\ldots(14)$$

From (13) we may derive an equivalent set of equations by use of the multiplier system

$$\left.\begin{array}{l} \dfrac{\partial y_1}{\partial \alpha_p}, -\dfrac{\partial x_1}{\partial \alpha_p}, \dfrac{\partial y_2}{\partial \alpha_p}, -\dfrac{\partial x_2}{\partial \alpha_p}, \dots \\ \dfrac{\partial y_1}{\partial \beta_p}, -\dfrac{\partial x_1}{\partial \beta_p}, \dfrac{\partial y_2}{\partial \beta_p}, -\dfrac{\partial x_2}{\partial \beta_p}, \dots \end{array}\right\} \quad (p = 1, \dots n);$$

in virtue of (14) we have

$$\frac{\partial x_1}{\partial t}\frac{\partial y_1}{\partial \alpha_p} - \frac{\partial y_1}{\partial t}\frac{\partial x_1}{\partial \alpha_p} + \frac{\partial x_2}{\partial t}\frac{\partial y_2}{\partial \alpha_p} - \frac{\partial y_2}{\partial t}\frac{\partial x_2}{\partial \alpha_p} + \dots = \frac{\partial H}{\partial \alpha_p},$$

and the desired equations are thus seen to reduce to

$$\left.\begin{array}{l} \dfrac{\partial H}{\partial \alpha_p} + \sum\limits_{s=1}^{n} \left\{\dfrac{\partial \alpha_s}{\partial t}[\alpha_s, \alpha_p] + \dfrac{\partial \beta_s}{\partial t}[\beta_s, \alpha_p]\right\} = 0 \\ \dfrac{\partial H}{\partial \beta_p} + \sum\limits_{s=1}^{n} \left\{\dfrac{\partial \alpha_s}{\partial t}[\alpha_s, \beta_p] + \dfrac{\partial \beta_s}{\partial t}[\beta_s, \beta_p]\right\} = 0 \end{array}\right\} (p = 1, \dots n), \quad (15)$$

where the symbol $[\gamma, \delta]$ denotes the Lagrange bracket

$$[\gamma, \delta] = \sum_{r=1}^{n} \left(\frac{\partial x_r}{\partial \gamma}\frac{\partial y_r}{\partial \delta} - \frac{\partial x_r}{\partial \delta}\frac{\partial y_r}{\partial \gamma}\right) \qquad \dots\dots(16)$$
$$= -[\delta, \gamma].$$

From these equations we shall show that $\dfrac{\partial}{\partial t}(\alpha_r \beta_r) = 0$.

(iii) We shall first consider the form of the Lagrange brackets. From (11) we see that

$$\left.\begin{array}{ll} \dfrac{\partial x_r}{\partial \alpha_s} = e^{\lambda_s t} P_{rs}, & \dfrac{\partial y_r}{\partial \alpha_s} = e^{\lambda_s t} P_{rs}' \\ \dfrac{\partial x_r}{\partial \beta_s} = e^{-\lambda_s t} P_{rs}'', & \dfrac{\partial y_r}{\partial \beta_s} = e^{-\lambda_s t} P_{rs}''' \end{array}\right\} (r = 1, \dots n),$$

where the P's are power series in the arguments (10); hence

$$\left.\begin{array}{l} [\alpha_r, \alpha_s] = e^{(\lambda_r + \lambda_s)t} Q \\ [\alpha_r, \beta_s] = e^{(\lambda_r - \lambda_s)t} Q' \\ [\beta_r, \beta_s] = e^{-(\lambda_r + \lambda_s)t} Q'' \end{array}\right\}, \qquad \dots\dots(17)$$

where the Q's are power series in these same arguments. We know, however, from general theory* that each of the Lagrange brackets is independent of t, i.e. is a function only of the constants of integration $\alpha_1, \dots \alpha_n, \beta_1, \dots \beta_n$. Thus from (17) we must have

$$\left.\begin{array}{l} [\alpha_r, \alpha_s] = e^{(\lambda_r + \lambda_s)t} \,.\, \beta_r e^{-\lambda_r t} \,.\, \beta_s e^{-\lambda_s t} \,.\, R \\ [\beta_r, \beta_s] = e^{-(\lambda_r + \lambda_s)t} \,.\, \alpha_r e^{\lambda_r t} \,.\, \alpha_s e^{\lambda_s t} \,.\, R' \\ [\alpha_r, \beta_s] = e^{(\lambda_r - \lambda_s)t} \,.\, \beta_r e^{-\lambda_r t} \,.\, \alpha_s e^{\lambda_s t} \,.\, R'' \quad (r \neq s) \\ [\alpha_r, \beta_r] = R''' \end{array}\right\}, \dots(18)$$

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* Whittaker, *Analytical Dynamics* (2nd ed.), § 146.

where the R's are power series in the arguments (10), which must reduce to functions of $\alpha_1, \dots \alpha_n, \beta_1, \dots \beta_n$ only, so that the terms which vary with t must vanish identically. Now the general term of these series is of the form

$$At^m \,.\, (\alpha_1 e^{\lambda_1 t})^{p_1} \,.\, (\beta_1 e^{-\lambda_1 t})^{q_1} \dots (\alpha_n e^{\lambda_n t})^{p_n} \,.\, (\beta_n e^{-\lambda_n t})^{q_n},$$

where A is a constant and $m, p_1, q_1, \dots p_n, q_n$ are positive integers, and, since $\lambda_1, \dots \lambda_n$ are connected by no relation of the form (2) with integral coefficients not all zero, this can be independent of t only if

$$m = 0,\ p_1 = q_1,\ p_2 = q_2, \dots p_n = q_n;$$

any term independent of t is thus to a constant multiple the product of powers of $\alpha_1\beta_1, \alpha_2\beta_2, \dots \alpha_n\beta_n$. We shall write

$$\alpha_1\beta_1 = u_1, \dots \alpha_n\beta_n = u_n \qquad \text{......(19)}$$

so that the R's in (18) are power series in $u_1, \dots u_n$ and do not involve the α's and β's except in these combinations. Since the linear terms in f_r, g_r (equations (11)) are respectively $\alpha_r e^{\lambda_r t}$, $\beta_r e^{-\lambda_r t}$ it is easily seen that when $u_1 = u_2 = \dots = u_n = 0$ we have $R'' = 1$. We write therefore

$$\left.\begin{aligned}
[\alpha_r, \alpha_s] &= \beta_r\beta_s A_{rs}(u_1, \dots u_n)\\
[\beta_r, \beta_s] &= \alpha_r\alpha_s B_{rs}(u_1, \dots u_n)\\
[\alpha_r, \beta_s] &= \beta_r\alpha_s C_{rs}(u_1, \dots u_n) \quad (r \neq s)\\
[\alpha_r, \beta_r] &= 1 + D_r(u_1, \dots u_n)
\end{aligned}\right\}. \qquad \text{......(20)}$$

(iv) Since H is independent of t the equations (1) possess the integral

$$H = \text{constant},$$

and thus if by means of (11) we express H as a function of t, $\alpha_1, \dots \alpha_n, \beta_1, \dots \beta_n$ the result must be independent of t. It may be seen by an argument precisely similar to the above that H must thereby become a power series in the arguments $u_1, \dots u_n$.

(v) The functions A_{rs}, B_{rs}, C_{rs} occurring in (20) are connected by certain simple linear relations. To establish these we use the identities

$$\frac{\partial}{\partial\beta_s}[\alpha_r, \alpha_s] + \frac{\partial}{\partial\alpha_r}[\alpha_s, \beta_s] + \frac{\partial}{\partial\alpha_s}[\beta_s, \alpha_r] = 0, \quad \text{......(21)}$$

$$\frac{\partial}{\partial\alpha_s}[\beta_r, \beta_s] + \frac{\partial}{\partial\beta_r}[\beta_s, \alpha_s] + \frac{\partial}{\partial\beta_s}[\alpha_s, \beta_r] = 0, \quad \text{......(22)}$$

which are easily proved by direct differentiation from the definition (16). Substituting for the Lagrange brackets their values (20) and remembering that $u_r = \alpha_r\beta_r$ the identity (21) becomes

$$\beta_r A_{rs} + \beta_r\beta_s \,.\, \alpha_s \frac{\partial A_{rs}}{\partial u_s} + \beta_r \frac{\partial D_s}{\partial u_r} - \beta_r C_{rs} - \beta_r\alpha_s \,.\, \beta_s \frac{\partial C_{rs}}{\partial u_s} = 0,$$

i.e. $$A_{rs} - C_{rs} + u_s \frac{\partial}{\partial u_s}(A_{rs} - C_{rs}) + \frac{\partial D_s}{\partial u_r} = 0.$$

From (22) we obtain similarly

$$B_{rs} + C_{sr} + u_s \frac{\partial}{\partial u_s}(B_{rs} + C_{sr}) - \frac{\partial D_s}{\partial u_r} = 0,$$

whence by addition

$$\frac{\partial}{\partial u_s}\{u_s(A_{rs} + B_{rs} - C_{rs} + C_{sr})\} = 0.$$

Thus $u_s(A_{rs} + B_{rs} - C_{rs} + C_{sr})$, which is a power series in $u_1, \dots u_n$ vanishing with u_s, must be independent of u_s, and this is possible only if

$$A_{rs} + B_{rs} - C_{rs} + C_{sr} = 0. \qquad \text{......(23)}$$

There is a relation of this sort for each unequal pair of suffixes from 1, 2, ... n.

(vi) Now return to the equations (15); multiplying the first two by α_1, β_1 respectively and subtracting we have, on introducing the values (20) for the Lagrange brackets,

$$\alpha_1 \frac{\partial H}{\partial \alpha_1} - \beta_1 \frac{\partial H}{\partial \beta_1} = \beta_1(1 + D_1)\frac{\partial \alpha_1}{\partial t} - \sum_{s=2}^{n}(\alpha_1\beta_s\beta_1 A_{s1} - \beta_1\beta_s\alpha_1 C_{s1})\frac{\partial \alpha_s}{\partial t}$$
$$+ \alpha_1(1 + D_1)\frac{\partial \beta_1}{\partial t} + \sum_{s=2}^{n}(\alpha_1\beta_1\alpha_s C_{1s} + \beta_1\alpha_s\alpha_1 B_{s1})\frac{\partial \beta_s}{\partial t}.$$

Since H is a function only of $u_1, \dots u_n$ the left-hand side has the value

$$\alpha_1 \,.\, \beta_1 \frac{\partial H}{\partial u_1} - \beta_1 \,.\, \alpha_1 \frac{\partial H}{\partial u_1}, = 0;$$

and since from (23) $B_{s1} + C_{1s} = C_{s1} - A_{s1}$, we obtain

$$(1 + D_1)\left(\beta_1 \frac{\partial \alpha_1}{\partial t} + \alpha_1 \frac{\partial \beta_1}{\partial t}\right) + \sum_{s=2}^{n} \alpha_1\beta_1(B_{s1} + C_{1s})\left(\alpha_s \frac{\partial \beta_s}{\partial t} + \beta_s \frac{\partial \alpha_s}{\partial t}\right) = 0,$$

i.e.

$$(1 + D_1)\frac{\partial u_1}{\partial t} + u_1(B_{21} + C_{12})\frac{\partial u_2}{\partial t} + \dots + u_1(B_{n1} + C_{1n})\frac{\partial u_n}{\partial t} = 0.$$

From the remainder of equations (15) we may derive similarly

$$u_2(B_{12} + C_{21})\frac{\partial u_1}{\partial t} + (1 + D_2)\frac{\partial u_2}{\partial t} + \dots + u_2(B_{n2} + C_{2n})\frac{\partial u_n}{\partial t} = 0,$$

....

$$u_n(B_{1n} + C_{n1})\frac{\partial u_1}{\partial t} + u_n(B_{2n} + C_{n2})\frac{\partial u_2}{\partial t} + \dots + (1 + D_n)\frac{\partial u_n}{\partial t} = 0.$$

We thus have a set of n homogeneous linear equations in

$$\frac{\partial u_1}{\partial t}, \dots \frac{\partial u_n}{\partial t},$$

whose determinant has the value 1 when $u_1 = \ldots = u_n = 0$ and is therefore not identically zero. We can thus infer that

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t} = \ldots = \frac{\partial u_n}{\partial t} = 0.$$

This is the required result; it shows that the n quantities

$$\phi_1\psi_1 = \alpha_1\beta_1 = u_1, \ldots \phi_n\psi_n = u_n,$$

which from their mode of derivation from (11) are integrals of the equations and are power series in $x_1, \ldots x_n, y_1, \ldots y_n, t$, are in fact integrals independent of t.

It may be verified without excessive labour that as far as the fourth degree terms in the x's and y's the terms in t disappear from the products $\phi_1\psi_1, \ldots \phi_n\psi_n$, and that the resulting series are identical with the series whose explicit expression is found below. In particular the terms of lowest degree depend only on the first degree terms in f_r, g_r, and may immediately be seen to be $x_1y_1, \ldots x_ny_n$ respectively.

§ 4. *Direct construction of the integrals.*

We have seen in § 2 that there are n and only n independent integrals whose expansions begin with second degree terms, and writing these in the form (8) we have

$$Z_k^{(2)} = x_k y_k \quad (k = 1, \ldots n). \qquad \ldots\ldots(24)$$

The terms of successive degrees are to be determined from the partial differential equations

$$(Z_k^{(2)}, H_2) = 0,$$

$$(Z_k^{(3)}, H_2) + (Z_k^{(2)}, H_3) = 0, \qquad \ldots\ldots(25)$$

$$(Z_k^{(4)}, H_2) + (Z_k^{(3)}, H_3) + (Z_k^{(2)}, H_4) = 0, \qquad \ldots\ldots(26)$$

....

(of which the first is satisfied by (24)) which are derived by substituting in $(Z_k, H) = 0$ the expansions (8), (5). The result of § 3 assures us that the critical terms vanish from all these equations, and therefore that they may be solved in succession by polynomials, as desired.

(i) *Determination of* $Z_k^{(3)}$. Write

$$H_3 = \Sigma A^{(3)}_{r_1 s_1 \ldots r_n s_n} x_1^{r_1} y_1^{s_1} \ldots x_n^{r_n} y_n^{s_n}, \qquad \ldots\ldots(27)$$

where $r_1, \ldots s_n$ are positive or zero integers whose sum is 3. Then since $Z_k^{(2)} = x_k y_k$ we have

$$(Z_k^{(2)}, H_3) = y_k \frac{\partial H_3}{\partial y_k} - x_k \frac{\partial H_3}{\partial x_k}$$
$$= -\Sigma A^{(3)}_{r_1 s_1 \ldots r_n s_n} (r_k - s_k)\, x_1^{r_1} y_1^{s_1} \ldots x_n^{r_n} y_n^{s_n},$$

and the equation (25) is

$$\sum_{r=1}^{n} \lambda_r \left(x_r \frac{\partial Z_k^{(3)}}{\partial x_r} - y_r \frac{\partial Z_k^{(3)}}{\partial y_r} \right) = \Sigma A^{(3)}{}_{r_1 s_1 \dots r_n s_n} (r_k - s_k)\, x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}.$$

Since $r_1 + s_1 + \dots + r_n + s_n$ is odd it is impossible that we should have simultaneously

$$r_1 = s_1, \quad r_2 = s_2, \dots \; r_n = s_n$$

and thus there are no critical terms on the right. As has been seen in § 2, we may write down the solution of this equation from the polynomial on its right by introducing in each term a denominator, that appropriate to the term $x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}$ being

$$\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n).$$

Hence

$$Z_k^{(3)} = \Sigma A^{(3)}{}_{r_1 s_1 \dots r_n s_n} \cdot \frac{r_k - s_k}{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)} \cdot x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}. \qquad \dots\dots(28)$$

(ii) *Determination of* $Z_k^{(4)}$. Write

$$H_4 = \Sigma A^{(4)}{}_{r_1 s_1 \dots r_n s_n} x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}, \qquad \dots\dots(29)$$

where the summation is over all sets of positive or zero integers $r_1, \dots s_n$ whose sum is 4. We then have

$$(Z_k^{(2)}, H_4) = -\Sigma A^{(4)}{}_{r_1 s_1 \dots r_n s_n} (r_k - s_k)\, x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}.$$

Since $Z_k^{(3)}$ is a homogeneous linear function of the coefficients in the polynomial H_3 the Poisson bracket $(Z_k^{(3)}, H_3)$ consists of the sum of terms of which each contains as factor the product of two coefficients from H_3; the terms which depend on the typical product $A^{(3)}{}_{r_1 s_1 \dots r_n s_n} \cdot A^{(3)}{}_{r_1' s_1' \dots r_n' s_n'}$ are evidently

$$A^{(3)}{}_{r_1 s_1 \dots r_n s_n} A^{(3)}{}_{r_1' s_1' \dots r_n' s_n'} \left[\frac{(r_k - s_k)\,(x_1^{r_1} \dots y_n^{s_n},\ x_1^{r_1'} \dots y_n^{s_n'})}{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)} + \frac{(r_k' - s_k')\,(x_1^{r_1'} \dots y_n^{s_n'},\ x_1^{r_1} \dots y_n^{s_n})}{\lambda_1 (r_1' - s_1') + \dots + \lambda_n (r_n' - s_n')} \right].$$

Thus the equation (26) is

$$(Z_k^{(4)}, H_2) = \Sigma A^{(4)}{}_{r_1 \dots s_n} (r_k - s_k)\, x_1^{r_1} \dots y_n^{s_n}$$
$$- \Sigma \frac{A^{(3)}{}_{r_1 \dots s_n} A^{(3)}{}_{r_1' \dots s_n'} (r_k - s_k)}{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)} \cdot (x_1^{r_1} \dots y_n^{s_n},\ x_1^{r_1'} \dots y_n^{s_n'}), \quad (30)$$

where the first summation is over all coefficients from H_4 and the second over all *permutations* of pairs of coefficients from H_3. We must show that the coefficients of the critical terms $x_1^2 y_1^2$, $x_1 y_1 x_2 y_2$, ... vanish.

(*a*) *Critical terms depending on a coefficient* $A^{(4)}{}_{r_1 \dots s_n}$. If such a term is critical we have $r_k = s_k$, and hence from (30) its coefficient vanishes.

(*b*) *Critical terms depending on a product of two coefficients from* H_3. The Poisson bracket $(x_1^{r_1} \dots y_n^{s_n},\ x_1^{r_1'} \dots y_n^{s_n'})$ has the value

$$(r_1 s_1' - r_1' s_1)\, x_1^{r_1+r_1'-1}\, y_1^{s_1+s_1'-1}\, x_2^{r_2+r_2'}\, y_2^{s_2+s_2'} \dots x_n^{r_n+r_n'}\, y_n^{s_n+s_n'}$$
$$+ (r_2 s_2' - r_2' s_2)\, x_1^{r_1+r_1'}\, y_1^{s_1+s_1'}\, x_2^{r_2+r_2'-1}\, y_2^{s_2+s_2'-1} \dots x_n^{r_n+r_n'}\, y_n^{s_n+s_n'} + \dots . \qquad \dots\dots(31)$$

Hence critical terms arise only for those systems of indices $r_1, \dots s_n$; $r_1', \dots s_n'$ for which

$$r_1 + r_1' = s_1 + s_1',\ r_2 + r_2' = s_2 + s_2',\ \dots\ r_n + r_n' = s_n + s_n'.$$

From these equalities it follows that

$$r_k - s_k = -(r_k' - s_k')$$
$$\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n) = -\{\lambda_1 (r_1' - s_1') + \dots + \lambda_n (r_n' - s_n')\}.$$

Hence in (30) the coefficient of $A^{(3)}{}_{r_1 \dots s_n} A^{(3)}{}_{r_1' \dots s_n'}$ is

$$(x_1^{r_1} \dots y_n^{s_n},\ x_1^{r_1'} \dots y_n^{s_n'}) \left[\frac{r_k - s_k}{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)} - \frac{r_k' - s_k'}{\lambda_1 (r_1' - s_1') + \dots + \lambda_n (r_n' - s_n')}\right] = 0.$$

Thus on the right of (30) the coefficients of all critical terms vanish.

The expression for $Z_k^{(4)}$ is to be deduced from the polynomial on the right of (30) by introducing in each term the appropriate denominator. From (31) it follows that the denominator appropriate to each term arising from the Poisson bracket

$$(x_1^{r_1} \dots y_n^{s_n},\ x_1^{r_1'} \dots y_n^{s_n'})$$

is the same, and equal to

$$\{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)\} + \{\lambda_1 (r_1' - s_1') + \dots + \lambda_n (r_n' - s_n')\}.$$

Thus

$$Z_k^{(4)} = \Sigma A^{(4)}{}_{r_1 \dots s_n} \cdot \frac{r_k - s_k}{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)} \cdot x_1^{r_1} \dots y_n^{s_n} - \Sigma A^{(3)}{}_{r_1 \dots s_n} A^{(3)}{}_{r_1' \dots s_n'}$$
$$\times \frac{(r_k - s_k) \cdot (x_1^{r_1} \dots y_n^{s_n},\ x_1^{r_1'} \dots y_n^{s_n'})}{\{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)\}\{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n) + \lambda_1 (r_1' - s_1') + \dots + \lambda_n (r_n' - s_n')\}} \qquad \dots\dots(32)$$

where the summations are as in (30) and we make the convention that in any term with a vanishing denominator this denominator is to be replaced by an *arbitrary* finite coefficient.

Before proceeding further we shall introduce an abridged notation. A term

$$x_1^{r_1^a}\, y_1^{s_1^a} \dots x_n^{r_n^a}\, y_n^{s_n^a}$$

(a being a superscript, not an exponent) will be written ϕ_a, and the corresponding coefficient in H will be called $A_a^{(m)}$, where m denotes the degree of the term, viz.

$$r_1^a + s_1^a + \dots + r_n^a + s_n^a.$$

The appropriate denominator will be called D_α, so that

$$D_\alpha = \lambda_1 (r_1^\alpha - s_1^\alpha) + \dots + \lambda_n (r_n^\alpha - s_n^\alpha).$$

The denominator appropriate to the product $\phi_\alpha \phi_\beta$ will be called $D_{\alpha\beta}$, and we have evidently

$$D_{\alpha\beta} = D_\alpha + D_\beta; \qquad \dots\dots(33)$$

we have seen also that this same denominator is appropriate to each term arising from the Poisson bracket $(\phi_\alpha, \phi_\beta)$. In extension of the notation (33) we shall write

$$D_{\alpha\beta\gamma} = D_{\alpha\beta} + D_\gamma = D_\alpha + D_\beta + D_\gamma$$

and so on. With this notation we have

$$Z_k^{(3)} = \Sigma A_\alpha^{(3)} \cdot \frac{r_k^\alpha - s_k^\alpha}{D_\alpha} \cdot \phi_\alpha, \qquad \dots\dots(34)$$

$$Z_k^{(4)} = \Sigma A_\alpha^{(4)} \cdot \frac{r_k^\alpha - s_k^\alpha}{D_\alpha} \cdot \phi_\alpha - \Sigma A_\alpha^{(3)} A_\beta^{(3)} \cdot \frac{r_k^\alpha - s_k^\alpha}{D_\alpha D_{\alpha\beta}} (\phi_\alpha, \phi_\beta). \quad (35)$$

(iii) *Determination of* $Z_k^{(5)}$. We write

$$H_5 = \Sigma A^{(5)}_{r_1 s_1 \dots r_n s_n} x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n} = \Sigma A_\alpha^{(5)} \phi_\alpha, \quad \dots(36)$$

where the summation is over all sets of positive or zero integers $r_1, \dots s_n$ whose sum is 5.

Then we have from (24), (36)

$$(Z_k^{(2)}, H_5) = - \Sigma A_\alpha^{(5)} (r_k^\alpha - s_k^\alpha) \phi_\alpha, \qquad \dots\dots(37)$$

and from (29), (34)

$$(Z_k^{(3)}, H_4) = \Sigma A_\alpha^{(3)} A_\beta^{(4)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha}, \quad \dots\dots(38)$$

the summation being over all pairs of coefficients $A_\alpha^{(3)}$, $A_\beta^{(4)}$ from H_3, H_4 respectively. Also from (27), (35)

$$(Z_k^{(4)}, H_3) = \Sigma A_\alpha^{(4)} A_\beta^{(3)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha}$$

$$- \Sigma A_\alpha^{(3)} A_\beta^{(3)} A_\gamma^{(3)} (r_k^\alpha - s_k^\alpha) \frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_{\alpha\beta}}, \quad \dots\dots(39)$$

where the first summation is over all pairs of coefficients $A_\alpha^{(4)}$, $A_\beta^{(3)}$ from H_4, H_3 respectively, and the second over all third order permutations $A_\alpha^{(3)} A_\beta^{(3)} A_\gamma^{(3)}$ of coefficients from H_3; in these permutations the suffixes α, β, γ are not necessarily distinct, but every possible choice of the first suffix is to be taken with every possible choice of the second and every possible choice of the third, so that if there are m possible choices of any suffix there are m^3 terms in the summation.

It is easily seen that in each term of the triple Poisson bracket $((\phi_\alpha, \phi_\beta), \phi_\gamma)$ the indices of x_r and y_r (for $r = 1, 2, \dots n$) are lowered by the same amount from their values in the product $\phi_\alpha \phi_\beta \phi_\gamma$;

hence the denominator appropriate to each of these terms is the same, viz. $D_{\alpha\beta\gamma}$.

The equation for $Z_k^{(5)}$, viz.

$$(Z_k^{(5)}, H_2) + (Z_k^{(4)}, H_3) + (Z_k^{(3)}, H_4) + (Z_k^{(2)}, H_5) = 0,$$

becomes on use of (37), (38), (39)

$$\begin{aligned}(Z_k^{(5)}, H_2) = \Sigma A_\alpha^{(5)} (r_k^\alpha - s_k^\alpha)\,\phi_\alpha &- \Sigma A_\alpha^{(3)} A_\beta^{(4)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha} \\ &- \Sigma A_\alpha^{(4)} A_\beta^{(3)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha} \\ &+ \Sigma A_\alpha^{(3)} A_\beta^{(3)} A_\gamma^{(3)} (r_k^\alpha - s_k^\alpha) \frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_{\alpha\beta}}.\end{aligned}$$

Since the polynomial on the right is of fifth degree there are no critical terms, and we deduce

$$\begin{aligned}Z_k^{(5)} = \Sigma A_\alpha^{(5)} (r_k^\alpha - s_k^\alpha) \frac{\phi_\alpha}{D_\alpha} &- \Sigma A_\alpha^{(3)} A_\beta^{(4)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha D_{\alpha\beta}} \\ &- \Sigma A_\alpha^{(4)} A_\beta^{(3)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha D_{\alpha\beta}} \\ &+ \Sigma A_\alpha^{(3)} A_\beta^{(3)} A_\gamma^{(3)} (r_k^\alpha - s_k^\alpha) \frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_{\alpha\beta} D_{\alpha\beta\gamma}} \qquad (40)\end{aligned}$$

with the convention that for a term with a vanishing denominator that denominator is to be replaced by an arbitrary finite coefficient.

(iv) *Determination of* $Z_k^{(6)}$. Writing

$$H_6 = \Sigma A^{(6)}_{r_1 s_1 \dots r_n s_n} x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n} = \Sigma A_\alpha^{(6)} \phi_\alpha,$$

the equation for $Z_k^{(6)}$, viz.

$$(Z_k^{(6)}, H_2) + (Z_k^{(5)}, H_3) + (Z_k^{(4)}, H_4) + (Z_k^{(3)}, H_5) + (Z_k^{(2)}, H_6) = 0,$$

is found as above to reduce to

$$\begin{aligned}(Z_k^{(6)}, H_2) = \Sigma A_\alpha^{(6)} (r_k^\alpha - s_k^\alpha)\,\phi_\alpha &- \Sigma A_\alpha^{(5)} A_\beta^{(3)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha} \\ - \Sigma A_\alpha^{(4)} A_\beta^{(4)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha} &- \Sigma A_\alpha^{(3)} A_\beta^{(5)} (r_k^\alpha - s_k^\alpha) \frac{(\phi_\alpha, \phi_\beta)}{D_\alpha} \\ + \Sigma A_\alpha^{(3)} A_\beta^{(3)} A_\gamma^{(4)} (r_k^\alpha - s_k^\alpha) &\frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_{\alpha\beta}} \\ + \Sigma A_\alpha^{(3)} A_\beta^{(4)} A_\gamma^{(3)} (r_k^\alpha - s_k^\alpha) &\frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_{\alpha\beta}} \\ + \Sigma A_\alpha^{(4)} A_\beta^{(3)} A_\gamma^{(3)} (r_k^\alpha - s_k^\alpha) &\frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_{\alpha\beta}} \\ - \Sigma A_\alpha^{(3)} A_\beta^{(3)} A_\gamma^{(3)} A_\delta^{(3)} (r_k^\alpha - s_k^\alpha) &\frac{(((\phi_\alpha, \phi_\beta), \phi_\gamma), \phi_\delta)}{D_\alpha D_{\alpha\beta} D_{\alpha\beta\gamma}}, \quad \dots\dots(41)\end{aligned}$$

where as before the summations are over permutations (in which repetitions are allowed) of coefficients from the terms of H of the indicated degrees. We must show that the coefficients of the critical terms $x_1^3 y_1^3$, $x_1^2 y_1^2 x_2 y_2$, $x_1 y_1 x_2 y_2 x_3 y_3$, ... on the right-hand side vanish.

(*a*) *Critical terms depending on a coefficient* $A_\alpha^{(6)}$ *or a binary product* $A_\alpha A_\beta$. These vanish as in (ii) above.

(*b*) *Critical terms depending on a ternary product* $A_\alpha A_\beta A_\gamma$. From (41) the terms involving any three selected coefficients, two from H_3 and one from H_4, are (supposing the terms $\phi_\alpha, \phi_\beta, \phi_\gamma$ unequal*)

$$\begin{aligned}&+ A_\alpha A_\beta A_\gamma (r_k^\alpha - s_k^\alpha) \frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_{\alpha\beta}} + A_\beta A_\alpha A_\gamma (r_k^\beta - s_k^\beta) \frac{((\phi_\beta, \phi_\alpha), \phi_\gamma)}{D_\beta D_{\beta\alpha}}\\ &+ A_\beta A_\gamma A_\alpha (r_k^\beta - s_k^\beta) \frac{((\phi_\beta, \phi_\gamma), \phi_\alpha)}{D_\beta D_{\beta\gamma}} + A_\gamma A_\beta A_\alpha (r_k^\gamma - s_k^\gamma) \frac{((\phi_\gamma, \phi_\beta), \phi_\alpha)}{D_\gamma D_{\gamma\beta}}\\ &+ A_\gamma A_\alpha A_\beta (r_k^\gamma - s_k^\gamma) \frac{((\phi_\gamma, \phi_\alpha), \phi_\beta)}{D_\gamma D_{\gamma\alpha}} + A_\alpha A_\gamma A_\beta (r_k^\alpha - s_k^\alpha) \frac{((\phi_\alpha, \phi_\gamma), \phi_\beta)}{D_\alpha D_{\alpha\gamma}},\end{aligned} \quad ......(42)$$

where since the terms are critical we have

$$r_i^\alpha + r_i^\beta + r_i^\gamma = s_i^\alpha + s_i^\beta + s_i^\gamma \quad (i = 1, \ldots n), \quad ...(43)$$

so that

$$D_\alpha + D_\beta + D_\gamma = 0, \quad ......(44)$$

$$D_{\alpha\beta} = - D_\gamma, \; D_{\beta\gamma} = - D_\alpha, \; D_{\gamma\alpha} = - D_\beta. \quad ......(45)$$

We thus have

$$\begin{aligned}(r_k^\alpha - s_k^\alpha) D_\beta - (r_k^\beta - s_k^\beta) D_\alpha &= (D_\beta + D_\gamma)(r_k^\beta - s_k^\beta) + (r_k^\alpha - s_k^\alpha) D_\beta, \quad \text{from (44)}\\ &= D_\gamma (r_k^\beta - s_k^\beta) + D_\beta (r_k^\beta - s_k^\beta + r_k^\alpha - s_k^\alpha)\\ &= D_\gamma (r_k^\beta - s_k^\beta) - D_\beta (r_k^\gamma - s_k^\gamma), \quad \text{from (43)}\\ &= D_\alpha (r_k^\gamma - s_k^\gamma) - D_\gamma (r_k^\alpha - s_k^\alpha) \text{ by symmetry.} \quad ...(46)\end{aligned}$$

Combining the terms of (42) in pairs as they stand this expression has the value

$$\begin{aligned}&A_\alpha A_\beta A_\gamma \{(r_k^\alpha - s_k^\alpha) D_\beta - (r_k^\beta - s_k^\beta) D_\alpha\} \frac{((\phi_\alpha, \phi_\beta), \phi_\gamma)}{D_\alpha D_\beta D_{\alpha\beta}}\\ &\qquad + 2 \text{ terms derived by cyclical interchange of } \alpha, \beta, \gamma\\ &= - A_\alpha A_\beta A_\gamma \{(r_k^\alpha - s_k^\alpha) D_\beta - (r_k^\beta - s_k^\beta) D_\alpha\}\\ &\quad \times \frac{((\phi_\alpha, \phi_\beta), \phi_\gamma) + ((\phi_\beta, \phi_\gamma), \phi_\alpha) + ((\phi_\gamma, \phi_\alpha), \phi_\beta)}{D_\alpha D_\beta D_\gamma} \text{ from (45), (46)}\end{aligned}$$

$= 0$ from a well-known property of the Poisson bracket.

(*c*) *Critical terms depending on a quaternary product.* In (41) the terms involving the four coefficients A_1, A_2, A_3, A_4 from H_3 are†

$$A_1 A_2 A_3 A_4 \sum_{24} (r_k^\alpha - s_k^\alpha) \frac{(((\phi_\alpha, \phi_\beta), \phi_\gamma), \phi_\delta)}{D_\alpha D_{\alpha\beta} D_{\alpha\beta\gamma}}, \equiv A_1 A_2 A_3 A_4 S, \text{ say,}$$

---

* Here, and in (*c*) below, we consider only the general case; this is sufficient for illustrative purposes, since we are only verifying a result which we know from § 3, that the coefficients of all critical terms vanish.

† It is supposed that the terms $\phi_1, \phi_2, \phi_3, \phi_4$ are all different and that none of $D_1, D_2, D_{12}, D_{13}, \ldots$ vanishes.

the summation being over the 24 permutations, of which $\alpha\beta\gamma\delta$ is representative, of the suffixes 1, 2, 3, 4. We shall abbreviate by writing $(\alpha\beta\gamma\delta)$ for the multiple Poisson bracket $(((\phi_\alpha, \phi_\beta), \phi_\gamma)\,\phi_\delta)$ and u_α for $(r_k{}^\alpha - s_k{}^\alpha)/D_\alpha$, so that

$$S = \sum_{24} u_\alpha \frac{(\alpha\beta\gamma\delta)}{D_{\alpha\beta} D_{\alpha\beta\gamma}},$$

or, collecting the terms having the same denominator,

$$S = \sum_{12} (u_\alpha - u_\beta) \frac{(\alpha\beta\gamma\delta)}{D_{\alpha\beta} D_{\alpha\beta\gamma}}. \qquad \ldots\ldots(47)$$

Since the terms are critical we have

$$r_i{}^1 + r_i{}^2 + r_i{}^3 + r_i{}^4 = s_i{}^1 + s_i{}^2 + s_i{}^3 + s_i{}^4 \quad (i = 1, 2, \ldots n)$$

and thus

$$\begin{aligned} &D_1 + D_2 + D_3 + D_4 \\ &\quad = \sum_{i=1}^{n} \lambda_i (r_i{}^1 + r_i{}^2 + r_i{}^3 + r_i{}^4 - s_i{}^1 - s_i{}^2 - s_i{}^3 - s_i{}^4) = 0, \quad (48) \\ &u_1 D_1 + u_2 D_2 + u_3 D_3 + u_4 D_4 \\ &\quad = r_k{}^1 + r_k{}^2 + r_k{}^3 + r_k{}^4 - s_k{}^1 - s_k{}^2 - s_k{}^3 - s_k{}^4 = 0. \ldots\ldots(49) \end{aligned}$$

From the identity

$$((\phi_\alpha, \phi_\beta), \phi_\gamma) + ((\phi_\beta, \phi_\gamma), \phi_\alpha) + ((\phi_\gamma, \phi_\alpha), \phi_\beta) = 0 \quad \ldots(50)$$

we have, using the above abbreviations,

$$(\alpha\beta\gamma\delta) + (\beta\gamma\alpha\delta) + (\gamma\alpha\beta\delta) = 0.$$

Hence from (47)

$$3S = \sum_{12} (u_\alpha - u_\beta) \frac{2\,(\alpha\beta\gamma\delta) - (\beta\gamma\alpha\delta) - (\gamma\alpha\beta\delta)}{D_{\alpha\beta} D_{\alpha\beta\gamma}},$$

or on rearrangement

$$= \sum_{12} \frac{(\alpha\beta\gamma\delta)}{D_{\alpha\beta\gamma}} \left(2\,\frac{u_\alpha - u_\beta}{D_{\alpha\beta}} - \frac{u_\beta - u_\gamma}{D_{\beta\gamma}} - \frac{u_\gamma - u_\alpha}{D_{\gamma\alpha}} \right). \quad \ldots(51)$$

But since from (48) $D_{\alpha\beta} = -D_{\gamma\delta}$, $D_{\alpha\gamma} = -D_{\beta\delta}$, we have

$$\begin{aligned} \frac{u_\alpha - u_\beta}{D_{\alpha\beta}} - \frac{u_\gamma - u_\alpha}{D_{\alpha\gamma}} &= \frac{(u_\gamma - u_\alpha)\,D_{\gamma\delta} - (u_\alpha - u_\beta)\,D_{\beta\delta}}{D_{\alpha\beta} D_{\alpha\gamma}} \\ &= \frac{-u_\alpha (D_\beta + D_\gamma + 2D_\delta) + u_\beta D_\beta + u_\gamma D_\gamma + (u_\beta + u_\gamma)\,D_\delta}{D_{\alpha\beta} D_{\alpha\gamma}} \\ &= \frac{-u_\alpha (D_\alpha + D_\beta + D_\gamma + 2D_\delta) + D_\delta (u_\beta + u_\gamma - u_\delta)}{D_{\alpha\beta} D_{\alpha\gamma}}, \text{ from (49)} \\ &= \frac{D_\delta (u_\beta + u_\gamma - u_\delta - u_\alpha)}{D_{\alpha\beta} D_{\alpha\gamma}}, \text{ from (48).} \end{aligned}$$

Similarly

$$\frac{u_\alpha - u_\beta}{D_{\alpha\beta}} - \frac{u_\beta - u_\gamma}{D_{\beta\gamma}} = \frac{D_\delta\,(u_\beta - u_\gamma - u_\alpha + u_\delta)}{D_{\alpha\beta}D_{\beta\gamma}}.$$

Hence from (51), since $D_{\alpha\beta\gamma} = -D_\delta$,

$$3S = \sum_{12} (\alpha\beta\gamma\delta)\left(\frac{u_\alpha - u_\beta - u_\gamma + u_\delta}{D_{\alpha\beta}D_{\alpha\gamma}} - \frac{u_\beta - u_\alpha - u_\gamma + u_\delta}{D_{\beta\alpha}D_{\beta\gamma}}\right)$$

$$= \sum_{12} \left\{(\alpha\beta\gamma\delta)\frac{u_\alpha - u_\beta - u_\gamma + u_\delta}{D_{\alpha\beta}D_{\alpha\gamma}} + (\beta\alpha\gamma\delta)\frac{u_\beta - u_\alpha - u_\gamma + u_\delta}{D_{\beta\alpha}D_{\beta\gamma}}\right\}$$

$$= \sum_{24} (\alpha\beta\gamma\delta)\frac{u_\alpha - u_\beta - u_\gamma + u_\delta}{D_{\alpha\beta}D_{\alpha\gamma}}, \qquad \ldots\ldots(52)$$

where the summation is again over the 24 permutations $\alpha\beta\gamma\delta$ of the suffixes 1, 2, 3, 4.

Now since $D_{\alpha\beta} + D_{\gamma\delta} = D_{\alpha\gamma} + D_{\beta\delta} = \ldots = 0$ we have

$$\frac{u_\alpha - u_\beta - u_\gamma + u_\delta}{D_{\alpha\beta}D_{\alpha\gamma}} = \frac{u_\delta - u_\beta - u_\gamma + u_\alpha}{D_{\delta\beta}D_{\delta\gamma}}$$

$$= \frac{u_\beta - u_\alpha - u_\delta + u_\gamma}{D_{\beta\alpha}D_{\beta\delta}} = \frac{u_\gamma - u_\alpha - u_\delta + u_\beta}{D_{\gamma\alpha}D_{\gamma\delta}} = \ldots,$$

and thus all those multiple Poisson brackets $(\alpha\beta\gamma\delta)$ which have the same pair of elements as extremes or middles have equal coefficients in (52). Thus

$$3S = \frac{u_1 - u_2 - u_3 + u_4}{D_{12}D_{13}}\{(1234) + (1324) + (2143) + (2413)$$
$$+ (3142) + (3412) + (4231) + (4321)\}$$
$$+ \frac{u_1 - u_2 - u_4 + u_3}{D_{12}D_{14}}\{(1243) + \ldots\}$$
$$+ \frac{u_1 - u_3 - u_4 + u_2}{D_{13}D_{14}}\{(1342) + \ldots\}. \qquad \ldots\ldots(53)$$

Now from the identity (50) we have

$$(((1, 2), 3), 4) + ((4, (1, 2)), 3) + ((3, 4), (1, 2)) = 0,$$

i.e. $\quad (((1, 2), 3), 4) - (((1, 2), 4), 3) = -((3, 4), (1, 2)),$

i.e. $\quad (1234) + (2143) = ((12)\,(34)).$

Similarly $\quad (1324) + (3142) = ((13)\,(24)),$

$(2413) + (4231) = ((24)\,(13)),$

$(3412) + (4321) = ((34)\,(12)).$

Hence in (53) the coefficient of $\dfrac{u_1 - u_2 - u_3 + u_4}{D_{12}D_{13}}$ is

$$((12)\,(34)) + ((13)\,(24)) + ((24)\,(13)) + ((34)\,(12)),$$

which vanishes since $(\alpha, \beta) + (\beta, \alpha) = 0$. Similarly the other two terms in (53) vanish and we have

$$S = 0.$$

Thus all the critical terms vanish from the right of (41), and we may write down the expression for $Z_k^{(6)}$ by introducing in each term the appropriate denominator, which is D_α, $D_{\alpha\beta}$, $D_{\alpha\beta\gamma}$, or $D_{\alpha\beta\gamma\delta}$ according as the term is of first, second, third or fourth degree in the coefficients A.

(v) *Form of the complete series.* From the expressions found for $Z_k^{(3)}, \dots Z_k^{(6)}$ the mode of formation of the complete series for Z_k is sufficiently obvious, and it would be a reasonable induction to infer from the preceding work that the critical terms must vanish also from the equations for $Z_k^{(8)}$, $Z_k^{(10)}$, … . It is unnecessary to make this induction however, for from § 3 we are assured of the truth of the result.

To form the complete series, let

$$\begin{aligned} H &= \lambda_1 x_1 y_1 + \dots + \lambda_n x_n y_n + \Sigma A_{r_1 s_1 \dots r_n s_n} x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n} \\ &= \lambda_1 x_1 y_1 + \dots + \lambda_n x_n y_n + \Sigma A\phi, \text{ say,} \end{aligned}$$

the summation being over all sets of positive integral or zero values of $r_1, s_1, \dots r_n, s_n$ whose sum is greater than 2. Then

$$Z_k = x_k y_k + \Sigma\, (-)^{p+1} A_1 A_2 \dots A_p \frac{(r_k^1 - s_k^1)((\dots(\phi_1, \phi_2), \phi_3), \dots \phi_p)}{D_1 D_{12} D_{123} \dots D_{12\dots p}}, \qquad \dots\dots(54)$$

where $A_1\phi_1$, $A_2\phi_2$, … $A_p\phi_p$ are any p terms from H, and the summation is over all permutations, of all orders, of terms from H, the one written being representative; the terms in any permutation need not be all distinct, so that any group of m terms of H yields in the series (54) m^p terms of the pth degree in the coefficients A. The expression (54) must be interpreted with the convention that any vanishing denominator is to be replaced everywhere by a *finite arbitrary* coefficient.

Though in (54) we have an explicit expression for Z_k it must be remembered that we are fundamentally interested in Z_k as a power series in $x_1, \dots x_n, y_1, \dots y_n$, to obtain which form we must rearrange (54) after expanding the Poisson brackets. There does not appear to be any simple expression for the general term of the series as thus rearranged.

§ 5. *Properties of the integrals.*

(i) *There are no integrals developable in powers of $x_1, \dots x_n, y_1, \dots y_n$ independent of the n constructed in* § 4. The solution of the equations for $Z^{(3)}$, $Z^{(4)}$, … by polynomials is unique except for the terms with arbitrary coefficients which occur in $Z^{(4)}$, $Z^{(6)}$, … , and in each case these terms are arbitrary polynomials in $Z_1^{(2)}$, $Z_2^{(2)}, \dots Z_n^{(2)}$. This suggests that any solution of $(Z, H) = 0$ in power series can be expressed as a power series in n fundamental solutions of which the initial terms are respectively

$$x_1 y_1, x_2 y_2, \dots x_n y_n.$$

To establish this, let $Z_1, Z_2, \dots Z_n$ be n such solutions; these are derived by the process of § 4, where the arbitrary coefficients which occur in the terms of 4th, 6th, ... degrees are supposed given definite constant values, e.g. zero. Let

$$Z = Z^{(2)} + Z^{(3)} + Z^{(4)} + \dots \qquad \dots\dots(55)$$

be any other solution, the superscripts denoting as usual the degrees of the respective groups of terms; then $(Z^{(2)}, H_2) = 0$, and this requires that

$$Z^{(2)} = a_1 Z_1^{(2)} + \dots + a_n Z_n^{(2)},$$

where $a_1, \dots a_n$ are constants.

Now let $$U = Z - a_1 Z_1 - \dots - a_n Z_n; \qquad \dots\dots(56)$$

then evidently U is a solution, so that $(U, H) = 0$. The lowest terms in U are at least of third degree, and calling them $U^{(3)}$ we thus have

$$(U^{(3)}, H_2) = 0.$$

If $U^{(3)} = \Sigma A_{r_1 s_1 \dots r_n s_n} x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}$ we have

$$(U^{(3)}, H_2)$$
$$= \Sigma A_{r_1 s_1 \dots r_n s_n} \{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)\} x_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}.$$

Since this is identically zero we must have for each coefficient an equation like

$$A_{r_1 s_1 \dots r_n s_n} \{\lambda_1 (r_1 - s_1) + \dots + \lambda_n (r_n - s_n)\} = 0,$$

and therefore every coefficient must vanish except those of the critical terms, for which $r_1 = s_1, \dots r_n = s_n$. Since $U^{(3)}$ is of third degree it contains no critical terms, and thus must vanish identically. We then have

$$(U^{(4)}, H_2) = 0,$$

from which it follows by the above argument that only the critical terms in $U^{(4)}$ can have non-zero coefficients; thus

$$U^{(4)} = a_{11} Z_1^{(2)2} + a_{12} Z_1^{(2)} Z_2^{(2)} + \dots, \qquad \dots\dots(57)$$

where $a_{11}, a_{12}, \dots$ are constants.

Now let $$V = U - a_{11} Z_1^2 - a_{12} Z_1 Z_2 - \dots; \qquad \dots\dots(58)$$

then since V is a function of integrals it is itself an integral and satisfies $(V, H) = 0$, while since U begins with the terms (57), V begins with terms of the fifth degree at least. From $(V^{(5)}, H_2) = 0$ we deduce as above

$$V^{(5)} = 0,$$

and since then $(V^{(6)}, H_2) = 0$ we deduce

$$V^{(6)} = a_{111} Z_1^{(2)3} + a_{112} Z_1^{(2)2} Z_2^{(2)} + a_{123} Z_1^{(2)} Z_2^{(2)} Z_3^{(2)} + \dots,$$

where the a's are constants. This process may be continued indefinitely, and from (56), (58), ... we obtain

$$\begin{aligned} Z &= U + a_1 Z_1 + \dots + a_n Z_n \\ &= a_1 Z_1 + \dots + a_n Z_n + a_{11} Z_1^2 + a_{12} Z_1 Z_2 + \dots + V \\ &= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &= \text{power series in } Z_1, Z_2, \dots Z_n. \end{aligned}$$

Thus any integral developable in a power series in $x_1, \dots x_n$, $y_1, \dots y_n$ is expressible as a power series in $Z_1, \dots Z_n$, so that there is no such integral independent of $Z_1, \dots Z_n$.

(ii) *Connection of $Z_1, \dots Z_n$ with the energy integral.* As a particular case of what has just been proved H must be expressible as a power series in $Z_1, \dots Z_n$. The linear terms in this series are

$$\lambda_1 Z_1 + \lambda_2 Z_2 + \dots + \lambda_n Z_n,$$

and it is easily seen that when the arbitrary coefficients in $Z_1, \dots Z_{n-1}$ have been definitely assigned in any manner, those in Z_n may be so chosen that

$$H = \lambda_1 Z_1 + \dots + \lambda_n Z_n.$$

(iii) *The integrals $Z_1, \dots Z_n$ are in involution.* We must show that if Z_p and Z_q are any two of the integrals, $(Z_p, Z_q) = 0$. From the identity

$$\begin{aligned} -((Z_p, Z_q), H) &= ((Z_q, H), Z_p) + ((H, Z_p), Z_q) \\ &= 0, \end{aligned}$$

it follows that (Z_p, Z_q) is an integral, and this being obviously developable in powers of the x's and y's is from (i) developable in powers of $Z_1, \dots Z_n$. Thus the terms of *lowest* degree in (Z_p, Z_q) must be critical, and to show that $(Z_p, Z_q) = 0$ it is only necessary to show in succession that the critical terms of second, fourth, ... degrees vanish. The verification is easy up to the fourth degree, but as in the last section a general treatment does not appear to be possible. We proceed therefore by utilizing the relations between Lagrange and Poisson brackets.

With the notation of § 3, let us take for constants of integration quantities $u_1, \dots u_n, v_1, \dots v_n$ given by

$$\left.\begin{aligned} u_r &= \alpha_r \beta_r & v_r &= \log \alpha_r - \log \beta_r \\ \alpha_r^2 &= u_r e^{v_r} & \beta_r^2 &= u_r e^{-v_r} \end{aligned}\right\} \quad (r = 1, \dots n). \dots(59)$$

Then $$\frac{\partial x}{\partial v_r} = \frac{\partial x}{\partial \alpha_r}\frac{\partial \alpha_r}{\partial v_r} + \frac{\partial x}{\partial \beta_r}\frac{\partial \beta_r}{\partial v_r} = \frac{\partial x}{\partial \alpha_r}\frac{u_r e^{v_r}}{2\alpha_r} - \frac{\partial x}{\partial \beta_r}\cdot\frac{u_r e^{-v_r}}{2\beta_r}$$

$$= \tfrac{1}{2}\left(\alpha_r \frac{\partial x}{\partial \alpha_r} - \beta_r \frac{\partial x}{\partial \beta_r}\right),$$

and it is thence easily shown that

$$\begin{aligned}[v_r, v_s] &= \tfrac{1}{4}\{\alpha_r\alpha_s[\alpha_r, \alpha_s] - \beta_r\alpha_s[\beta_r, \alpha_s] - \alpha_r\beta_s[\alpha_r, \beta_s] + \beta_r\beta_s[\beta_r, \beta_s]\} \\ &= 0 \text{ from (20), (23).}\end{aligned}$$

Hence in the matrix of $4n^2$ elements represented by

$$\left\{\begin{matrix}[u_r, u_s], & [u_r, v_s] \\ [v_r, u_s], & [v_r, v_s]\end{matrix}\right\} \qquad \text{......(60)}$$

the n^2 elements in the bottom right-hand corner are zero, and since its determinant is non-zero[*] the determinant of the n^2 bottom left-hand corner elements must be non-zero, i.e.

$$|\,[v_r u_s]\,| \neq 0. \qquad \text{......(61)}$$

Now the matrix of Poisson brackets

$$\left\{\begin{matrix}(u_r, u_s), & (u_r, v_s) \\ (v_r, u_s), & (v_r, v_s)\end{matrix}\right\} \qquad \text{......(62)}$$

is well-known to be reciprocal to (60)[†], whence, compounding in succession the $(n+1)$th, $(n+2)$th, ... $2n$th rows of (60) with the pth row ($p \leqslant n$) of (62) we have

$$[v_r, u_1](u_p, u_1) + [v_r, u_2](u_p, u_2) + \dots + [v_r, u_n](u_p, u_n) = 0 \qquad (r = 1, \dots n).$$

This is a set of n homogeneous linear equations in (u_p, u_1), ... (u_p, u_n) whose determinant is, from (61), non-zero; thus

$$(u_p, u_1) = (u_p, u_2) = \dots = (u_p, u_n) = 0,$$

and similarly $(u_p, u_q) = 0$ for $p, q = 1, 2, \dots n$.

Thus the n integrals constructed in § 3 are in involution, and since by (i) above there are no independent developable integrals, any n integrals constructed as in § 4 must be in involution.

§ 6. *Remarks on the convergence of the series.*

The difficulty in discussing the convergence of the series (54) arises from the presence of the denominators, which are products of factors of the form

$$A_1\lambda_1 + A_2\lambda_2 + \dots + A_n\lambda_n, \qquad \text{......(63)}$$

where the A's are integers. Now in general when, as supposed, the λ's are such that none of these factors vanishes unless the A's are all zero, we can find integral values of the A's which make the quantity (63) less in absolute value than any assigned quantity; so that, though the series (54) contains no vanishing denominators, it will in general contain denominators whose moduli are arbitrarily small*.

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[*] This determinant has the value $\left\{\dfrac{\partial(x_1, \dots x_n, y_1, \dots y_n)}{\partial(u_1, \dots u_n, v_1, \dots v_n)}\right\}^2$, and is non-zero because $\dfrac{\partial(x_1, \dots y_n)}{\partial(\alpha_1, \dots \beta_n)} \neq 0$ and, from (59), $\dfrac{\partial(\alpha_1, \dots \beta_n)}{\partial(u_1, \dots v_n)} \neq 0$.

[†] Whittaker, *Analytical Dynamics* (2nd ed.), § 130.

That the presence of such "small divisors" does not necessarily make a series divergent was shown long ago by Bruns; it is in fact easily proved that the series

$$\sum_{m,n=1}^{\infty} \frac{x^m y^n}{m\lambda_1 - n\lambda_2}$$

is absolutely convergent if the ratio $\lambda_1 : \lambda_2$ is an algebraic number. We might therefore hope that the series (54) would be convergent for sufficiently small values of $|x_1|, \dots |y_2|$ if the ratios of the λ's are algebraic numbers. Unfortunately this is not in general the case *if the series be taken as it stands*, i.e. be not rearranged in powers of $x_1, \dots x_n, y_1, \dots y_n$.

To show this, let

$$A\phi_\alpha = Ax_1^{r_1} y_1^{s_1} \dots x_n^{r_n} y_n^{s_n}, \quad B\phi_\beta = Bx_1^2 y_1^2$$

be two terms of H, where $A \neq 0$, $B \neq 0$, $r_1 \neq s_1$, $r_k \neq s_k$, and consider the terms from (54) forming the "partial series"

$$\frac{A(r_k - s_k)\phi_\alpha}{D_\alpha} - \frac{AB(r_k - s_k)(\phi_\alpha, \phi_\beta)}{D_\alpha D_{\alpha\beta}} + \frac{AB^2(r_k - s_k)((\phi_\alpha, \phi_\beta), \phi_\beta)}{D_\alpha D_{\alpha\beta} D_{\alpha\beta\beta}}$$

$$- \frac{AB^3(r_k - s_k)(((\phi_\alpha, \phi_\beta), \phi_\beta), \phi_\beta)}{D_\alpha D_{\alpha\beta} D_{\alpha\beta\beta} D_{\alpha\beta\beta\beta}} + \dots.$$

We have

$$\begin{aligned}
(\phi_\alpha, \phi_\beta) &= 2(r_1 - s_1) x_1^{r_1+1} y_1^{s_1+1} x_2^{r_2} y_2^{s_2} \dots x_n^{r_n} y_n^{s_n} \\
&= 2(r_1 - s_1) x_1 y_1 \phi_\alpha, \\
((\phi_\alpha, \phi_\beta), \phi_\beta) &= 2(r_1 - s_1)\{x_1 y_1 (\phi_\alpha, \phi_\beta) + \phi_\alpha (x_1 y_1, \phi_\beta)\} \\
&= 2(r_1 - s_1) x_1 y_1 (\phi_\alpha, \phi_\beta) = 2^2 (r_1 - s_1)^2 x_1^2 y_1^2 \phi_\alpha, \\
(((\phi_\alpha, \phi_\beta), \phi_\beta), \phi_\beta) &= 2^2 (r_1 - s_1)^2 x_1^2 y_1^2 (\phi_\alpha, \phi_\beta) = 2^3 (r_1 - s_1)^3 x_1^3 y_1^3 \phi_\alpha,
\end{aligned}$$

and so on, and, since $D_\beta = 0$,

$$D_\alpha = D_{\alpha\beta} = D_{\alpha\beta\beta} = \dots,$$

and D_α is not zero since $r_1 \neq s_1$. Thus the "partial series" is

$$\frac{A(r_k - s_k)\phi_\alpha}{D_\alpha} \left\{1 - \frac{2B(r_1 - s_1) x_1 y_1}{D_\alpha} + \frac{2^2 B^2 (r_1 - s_1)^2 x_1^2 y_1^2}{D_\alpha^2} - \dots\right\},$$

which has the sum $\dfrac{A(r_k - s_k)\phi_\alpha}{D_\alpha + 2B(r_1 - s_1) x_1 y_1}$, and a region of convergence given by

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* If $n=2$ and the ratio $\lambda_1 : \lambda_2$ is not real the modulus of the expression (63) has a definite non-zero lower bound for all possible integral choices of A_1, A_2 except $A_1 = A_2 = 0$; if $n=2$ and $\lambda_1 : \lambda_2$ is real, or if $n > 2$, the lower bound is zero and the statement in the text is valid.

$$| x_1 y_1 | < \left| \frac{D_a}{2B(r_1 - s_1)} \right|.$$

We may pick out from (54) a similar series starting from any term in which the last element of the multiple Poisson bracket is not ϕ_β, and the series starting from the term

$$A_1 A_2 \dots A_p \frac{(r_k{}^1 - s_k{}^1)((\dots(\phi_1, \phi_2), \phi_3) \dots \phi_p)}{D_1 D_{12} \dots D_{12\dots p}} \quad \dots\dots(64)$$

is readily seen to have a region of convergence given by

$$| x_1 y_1 | < \left| \frac{D_{12\dots p}}{2B(r_1{}^1 + r_1{}^2 + \dots + r_1{}^p - s_1{}^1 - s_1{}^2 - \dots - s_1{}^p)} \right|.$$

Since in general (54) contains terms for which

$$\left| \frac{D_{12\dots p}}{r_1{}^1 + r_1{}^2 + \dots + r_1{}^p - s_1{}^1 - s_1{}^2 - \dots - s_1{}^p} \right|$$

is arbitrarily small, it follows that we can pick out from (54) partial series which are divergent for any given non-zero values of $| x_1 |$, $| y_1 |$, however small. *Hence in general the series* (54) *cannot be convergent for any given non-zero values of* $| x_1 |, \dots | x_n |, | y_1 |, \dots | y_n |$, *however small.* Moreover, no great complication is required in H for this result to hold; for it may be seen without difficulty that if H contains a term $Bx_1{}^2 y_1{}^2$ and only *two* terms which are not critical[*] and not functionally related, then the series (54) will always (unless $n = 2$ and $\lambda_1 : \lambda_2$ is unreal) contain terms such as (64) for which

$$\left| \frac{D_{12\dots p}}{r_1{}^1 + r_1{}^2 + \dots + r_1{}^p - s_1{}^1 - s_1{}^2 - \dots - s_1{}^p} \right|$$

is arbitrarily small.

If the Poisson brackets in (54) be expanded out, it will be found that in general any term $x_1{}^{r_1} y_1{}^{s_1} \dots x_n{}^{r_n} y_n{}^{s_n}$ occurs in more than one of the partial series whose regions of convergence we have been considering, and the higher the degree of the term the greater will be the number of partial series in which it occurs. It is evident, therefore, that the result just established has no bearing on the question of the convergence of the series (54) *when arranged as a power series in* $x_1, \dots x_n, y_1, \dots y_n$[†]; and of course it is this latter question which is of fundamental interest, since the discussion of

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[*] I.e. products of powers of $x_1 y_1, x_2 y_2, \dots x_n y_n$.

[†] Thus

$$1 + x + x^2 + \dots = 1 + 2x + 2^2 x^2 + \dots - (x + 3x^2 + 7x^3 + \dots),$$

but though the two sserie on the right are divergent for $(x) > \frac{1}{2}$ we cannot conclude the same range of divergence for the series on the left, which is in fact convergent for $| x | < 1$.

the integrals as functions of the coordinates must precede their discussion as functions of parameters on which H may depend.

It is of interest to give a case in which the series (54) is, as it stands, divergent for all finite non-zero values of $|x_1|, \dots |y_n|$, but is known to be convergent when arranged as a power series in these arguments. This circumstance arises with the system of equations* for which

$$H = \frac{1}{2A}\left(p\sin\psi - \frac{q + r\sin\theta}{\cos\theta}\cos\psi\right)^2 + \frac{1}{2B}\left(p\cos\psi + \frac{q + r\sin\theta}{\cos\theta}\sin\psi\right)^2 + \frac{r^2}{2C}, \quad \dots(65)$$

the conjugate variables being p, θ and r, ψ, while q, A, B, C are positive constants. The singular points, obtained by solving the equations

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial r} = \frac{\partial H}{\partial \theta} = \frac{\partial H}{\partial \psi} = 0$$

are given by $$p = r = \theta = 0,\ \psi = 0 \text{ or } \frac{\pi}{2}.$$

Expanding in the neighbourhood of the first of these, we have to the fourth order in the small quantities p, r, θ, ψ

$$\begin{aligned} H = {} & \frac{1}{2A}(q^2 + 2qr\theta - 2qp\psi + q^2\theta^2 - q^2\psi^2) + \frac{1}{2B}(p^2 + 2qp\psi + q^2\psi^2) + \frac{r^2}{2C} \\ & + \frac{\theta^2(r + q\theta)}{2A}(r + \tfrac{2}{3}q\theta) + \frac{\psi}{2}\left(\frac{1}{B} - \frac{1}{A}\right)(p + q\psi)\{q\theta^2 + 2r\theta - \psi(p + \tfrac{1}{3}q\psi)\} \\ = {} & \frac{q^2}{2A} + H_2 + H_4, \text{ say}\dagger. \end{aligned} \quad \dots\dots(66)$$

Now apply the contact transformation (changing conjugate pairs $\theta, p; \psi, r$ into $x_1, y_1; x_2, y_2$)

$$\left.\begin{aligned} 2\iota q\gamma x_1 &= \quad q\psi + p - \iota\gamma r \\ y_1 &= \quad q\psi + p + \iota\gamma r \\ 2\iota q x_2 &= \iota q\theta \qquad\quad + p + \iota r \\ y_2 &= -\iota q\theta \qquad + p - \iota r \end{aligned}\right\} \quad \dots\dots(67)$$

where $$\gamma^2 = \frac{AB(A - C)}{AC(A - B)},$$

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[*] Arising in the problem of the motion of a rigid body about a fixed point under no forces; q is the constant in an angular momentum integral by means of which the number of degrees of freedom has been reduced from 3 to 2.

[†] Since H enters the equations only through its derivatives, the presence of the constant term $q^2/2A$ is immaterial.

and we find that

$$H_2 = \iota q\gamma\left(\frac{1}{B}-\frac{1}{A}\right)x_1y_1+\frac{\iota q}{A}x_2y_2;$$

thus the exponents of the singular point are

$$\pm\,\iota q\gamma\left(\frac{1}{B}-\frac{1}{A}\right), = \pm\,\iota q\sqrt{\frac{(A-B)(A-C)}{A^2BC}},$$

and $\quad \pm\,\iota q/A.$

If the values of the constants A, B, C are such that A lies between B and C two exponents are real and two imaginary, and the process of § 4 can be applied to give a series in which the moduli of the divisors (63) have a positive lower bound. If, however, A lies outside the range B, C the exponents are all imaginary, and it is necessary that

$$\sqrt{\frac{(A-B)(A-C)}{BC}}$$

should be an irrational number for the process to be applicable; the resulting series then contains divisors whose moduli are arbitrarily small. On substituting from (67) in (66) the values of θ, ψ, p, q in terms of x_1, y_1, x_2, y_2 it is found that the coefficients in H_4 are not in general zero; for example the coefficient of

$$x_2{}^2y_2{}^2 \text{ is } -\frac{1}{2A},$$

that of $\quad x_2y_2{}^3$ is $-\dfrac{\iota}{6Aq}$,

and that of $\quad y_2{}^3y_1$ is $\dfrac{1}{32Aq^2\gamma}+\dfrac{1}{96q^2}\left(\dfrac{1}{B}-\dfrac{1}{A}\right).$

Hence by what has been seen, if A lies outside the range B, C the series (54) is, as it stands, divergent for all finite non-zero values of $|x_1|$, $|y_1|$, $|x_2|$, $|y_2|$.

Now it is easily verified from (65) that $(K, H) = 0$, where

$$K = p^2+\frac{q^2+r^2+2qr\sin\theta}{\cos^2\theta},$$

so that K is an integral of the system; and it is easily seen that K is developable in a convergent series of powers of x_1, y_1, x_2, y_2 of which the initial terms are $q^2+2\iota qx_2y_2$. Thus the series starting with x_2y_2 constructed as in § 4 must be convergent when arranged in powers of x_1, y_1, x_2, y_2.

The difficulty arising from the presence of "small divisors" is one which is familiar in Celestial Mechanics. In such series as those of Lindstedt the divisors are functions of the constants of integration, and the result is that the series are not in general valid over any range of these constants, however small. In the series constructed in § 4 however the divisors are, for any given characteristic function *H*, constants, so that the difficulty is far less acute. It is not unreasonable therefore to suppose that the complication of Lindstedt's series is to a certain extent artificial, and that in the problem of the convergence of the series of § 4 we have the essence of the difficulties of Celestial Mechanics without any unessential complications.

On a Case of steady flow of a Gas in Two Dimensions. By Mr S. LEES, St John's College.

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§ 1. The problem of finding general types of solution for the steady flow of gases in two dimensions, adiabatically and without friction, does not appear to have received a great deal of attention. An interesting and suggestive treatment of hydrodynamics applied to gases is given[*] in Riemann-Weber's *Partielle Differential-Gleichungen*, but the application is to pressure propagation. Reference must also be made to a paper by the late Lord Rayleigh, who gave[†] a general differential equation which must be satisfied by the velocity potential ϕ, but the problem of utilizing this equation does not appear easy.

In the following paper, the assumption is made that the velocity components are functions of the density only, and the consequences of this assumption are then discussed. No external forces are supposed to act on the gas.

§ 2. The notation used in this paper is the usual one, viz. u and v are the component velocities of the gas at a point (x, y) where the pressure is p and the density is ρ. The initial density and pressure (i.e. the density and pressure of the fluid at regions where it may be considered at rest) are taken as ρ_0 and p_0 respectively.

The equations of motion are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}=-F(\rho)\,.\,\frac{\partial \log \rho}{\partial x}\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}=-F(\rho)\,.\,\frac{\partial \log \rho}{\partial y}\end{aligned}\right\}, \quad \text{......(1)}$$

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[*] 1901 Edn, Bd. 2, p. 469 *et seq.*

[†] *Phil. Mag.*, 32 (1916), and *Scientific Papers*, 6, 402. With the notation used in this paper, the equation referred to for adiabatic flow is (for two dimensions)

$$\{2a_0^2-(\gamma-1)\,q^2\}\,\nabla^2\phi=\frac{\partial \phi}{\partial x}\,\frac{\partial q^2}{\partial x}+\frac{\partial \phi}{\partial y}\,\frac{\partial q^2}{\partial y},$$

which may also be written

$$\left\{2a_0^2-\gamma\left[\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2\right]\right\}\nabla^2\phi=\left[\left(\frac{\partial \phi}{\partial x}\right)^2-\left(\frac{\partial \phi}{\partial y}\right)^2\right]\left[\frac{\partial^2 \phi}{\partial x^2}-\frac{\partial^2 \phi}{\partial y^2}\right]+4\,\frac{\partial \phi}{\partial x}\,\frac{\partial \phi}{\partial y}\,\frac{\partial^2 \phi}{\partial x\,\partial y},$$

which does not appear to be utilized easily.

For two-dimensional flow, a stream function exists, and it may readily be verified that it satisfies

$$\{2a_0^2-(\gamma-1)\,q^2\}\,\nabla^2\psi=-\left[\frac{\partial \psi}{\partial x}\,\frac{\partial q^2}{\partial x}+\frac{\partial \psi}{\partial y}\,\frac{\partial q^2}{\partial y}\right].$$

This would not appear any more easy to handle than Rayleigh's equation for ϕ.

In the above equations, a_0 is the velocity of sound for regions where the gas may be considered at rest, whilst q is the absolute velocity at (x, y).

where for adiabatic expansion (B representing a constant)

$$p = B\rho^{\gamma}, \qquad \ldots\ldots(2)$$

and

$$dp/\rho = \gamma B\rho^{\gamma-1} d\rho/\rho;$$

thus

$$F(\rho) = \gamma B\rho^{\gamma-1} = \gamma \frac{p}{\rho}. \qquad \ldots\ldots(3)$$

The equation of continuity is $\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0$,

or

$$u\frac{\partial \log \rho}{\partial x} + v\frac{\partial \log \rho}{\partial y} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \qquad \ldots\ldots(4)$$

In what follows, we shall employ z for $\log(\rho/\rho_0)$.

We now assume that u and v are functions of ρ only, with ρ depending on both x and y, so that e.g.

$$\frac{\partial u}{\partial x} = \frac{du}{dz} \cdot \frac{\partial z}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{du}{dz} \cdot \frac{\partial z}{\partial y}.$$

The equations (1) become

$$\left.\begin{aligned} \frac{du}{dz}\left(u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right) &= -F(z)\frac{\partial z}{\partial x} \\ \frac{dv}{dz}\left(u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right) &= -F(z)\frac{\partial z}{\partial y} \end{aligned}\right\}, \qquad \ldots\ldots(5)$$

where $F(z)$ denotes $\gamma B\rho^{\gamma-1} = \gamma \frac{p_0}{\rho_0} e^{(\gamma-1)z}$. Also (4) becomes

$$u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} = -\left(\frac{du}{dz}\frac{\partial z}{\partial x} + \frac{dv}{dz}\frac{\partial z}{\partial y}\right). \qquad \ldots\ldots(6)$$

From (5), we derive

$$(du/dz)/(dv/dz) = (\partial z/\partial x)/(\partial z/\partial y), \qquad \ldots\ldots(7)$$

i.e.

$$\frac{du}{dz}\frac{\partial z}{\partial y} = \frac{dv}{dz}\frac{\partial z}{\partial x}, \quad \text{or} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \qquad \ldots\ldots(8)$$

From (8) follows the existence of the velocity potential ϕ, such that

$$u = \partial\phi/\partial x, \text{ and } v = \partial\phi/\partial y. \qquad \ldots\ldots(9)$$

Such motions as we consider are therefore irrotational.

§ 3. If we substitute the values of $(\partial z/\partial x)/(\partial z/\partial y)$ from (5), (6) and (7) we obtain

$$\frac{-\dfrac{\partial z}{\partial x}}{\dfrac{\partial z}{\partial y}} = \frac{v\dfrac{du}{dz}}{u\dfrac{du}{dz} + F(z)} = \frac{v\dfrac{dv}{dz} + F(z)}{u\dfrac{dv}{dz}} = \frac{v + \dfrac{dv}{dz}}{u + \dfrac{du}{dz}} = \frac{-\dfrac{du}{dz}}{\dfrac{dv}{dz}}. \quad \ldots(10)$$

Equating the first and second of these expressions for $(\partial z/\partial x)/(\partial z/\partial y)$, we get (assuming that $F(z) \neq 0$)

$$u\frac{du}{dz} + v\frac{dv}{dz} = -F(z). \qquad \ldots\ldots(11)$$

The last expression of (10) therefore follows from the two just considered, and is not independent. From the last two expressions of (10), we have

$$\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 = -\left(u\frac{du}{dz} + v\frac{dv}{dz}\right) = F(z). \quad \text{......(12)}$$

Equation (11) is immediately integrable, and gives

$$q^2 = u^2 + v^2 = -2\int F(z)\,dz = \frac{-2\gamma p_0}{\rho_0}\int e^{(\gamma-1)z}\,dz$$

$$= \frac{-2\gamma p_0}{(\gamma-1)\rho_0}e^{(\gamma-1)z} + \text{const.}$$

Since when $z = 0$ and $\rho/\rho_0 = 1$, the resultant velocity q is zero, the constant is determined; we get in fact*

$$q^2 = u^2 + v^2 = \frac{2a_0^2}{\gamma - 1}\left[1 - e^{(\gamma-1)z}\right] = \psi_1(z), \text{ say.} \quad \text{...(13)}$$

For simplicity, we have replaced here the term $\gamma p_0/\rho_0$ by a_0^2, where a_0 is the velocity of sound in the gas under the initial conditions. Equation (13) is, of course, merely the energy equation.

We next eliminate one of the two variables u and v (say v) from (12) and (13). From (11), employing F for $F(z)$, we have

$$\frac{dv}{dz} = \frac{-\left(F + u\frac{du}{dz}\right)}{v} = \frac{-\left(F + u\frac{du}{dz}\right)}{\pm(\psi_1 - u^2)^{\frac{1}{2}}}.$$

Hence from (12), $$\left(\frac{du}{dz}\right)^2 + \frac{\left(F + u\frac{du}{dz}\right)^2}{\psi_1 - u^2} = F,$$

or $$(\psi_1 - u^2)\left(\frac{du}{dz}\right)^2 + \left(F + u\frac{du}{dz}\right)^2 = F(\psi_1 - u^2).$$

Thus $$\frac{du}{dz} = \frac{-Fu}{\psi_1} \pm \left[\frac{F^2u^2}{\psi_1^2} + F\left(1 - \frac{u^2}{\psi_1}\right) - \frac{F^2}{\psi_1}\right]^{\frac{1}{2}}, \quad \text{......(14)}$$

and after some reduction (noting that $F = a_0^2 e^{(\gamma-1)z}$)

$$u\frac{du}{dz} = \frac{-(\gamma-1)u^2}{2f} \pm \left\{u^2\left(\frac{\gamma-1}{2} - f\right)\left[\frac{(\gamma-1)u^2}{2f^2} - \frac{a_0^2}{f(1+f)}\right]\right\}^{\frac{1}{2}},$$

where f stands for $e^{-(\gamma-1)z} - 1$. Writing U for u^2, and noting that

$$\frac{dU}{dz} = \frac{dU}{df}\frac{df}{dz} = \frac{-dU}{df}(\gamma-1)e^{-(\gamma-1)z} = -(\gamma-1)(1+f)\frac{dU}{df},$$

---

* $a_0^2 e^{(\gamma-1)z}$ is the velocity of sound at (x, y). See Lamb, *Hydrodynamics*, 1916 Edn, §§ 23, 25. See also § 6 of this paper.

we obtain

$$-\frac{(\gamma-1)}{2}\frac{d}{df}\left[U\left(1+\frac{1}{f}\right)\right] = \pm\frac{1}{f}\left\{U\left[\frac{\gamma-1}{2}-f\right]\left[\frac{(\gamma-1)\,U}{2f^2}-\frac{a_0^2}{f(1+f)}\right]\right\}^{\frac{1}{2}}.$$

Putting $\mu = U\left(1+\frac{1}{f}\right)$, i.e. $U = \mu f/(1+f)$, we get

$$\frac{(\gamma-1)}{2}\,\frac{d\mu}{\mu^{\frac{1}{2}}\left[\frac{(\gamma-1)\,\mu}{2}-a_0^2\right]^{\frac{1}{2}}} = \mp\frac{df\left[\frac{(\gamma-1)}{2}-f\right]^{\frac{1}{2}}}{f(1+f)}. \quad (15)$$

This is in an integrable form with the variables separated. On integration, we find

$$\sqrt{\frac{\gamma-1}{2}}\cosh^{-1}\left(\frac{2\mu-\alpha}{\alpha}\right)$$
$$= \pm 2\int\frac{\lambda^2 d\lambda}{\left(\frac{\gamma-1}{2}-\lambda^2\right)\left(\frac{\gamma+1}{2}-\lambda^2\right)} = \mp\int\frac{(\gamma+1)\,d\lambda}{\frac{\gamma+1}{2}-\lambda^2}\pm\int\frac{(\gamma-1)\,d\lambda}{\frac{\gamma-1}{2}-\lambda^2},$$

where $\alpha = 2a_0^2/(\gamma-1)$ and $\lambda^2 = \frac{\gamma-1}{2}-f$. We get therefore

$$\sqrt{\frac{\gamma-1}{2}}\cosh^{-1}\left(\frac{2\mu-\alpha}{\alpha}\right)$$
$$= \mp\left[\sqrt{\frac{\gamma+1}{2}}\log\frac{\sqrt{\frac{\gamma+1}{2}}+\lambda}{\sqrt{\frac{\gamma+1}{2}}-\lambda}-\sqrt{\frac{\gamma-1}{2}}\log\frac{\sqrt{\frac{\gamma-1}{2}}+\lambda}{\sqrt{\frac{\gamma-1}{2}}-\lambda}\right]+\text{const.} \quad \ldots\ldots(16)$$

Taking the cosh of both sides of (16), after first dividing through by $\sqrt{\frac{\gamma-1}{2}}$, we get

$$\frac{2\,(2\mu-\alpha)}{\alpha} = M + M^{-1}, \quad \ldots\ldots(17)$$

where $$M = C\left[\frac{\sqrt{\frac{\gamma+1}{2}}+\lambda}{\sqrt{\frac{\gamma+1}{2}}-\lambda}\right]^{\sqrt{\frac{\gamma+1}{\gamma-1}}}\Bigg/\left[\frac{\sqrt{\frac{\gamma-1}{2}}+\lambda}{\sqrt{\frac{\gamma-1}{2}}-\lambda}\right], \ldots(18)$$

and C is a constant which may be complex.

Writing $M + M^{-1} = (M^{\frac{1}{2}} + M^{-\frac{1}{2}})^2 - 2$, equation (17) may also be written

$$\pm\left(\frac{4\mu}{\alpha}\right)^{\frac{1}{2}} = M^{\frac{1}{2}} + M^{-\frac{1}{2}}, \qquad \text{......(19)}$$

which can be solved as

$$\left.\begin{aligned} M^{\frac{1}{2}} &= \pm 2\sqrt{\frac{\mu}{\alpha}} \pm 2\sqrt{\frac{\mu}{\alpha} - \frac{1}{4}} \\ M^{-\frac{1}{2}} &= \pm 2\sqrt{\frac{\mu}{\alpha}} \mp 2\sqrt{\frac{\mu}{\alpha} - \frac{1}{4}} \end{aligned}\right\}. \qquad \text{......(20)}$$

It is convenient to write (19) in the form

$$4\mu = 4u^2\left(1 + \frac{1}{f}\right) = \alpha\,[M + M^{-1} + 2]. \qquad \text{......(21)}$$

Since from (13)

$$q^2 = u^2 + v^2 = \frac{2a_0{}^2}{\gamma - 1}[1 - e^{(\gamma-1)z}] = \alpha\left[1 - \frac{1}{1+f}\right] = \frac{\alpha f}{1+f}, \quad \text{...(22)}$$

it follows that

$$v^2 = \frac{\alpha f}{1+f} - u^2 = \frac{-\alpha f\,[M + M^{-1} - 2]}{4\,(1+f)}.$$

Thus u and v may be exhibited* in the forms ($i \equiv \sqrt{-1}$)

$$\left.\begin{aligned} 2u &= (M^{\frac{1}{2}} + M^{-\frac{1}{2}})\sqrt{\frac{\alpha f}{1+f}} \\ 2v &= i\,(M^{\frac{1}{2}} - M^{-\frac{1}{2}})\sqrt{\frac{\alpha f}{1+f}} \end{aligned}\right\}. \qquad \text{......(23)}$$

§ 4. We now proceed to discuss the dependence of z and f on x and y. From (8), we have

$$\frac{\dfrac{\partial z}{\partial y}}{\dfrac{\partial z}{\partial x}} = \frac{\dfrac{dv}{dz}}{\dfrac{du}{dz}} = \frac{\dfrac{dv}{df}}{\dfrac{du}{df}} = \Psi(f) \text{ say.} \qquad \text{......(24)}$$

We therefore have the partial differential equation

$$\frac{\partial z}{\partial y} - \Psi(f)\frac{\partial z}{\partial x} = 0, \qquad \text{......(25)}$$

or since always $df = -(\gamma - 1)(1 + f)\,dz$,

$$\frac{\partial f}{\partial y} - \Psi(f)\frac{\partial f}{\partial x} = 0. \qquad \text{......(26)}$$

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* The elimination of one of the variables u and v may also be conducted in another manner, for which I am indebted to Prof. W. McF. Orr, F.R.S. We write $u + iv = u_1$, $u - iv = v_1$. Equations (12) and (13) then become $\frac{du_1}{dz}\frac{dv_1}{dz} = F(z)$ and $u_1 v_1 = \psi_1(z)$ respectively. On eliminating v_1 (say) from these, we are ultimately led to a result agreeing with (23), but the method is neater than that used by the writer.

The general solution of this is

$$x + y\Psi(f) = \Phi(f), \qquad \text{......(27)}$$

where $\Phi(f)$ is an arbitrary function of f.

In the present case,

$$'(f) = \frac{\dfrac{dv}{df}}{\dfrac{du}{df}} = \frac{\dfrac{i}{2}\left[\sqrt{\dfrac{\alpha f}{1+f}}\left(M^{-\frac{1}{2}} + M^{-\frac{3}{2}}\right)\dfrac{dM}{df} + \dfrac{\left(M^{\frac{1}{2}} - M^{-\frac{1}{2}}\right)}{(1+f)^2}\sqrt{\dfrac{\alpha(1+f)}{f}}\right]}{\dfrac{1}{2}\left[\sqrt{\dfrac{\alpha f}{1+f}}\left(M^{-\frac{1}{2}} - M^{-\frac{3}{2}}\right)\dfrac{dM}{df} + \dfrac{\left(M^{\frac{1}{2}} + M^{-\frac{1}{2}}\right)}{(1+f)^2}\sqrt{\dfrac{\alpha(1+f)}{f}}\right]}$$

or

$$\Psi(f) = \frac{i\left[f(1+f)(M+1)\dfrac{dM}{df} + M(M-1)\right]}{f(1+f)(M-1)\dfrac{dM}{df} + M(M+1)}. \qquad \text{...(28)}$$

Now

$$\log M = \log C + \sqrt{\frac{\gamma+1}{\gamma-1}}\left[\log\left(\sqrt{\frac{\gamma+1}{2}} + \lambda\right) - \log\left(\sqrt{\frac{\gamma+1}{2}} - \lambda\right)\right]$$
$$- \left[\log\left(\sqrt{\frac{\gamma-1}{2}} + \lambda\right) - \log\left(\sqrt{\frac{\gamma-1}{2}} - \lambda\right)\right],$$

hence $\dfrac{1}{M}\dfrac{dM}{df} = -2\lambda^2\dfrac{d\lambda}{df}\sqrt{\dfrac{2}{\gamma-1}}\Big/\left(\dfrac{\gamma+1}{2} - \lambda^2\right)\left(\dfrac{\gamma-1}{2} - \lambda^2\right).$

But $\lambda^2 = \dfrac{\gamma-1}{2} - f$, $d\lambda/df = -\tfrac{1}{2}\lambda$, therefore

$$\frac{1}{M}\frac{dM}{df} = \lambda\sqrt{\frac{2}{\gamma-1}}\Big/f(f+1). \qquad \text{......(29)}$$

Thus (28) becomes

$$\Psi(f) = \frac{i\left[(M+1)\sqrt{1 - \dfrac{2f}{\gamma-1}} + M - 1\right]}{(M-1)\sqrt{1 - \dfrac{2f}{\gamma-1}} + M + 1}, \qquad \text{...(30)}$$

and accordingly from (27) and (18)

$$\frac{x - \Phi - iy}{x - \Phi + iy} = M\frac{\sqrt{1 - \dfrac{2f}{\gamma-1}} + 1}{\sqrt{1 - \dfrac{2f}{\gamma-1}} - 1}$$

$$= -C\left[\frac{\sqrt{\dfrac{\gamma+1}{\gamma-1}} + \sqrt{1 - \dfrac{2f}{\gamma-1}}}{\sqrt{\dfrac{\gamma+1}{\gamma-1}} - \sqrt{1 - \dfrac{2f}{\gamma-1}}}\right]^{\sqrt{\frac{\gamma+1}{\gamma-1}}}. \qquad \text{...(31)}$$

Equations (23) and (31) for this method are to be regarded as fundamental for working out cases of steady gas flow under the given conditions.

§ 5. A restriction is placed on $M^{\frac{1}{2}}$ of equations (23) when we insist on physically real values of u and v arising from real values of f; for then both $(M^{\frac{1}{2}} + M^{-\frac{1}{2}})$ and $i(M^{\frac{1}{2}} - M^{-\frac{1}{2}})$ must be real. Let $M^{\frac{1}{2}} = M_0 + iM_1$, where M_0 and M_1 are both real. Then

$$M^{\frac{1}{2}} + M^{-\frac{1}{2}} = \left(M_0 + \frac{M_0}{M_0^2 + M_1^2}\right) + i\left(M_1 - \frac{M_1}{M_0^2 + M_1^2}\right),$$

and

$$i(M^{\frac{1}{2}} - M^{-\frac{1}{2}}) = -\left(M_1 + \frac{M_1}{M_0^2 + M_1^2}\right) + i\left(M_0 - \frac{M_0}{M_0^2 + M_1^2}\right).$$

For the imaginary portions of both these expressions to vanish, we must have clearly $M_0^2 + M_1^2 = 1$.

Hence $$|M^{\frac{1}{2}}| = 1. \qquad \text{......(32)}$$

Referring to (18), it will be seen that (32) cannot be satisfied with real values of λ, unless λ itself (and therefore f and ρ) be constant. Since $\lambda^2 = \frac{\gamma - 1}{2} - f$, we conclude that if $f < (\gamma - 1)/2$, the only solution arising out of our equations is $f =$ constant, i.e. the density is everywhere the same, and so also is the velocity of the gas.

If, however, λ be a pure imaginary, i.e. if $f > (\gamma - 1)/2$, then the moduli of the bracketed expressions of (18) are each unity, and to satisfy (32) we have merely to make

$$|C^{\frac{1}{2}}| = 1, \text{ or } C = e^{2i\delta}, \qquad \text{......(33)}$$

where δ is a constant (real) phase angle.

§ 6. We proceed to discuss the case of $f > (\gamma - 1)/2$ in some detail. In the first place we notice that if a be the velocity of sound at the point (x, y), then

$$a^2 = \frac{\gamma p}{\rho} = \gamma B \rho^{\gamma - 1} = \frac{\gamma p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma - 1} = a_0^2 e^{(\gamma - 1)s} = \frac{a_0^2}{1 + f}. \qquad \text{...(34)}$$

Also from (22),

$$q^2 = \frac{2a_0^2}{\gamma - 1} \cdot \frac{f}{1 + f}.$$

Thus $q^2 > a^2$ when (for f positive) $f > (\gamma - 1)/2$. In other words, the case we are discussing is that in which the velocity of the gas exceeds the velocity of sound at the points considered. In the theory of nozzle flow, the expansion would be said to be beyond the critical pressure, under these conditions.

To simplify the details of what follows, we shall put

$$\sqrt{\frac{\gamma+1}{\gamma-1}} = A, \quad \sqrt{\frac{2f}{\gamma-1} - 1} = \zeta. \qquad \text{......(35)}$$

From (18), we have

$$M = C\left[\frac{A+i\zeta}{A-i\zeta}\right]^{A} \Big/ \left[\frac{1+i\zeta}{1-i\zeta}\right]. \qquad \text{......(36)}$$

If we put

$$\left.\begin{aligned} \tan\theta_1 &= \zeta \\ \tan\theta_2 &= \frac{\zeta}{A} = \sqrt{\frac{2f-\gamma+1}{\gamma+1}} \end{aligned}\right\}, \qquad \text{......(37)}$$

we get from (33) and (37)

$$M = e^{2i(\delta-\theta_1+A\theta_2)} = e^{2i\Theta} \text{ say.} \qquad \text{......(38)}$$

Hence from (23), we get the real values of u and v as follows:

$$\left.\begin{aligned} u &= \frac{e^{i\Theta}+e^{-i\Theta}}{2}\sqrt{\frac{af}{1+f}} = \sqrt{\frac{af}{1+f}}\cos\Theta \\ v &= \frac{i\,(e^{i\Theta}-e^{-i\Theta})}{2}\sqrt{\frac{af}{1+f}} = -\sqrt{\frac{af}{1+f}}\sin\Theta \end{aligned}\right\} \text{....(39)}$$

These equations give u and v as functions of the density only.

We may similarly find the real value of $\Psi(f)$ from (30). We write

$$\begin{aligned} \Psi(f) &= \frac{i\,[(M^{\frac{1}{2}}+M^{-\frac{1}{2}})\,i\zeta + (M^{\frac{1}{2}}-M^{-\frac{1}{2}})]}{(M^{\frac{1}{2}}-M^{-\frac{1}{2}})\,i\zeta + (M^{\frac{1}{2}}+M^{-\frac{1}{2}})} \\ &= \frac{i\left[\left(\dfrac{e^{i\Theta}+e^{-i\Theta}}{2i}\right)i\zeta + \left(\dfrac{e^{i\Theta}-e^{-i\Theta}}{2i}\right)\right]}{\left(\dfrac{e^{i\Theta}-e^{-i\Theta}}{2i}\right)i\zeta + \left(\dfrac{e^{i\Theta}-e^{-i\Theta}}{2i}\right)} \\ &= \frac{\zeta\cos\Theta+\sin\Theta}{\zeta\sin\Theta-\cos\Theta} = -\tan(\Theta+\theta_1) \\ &= -\tan(\delta+A\theta_2). \end{aligned} \qquad \text{......(40)}$$

Equation (27) thus becomes

$$x - y\tan(\delta+A\theta_2) = \Phi(f). \qquad \text{......(41)}$$

If we decide on the arbitrary function Φ, (41) then gives (in theory) the value of the density at (x, y), whilst (39) then settles the component velocities.

The conditions for a stream line are interesting. For such a line, we may put $dy/dx = \tan\chi$, where χ is the angle made by the tangent to the stream line at the point (x, y) with Ox. But from (39),

$$dy/dx = v/u = -\tan\Theta. \qquad \text{......(42)}$$

Hence $\tan\chi = -\tan\Theta$. Thus if n be any integer,

$$\chi = -\Theta + n\pi. \qquad \text{......(43)}$$

Referring to the definition of Θ in (38), we see that δ depends merely on the orientation of the reference axes Ox, Oy, and can be neglected with a suitable choice of these axes.

In general, the stream lines are obtained by eliminating f between (41) and (42).

§ 7. We proceed to the particular case of $\Phi(f) =$ constant, i.e. Φ independent of f. By a suitable displacement of the origin, we can by (41) treat this constant as zero, without loss of generality. We shall suppose the axes so orientated that $\delta = 0$. Our equations then become

$$\left.\begin{array}{r} \tan(A\theta_2) = x/y, \\ dy/dx = -\tan\Theta = -\tan(A\theta_2 - \theta_1) \end{array}\right\}. \qquad \text{......(44)}$$

To solve these equations, let $x = y\tan\theta$, so that

$$\frac{dx}{dy} = \tan\theta + y\sec^2\theta\frac{d\theta}{dy}.$$

Since $\tan(A\theta_2) = \tan\theta$, we shall consider the case of $A\theta_2 = \theta$, i.e. $\theta_2 = \theta/A$. We then have from (37)

$$\tan\theta_1 = \zeta = A\tan\theta_2 = A\tan(\theta/A).$$

From the second equation of (44), we find

$$\frac{1}{\tan\theta + y\dfrac{d\theta}{dy}\sec^2\theta} = \frac{-[\tan\theta - A\tan(\theta/A)]}{1 + A\tan\theta\tan(\theta/A)},$$

which reduces to

$$\frac{dy}{y} + d\theta\,[\tan\theta - A\tan(\theta/A)] = 0. \qquad \text{......(45)}$$

On integration, this gives (where K is a constant)

$$y = K\cos\theta\,[\cos(\theta/A)]^{-A^2}. \qquad \text{......(46)}$$

If we use polar coordinates (r, θ), with θ measured as above from the axis Oy, we have $y = r\cos\theta$, and hence the polar equation to the lines of flow is

$$r = K[\cos(\theta/A)]^{-A^2}. \qquad \text{......(47)}$$

If in the above working, we take instead of $\theta_2 = \theta/A$, the more general expression $\theta_2 = (\theta + n\pi)/A$, we get

$$r = K\{\cos[(\theta + n\pi)/A]\}^{-A^2}, \qquad \text{......(48)}$$

and this is seen to be equivalent to a simple rotation of (47) through an angle $n\pi/A$ about the origin. Nothing fresh thus arises.

Referring to (47), we see that the lines of flow make with a radius vector from O an angle which is

$$\tan^{-1}\frac{r\,d\theta}{dr} = \tan^{-1}\left(\frac{1}{A}\cot\frac{\theta}{A}\right) = \tan^{-1}\left(\frac{1}{\zeta}\right) = \frac{\pi}{2} - \theta_1. \quad (49)$$

The stream lines therefore always cut a given radius vector at a constant angle.

§ 8. The results (47) and (48) of § 7 may be confirmed and the value of K for a given stream line determined by calculating directly the stream function ψ, reckoned as per unit thickness the constant mass per second flowing from left to right over the radius vector joining O to the point (x, y).

Since the velocity and density are constant along this radius vector, we have

$$\psi = \rho r\,(u\cos\theta - v\sin\theta). \qquad \ldots\ldots(50)$$

Using (39) and remembering that $\rho = \rho_0 e^{s} = \rho_0\,(1+f)^{\frac{-1}{\gamma-1}}$, we have

$$\psi = \rho_0 r\,(1+f)^{\frac{-1}{\gamma-1}}\sqrt{\frac{\alpha f}{1+f}}\cos(\Theta - \theta)$$

$$= \rho_0 r\,(1+f)^{\frac{-1}{\gamma-1}}\sqrt{\frac{\alpha f}{1+f}}\cos\theta_1, \qquad \ldots\ldots(51)$$

assuming that $A\theta_2 = \theta$, and that $\delta = 0$.

Now $\quad \tan\theta_2 = \tan(\theta/A) = \sqrt{\dfrac{2f-\gamma+1}{\gamma+1}}$, by (37).

Hence
$$f = \frac{\gamma+1}{2}\sec^2\frac{\theta}{A} - 1. \qquad \ldots\ldots(52)$$

From (35) and (37)

$$\tan\theta_1 = \sqrt{\frac{2f-\gamma+1}{\gamma-1}} = \sqrt{\frac{(\gamma+1)\sec^2\dfrac{\theta}{A} - \gamma - 1}{\gamma-1}}, \quad (53)$$

from which we derive

$$\cos\theta_1 = \sqrt{\frac{\gamma-1}{(\gamma+1)\sec^2\dfrac{\theta}{A} - 2}}. \qquad \ldots\ldots(54)$$

Substituting the values (52) and (54) for f and $\cos\theta_1$ respectively in (51), and remembering that $\alpha = \sqrt{\dfrac{2a_0^2}{\gamma-1}}$, we find after some reduction

$$\psi = \rho_0 r a_0\left(\frac{\gamma+1}{2}\right)^{-\frac{A^2}{2}}\left(\cos\frac{\theta}{A}\right)^{A^2}. \qquad \ldots\ldots(55)$$

A comparison of (47) and (55) shows that for the stream line $\psi =$ constant, the value of K in (47) is

$$K = \frac{\psi}{\rho_0 a_0} \left(\frac{\gamma+1}{2}\right)^{\frac{A^2}{2}}. \qquad \text{......(56)}$$

The same value will clearly hold for K in (48).

§ 9. We proceed to find an expression for ϕ, the velocity potential, and confine ourselves to the case of $A\theta_2 = \theta$.

In our polar coordinates, with θ measured clockwise from Oy, we have

$$d\psi = (-v_r r\,d\theta + v_t\,dr), \qquad \text{......(57)}$$

where v_r and v_t are the component velocities respectively along and perpendicular to the radius vector.

From (52) we have

$$e^{-(\gamma-1)z} = 1 + f = \left(\frac{\gamma+1}{2}\right)\sec^2(\theta/A).$$

Hence
$$\rho = \rho_0 e^z = \rho_0 \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\cos\frac{\theta}{A}\right)^{\frac{2}{\gamma-1}}. \qquad \text{......(58)}$$

Thus from (55) and (57),

$$\left.\begin{aligned} v_r &= \frac{-1}{\rho r}\frac{\partial\psi}{\partial\theta} = A a_0 \left(\frac{\gamma+1}{2}\right)^{-\frac{1}{2}} \sin(\theta/A) \\ v_t &= \frac{1}{\rho}\frac{\partial\psi}{\partial r} = a_0 \left(\frac{\gamma+1}{2}\right)^{-\frac{1}{2}} \cos(\theta/A) \end{aligned}\right\}. \qquad \text{......(59)}$$

For the velocity potential ϕ, we have

$$d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{r\partial\theta}r\,d\theta, \qquad \text{......(60)}$$

so that

$$\left.\begin{aligned} v_r &= \frac{\partial\phi}{\partial r} = A a_0 \left(\frac{\gamma+1}{2}\right)^{-\frac{1}{2}} \sin(\theta/A) \\ v_t &= \frac{\partial\phi}{r\partial\theta} = a_0 \left(\frac{\gamma+1}{2}\right)^{-\frac{1}{2}} \cos(\theta/A) \end{aligned}\right\}. \qquad \text{......(61)}$$

On integrating equations (61), we find (neglecting an arbitrary constant)

$$\phi = A a_0 r \left(\frac{\gamma+1}{2}\right)^{-\frac{1}{2}} \sin(\theta/A). \qquad \text{......(62)}$$

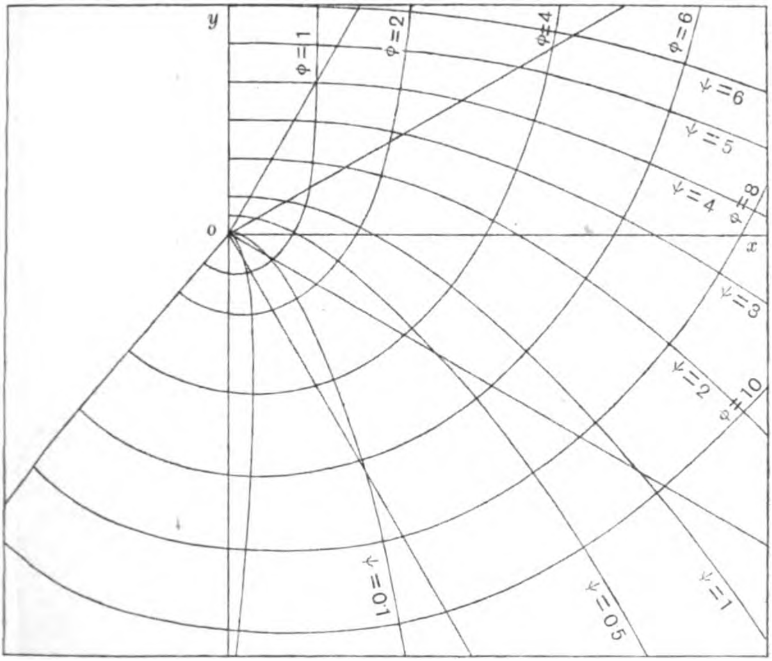
The results (61) may also be expressed as

$$\left.\begin{aligned} v_r &= a_0 \left(\frac{2}{\gamma-1}\right)^{\frac{1}{2}} \sin\frac{\theta}{A} \\ v_t &= a_0 \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \cos\frac{\theta}{A} \end{aligned}\right\}, \qquad \text{......(63)}$$

so that $$q^2 = v_r^2 + v_t^2 = 2a_0^2 \left[\frac{\sin^2 \frac{\theta}{A}}{\gamma - 1} + \frac{\cos^2 \frac{\theta}{A}}{\gamma + 1}\right]. \quad \ldots\ldots(64)$$

Under the conditions assumed, the least velocity q occurs at $\theta = 0$, and is then equal to $a_0 \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{2}}$. It may easily be verified that this corresponds to a velocity equal to the velocity of sound for $\theta = 0$. Incidentally, we note that ρ and v_t vanish when $\theta = A\pi/2$.

The annexed diagram shows the general character of the solution we have obtained, when the fluid is air (say), with γ taken as 1·4.



Conclusion.

§ 10. By assuming that the velocity components for steady adiabatic flow of a gas are functions only of the density (and pressure), we have obtained the following results:

(i) Real solutions are only possible when the velocity of the gas exceeds the velocity of sound at any point.

(ii) The general solution for real steady flows is covered by equations (39) and (41), but these results take a particularly simple form when the arbitrary function Φ is taken as constant. In this case, we have:

(α) Initially, a uniform flow over and normal to the plane Oy, the velocity then being that of sound.

(β) The lines of constant density, constant pressure and constant total absolute velocity q, are radial straight lines* from O.

(γ) The stream lines are given by putting $\psi =$ constant in (55). They are similar curves, and cut any particular radius vector at the same angle, which angle varies from $\pi/2$ at $\theta = 0$ to 0 at $\theta = A\pi/2$.

(δ) The motion is irrotational, and the lines of constant velocity potential are obtained from (62). These are again similar curves, cutting the stream lines normally (of course) but not (in general) the lines of constant pressure, in this way.

(ϵ) Provided the initial uniform parallel flow can be obtained (with the expansion down to the critical value as ordinarily understood in nozzle theory) the above points show that we can design nozzles (their boundaries coinciding with stream lines) for expansion beyond the critical condition. Such nozzles would have the practical advantages:

(ζ) Oblique plane sections of the nozzles properly taken would be constant pressure lines. In practice, most turbine nozzles end obliquely.

(η) The lines of flow across such a properly chosen section would be at a constant angle to the section. This constant angle, however, from (49) settles the pressure ratio p/p_0 at the section, and this may not be a practicable one. Thus if we take $\gamma = 1{\cdot}3$ for superheated or supercooled steam, and the constant angle as 20°, p/p_0 works out to about $1/35{\cdot}7$, a very large expansion.

It scarcely seems necessary to state that the above conclusions are liable to modification if fluid friction be taken into account.

This paper originally arose out of a correspondence between Prof. W. McF. Orr and the writer on the subject of nozzles. Although the writer accepts full responsibility for the conclusions arrived at in this paper, he is desirous of putting on record the stimulating criticisms of Prof. Orr, and of expressing the writer's thanks for these.

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* Lines of constant pressure (and density) are straight in the general case. See equation (27).

The Thallous Thallic Halides. By Mr A. J. BERRY, Fellow of Downing College.

[*Read* 4 February 1924.]

In addition to the simple thallous and thallic chlorides and bromides, a considerable number of intermediate and mixed halides have been described by various investigators. Most of the modern work on these substances has been done by Thomas*, by Meyer† and by Cushman‡. There is general agreement as regards the existence of substances of the type $TlX_3.3TlX$, but considerable divergence exists with regard to the mixed chlorobromides. Compounds of the type $TlX_3.TlX$ have also been described, the best known of which appears to be the dibromide $TlBr_3.TlBr$.

No definite proof has been given hitherto that the sesquihalides, $TlX_3.3TlX$, are definite compounds. Indeed Meyer (*loc. cit.*) has ventured the opinion that thallic chloride can crystallize with varying quantities of thallous chloride up to three molecules of the latter to one of the former; in other words that the well known substance $TlCl_3.3TlCl$ may be the end member of a series of solid solutions. Others have considered the sesquichloride to be a definite compound. Benrath§ has stated that there are four well defined compounds, viz., the dichloride and dibromide belonging to the $TlX_3.TlX$ type, and the sesquichloride and sesquibromide which belong to the $TlX_3.3TlX$ type; the dihalides being produced from concentrated and the sesquihalides from dilute solutions of thallic halides. Benrath has also stated that the sesquichloride and sesquibromide can form a continuous series of mixed crystals with each other, but does not admit the existence of chlorobromides as chemical individuals. It must however be pointed out that the experimental evidence which Benrath has brought forward is exceedingly scanty, and it does not include one single quantitative analysis.

Experiments were accordingly made to decide this point in the following manner. Solutions of thallous and thallic chlorides containing the two halides in varying proportions were prepared and the composition of the crystals deposited from such solutions determined. It was found that while the composition of the liquid phase could be varied over a wide range, the composition of the solid phase deposited was practically constant.

A standard solution of thallic chloride was prepared by dissolving thallic oxide, prepared from the recrystallized sulphate by oxidation with alkaline potassium ferricyanide as already described by the author* in normal hydrochloric acid, the oxide being present in excess. After prolonged heating of the mixture, the excess of thallic oxide was removed by filtering, and the resulting solution diluted to a suitable volume. The concentration of thallic chloride in this solution was determined by the method of analysis with standard sodium arsenite devised by the author†, and found to be 33·85 grammes of thallic chloride (calculated as the anhydrous salt) per litre. The ratio of thallium to chlorine was also determined and found to be 1 : 3·1. Chlorine in thallic chloride cannot be determined by Volhard's method because of the oxidizing action of thallic ions upon thiocyanate. It was found, however that results of sufficient accuracy could be obtained by direct titration with standard silver nitrate using potassium chromate as indicator if ammonium acetate was added to the solution of thallic chloride. Ammonium acetate acts in two ways, firstly in diminishing the acidity of the solution due to hydrolysis of the thallic salt, and secondly, by preventing the precipitation of thallic hydroxide, thereby making the employment of potassium chromate as indicator practicable.

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* *Ann. Chim. Phys.* (8), 11, 204 (1907).

† *Zeitschr. anorg. Chem.* (1900), 24, 321.

‡ *Amer. Chem. Journ.* (1900), 24, 222.

§ *Zeitschr. anorg. Chem.* (1915), 93, 161.

The experiments were carried out in the following manner. Definite quantities of thallous chloride were added to the same quantity of thallic chloride solution, and the mixture heated. Sufficient water was added for solution to take place at the boiling point. The liquid was then cooled in ice, and the crystals which separated, removed by filtering, and dried by exposure over concentrated sulphuric acid in a vacuum desiccator. The crystals were then analysed by the sodium arsenite method, quantities of the order of 1 gramme and 50 c.c. of a standard solution (approximately *N*/10) being employed in each determination, the excess of arsenite being determined by sodium hypochlorite. The following results were obtained:

Weight of thallous chloride taken to 1·69 grm. of $TlCl_3$	Equivalent weight of the crystals deposited	Percentage of $TlCl_3$ in the crystals calculated from the formula $TlCl_3 . 3TlCl$
1·615 grammes	501	31·0
2·12 ,,	519	29·9
2·82 ,,	507	30·6
3·21 ,,	520	29·8
3·62 ,,	504	30·8
4·175 ,,	502	30·9

The mean equivalent weight of these various preparations is 509, the percentage of thallic chloride being 30·5. The calculated equivalent weight from the formula $TlCl_3 . 3TlCl$ is 514·5 corresponding to a percentage of thallic chloride of 30·2. It is clear that while the ratio of thallous to thallic salt may be varied in the liquid phase, the composition of the crystals which are deposited is sensibly constant. As the composition of the solid phase does not vary continuously with that of the liquid from which it is deposited, it is obvious that the sesquichloride cannot be the end member of a series of mixed crystals, but must be a definite compound. Further, it was found that when thallous thallic chloride was recrystallized from very dilute ($N/100$) hydrochloric acid, it was unaltered in composition.

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* *Trans. Chem. Soc.* (1922), 121, 394.

† *Ibid.* (1923), 123, 1110.

The possibility of isomerism among the chlorobromides has been recognized, the most interesting example of which has reference to compounds of the formula $Tl_4Cl_3Br_3$ discussed by Cushman (*loc. cit.*). Obviously there should be two compounds of this formula according as the thallous part of the molecule consists of the chloride and the thallic part of the bromide or *vice versa*, the formulae being $TlBr_3 . 3TlCl$ and $TlCl_3 . 3TlBr$ respectively. Cushman considers that these compounds may be prepared by dissolving the thallous halide in solutions of the required thallic halide and crystallizing out the resulting products. Meyer (*loc. cit.*), however, has concluded that when thallous bromide is dissolved in solutions of thallic chloride, the compound produced corresponds with the formula

$$TlCl_3 . TlCl . 2TlBr.$$

In deciding the question of isomerism in these compounds, it is necessary to divide the molecule into its thallous and thallic constituents, and a method for so doing has been worked out by extraction of the thallic constituents with organic solvents. Methyl alcohol has been found the most convenient solvent for this purpose. A large number of experiments were carried out upon thallium sesquichloride using an ordinary Soxhlet apparatus. It was found that methyl alcohol dissolved out all the thallic chloride, the insoluble residue consisting of thallous chloride without a trace of thallic salt. The methyl alcohol extract always deposited a small quantity of the sesquichloride on cooling, but gravimetric determinations of the ratio of thallium to chlorine gave values of 1 : 3·1 and 1 : 3·2 for the contents of the methyl alcohol solutions. Very careful qualitative experiments were made to ascertain whether the separation of the thallous and thallic constituents was accompanied with any oxidation of the methyl alcohol. These experiments consisted in distilling the solutions of thallic chloride the vapour being absorbed in water. These distillates were neutral to litmus, and sensitive tests for formaldehyde* gave uniformly negative results. It must therefore be concluded that the action of methyl alcohol is simply that of a solvent, and is not in any way complicated with reduction of the thallic salt. Experiments with other organic solvents, such as acetone and pyridine, were also carried out, but as these solvents were not found to possess any advantage over methyl alcohol, their use was abandoned.

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* Compare Fenton, *Trans. Chem. Soc.* (1907), 91, 693.

Action of thallous chloride on solutions of thallic bromide. A number of experiments were made with the object of preparing a chlorobromide of the formula $TlBr_3.3TlCl$. Thallous chloride was added to hot aqueous solutions of thallic bromide, prepared by oxidizing thallous bromide with excess of bromine water and expelling the excess of bromine by evaporation. The products were then crystallized.

The equivalent weights of the preparations, which were fine orange red crystalline products, were determined with reference to sodium arsenite. The calculated equivalent weight of the chlorobromide $TlBr_3.3TlCl$ is 581·2, or expressing the result as a percentage of thallic bromide for a substance having this formula, it is 38·2 per cent. In different preparations, values for the equivalent weights were found to be 564, 579, 588, and 590, or expressed as percentages of thallic bromide, the results become 39·3, 38·4, 37·6, and 37·6 per cent. respectively.

Extraction experiments with methyl alcohol showed that the soluble constituent consisted essentially of thallic bromide, a determination of the ratio of thallium to bromine in the solution giving the value 1 : 3·2. Qualitative tests applied to the insoluble portion showed that it consisted essentially of thallous chloride; no thallic salt was present, and thallous bromide could not be detected with certainty. The methyl alcohol solution deposited a small quantity of a white insoluble substance which was found to be thallous chloride, but no separation of the double halide occurred. We may therefore conclude that the product obtained by dissolving thallous chloride in solutions of thallic bromide consists essentially of a chlorobromide having the formula $TlBr_3.3TlCl$. These results are in agreement with those of Thomas and of Cushman, but at variance with those of Meyer.

Action of thallous bromide on solutions of thallic chloride. Numerous experiments were made with the object of obtaining a chlorobromide having the formula $TlCl_3.3TlBr$, but in every case yellow crystals, similar in appearance to thallium sesquichloride but slightly darker in colour, were obtained. In six different preparations, values for the equivalent weight with reference to sodium arsenite varying between the limits of 524 and 534 were obtained. The same material was obtained whether the thallous bromide was added to the solution of thallic chloride or *vice versa*. Extraction experiments with methyl alcohol showed the presence of small quantities of bromide present in both the thallous and thallic constituents, but it was clearly evident that the halogen which greatly predominated was chlorine. This was confirmed by direct gravimetric determination of the thallium in two preparations whose equivalent weights were 530 and 526 respectively. The determinations were carried out by oxidizing the substance with bromine water, and weighing the thallium as oxide after precipitation with sodium carbonate. The values obtained in both cases were 79·0 per cent. The calculated values are for Tl_2Cl_3 79·3 per cent., and for Tl_4Cl_5Br 76·0 per cent. of the metal. It must therefore be concluded that the products consist of the sesquichloride containing small quantities of bromide probably present in solid solution. When the substance was recrystallized from $N/100$ hydrochloric acid, it was found to be unaltered in composition. It is clear therefore, that while the existence of a chlorobromide having the formula $TlBr_3.3TlCl$ has been established, there is no evidence of the formation of the isomeric substance $TlCl_3.3TlBr$.

Thallous thallic bromides. A few experiments were carried out on the preparation and analysis of these substances. In the first place, it was found that when thallic oxide is dissolved in hydrobromic acid prepared by Scott's method* a solution of thallic bromide is obtained without any evolution of bromine. In this respect the action of hydrobromic acid is precisely similar to that of hydrochloric acid†. The dibromide $TlBr_3.TlBr$, which separates from very concentrated solutions in fine yellow needles, was easily obtained. Determinations of the equivalent weight with reference to sodium arsenite in two different preparations gave values of 370 and 364, the value calculated from the formula being 364. The dark red sesquibromide $TlBr_3.3TlBr$ was found to be difficult to prepare in a state of purity corresponding with its formula, products of lower equivalent weight being for the most part obtained. Previous investigators have stated that the dibromide is decomposed by water into the sesquibromide and thallic bromide, and that the sesquibromide is in its turn decomposed into thallous and thallic bromides. These observations have been confirmed. The dibromide was found to be very sensitive to traces of moisture, exposure to atmospheric moisture causing the yellow crystals to turn red.

Reactions of the thallous thallic halides in aqueous solution. So far as their behaviour towards reagents is concerned, these compounds exhibit the properties of their simple ions only, no evidence of the formation of any stable complex ions having been obtained. Contradictory statements have however been published by Meyer and Cushman with reference to the behaviour of these substances towards silver nitrate. Meyer has stated that when thallic chloride and silver nitrate react together, the resulting thallic nitrate has a marked solvent action on the resulting silver chloride[*]. Cushman[†] on the other hand has found that when sufficient nitric acid is present to overcome hydrolysis, and a sufficient excess of silver nitrate is added, the gravimetric determination of the halogen can be carried out accurately.

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* *Trans. Chem. Soc.* (1900), 77, 648.

† Compare Berry, *Trans. Chem. Soc.* (1922), 121, 395.

The author has made determinations of the halogens in the various preparations, and results of sufficient accuracy have been obtained to verify their composition. In particular the yield of silver chloride and bromide obtained from the chlorobromide $TlBr_3.3TlCl$ was in accordance with the value calculated from the formula.

Effect of alteration of temperature on the colour of the thallium sesquihalides. The intermediate thallium halides exhibit remarkable colour changes with alteration of temperature. In all cases, exposure to low temperatures causes a very marked diminution in the intensity of the colour. When cooled to the temperature of liquid air, the sesquichloride loses its colour completely and assumes a snow white appearance; the red sesquibromide and the chlorobromide when treated in the same manner become yellow in colour. These effects are purely temporary however. This phenomenon has been observed with other similarly coloured substances, but attention is drawn to it as it is so strongly marked in the case of these thallium halides.

The author desires to express his sincere thanks to Sir William Pope and Mr Heycock for their interest in this investigation.

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[*] *Zeitschr. anorg. Chem.* (1902), 32, 72.

[†] *Loc. cit.* and *Amer. Chem. Journ.* (1901), 26, 505.

The High Energy Groups in the Magnetic Spectrum of the Radium C β-Rays. By Dr C. D. Ellis, Trinity College.

[*Read* 19 May 1924.]

Many years ago Rutherford remarked that the high energy lines of the radium C β-ray spectrum showed the existence of γ-rays of very high frequency, and later he pointed out that their frequencies must be closely connected with the β-ray energies through the quantum relation*. Although the truth of Rutherford's point of view has become more and more obvious, no advance in the detailed knowledge of these lines has been made in the last seven years. This spectrum is very complicated and the lines faint.

It might perhaps be thought that it was not very important whether a few more β-ray lines were found to conform to the quantum relation or not, but the lines I am considering in this paper occupy a special position owing to their high energies. β-ray lines of over two million volts energy occur in this spectrum, and information about the way they originate has reference to the details of the interaction of atoms with radiation of frequency far greater than any that can be produced in the laboratory.

The measurements of this spectrum that I describe here show that the absorption of these high frequencies is still quite normal and no fresh levels in the atom are found. This is in itself important, but further it appears that some evidence on the origin of the γ-rays and their connection with the disintegration process can be obtained from these experiments.

Experimental Method and Results.

The radium C β-ray lines have been very thoroughly investigated by Rutherford and Robinson†, but no high accuracy in the absolute energies was possible or was claimed with their small apparatus. A recent determination of the absolute energies of some of the stronger radium B lines has shown that while the relative values were accurate their entire energy scale needed a slight increase‡.

The interpretation of β-ray spectra depends on a comparison of the energies of the lines with X-ray data, and this is impossible unless the absolute energies are known accurately. The absolute energies of the main radium C lines were not determined at the same time as those of the radium B lines, since, as these lines lie in a different region fresh apparatus would have been needed, and in addition there is a simple comparison method which leads to sufficiently accurate results.

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* Rutherford, *Phil. Mag.*, vol. 34, p. 153 (1917).

† Rutherford and Robinson, *Phil. Mag.*, vol. 26, p. 717 (1913).

‡ Ellis and Skinner, *Roy. Soc. Proc.*, A., vol. 105, p. 60 (1924).

The details of the experimental method for measuring β-ray spectra have been often described*. A uniform magnetic field H is used and the β-ray groups give sharp tracks on a photographic plate from whose position the radii of curvature $\rho_1, \rho_2 \ldots$ of the paths of the various electrons can be calculated. The experiments determine the quantities $H\rho_1, H\rho_2 \ldots$ from which the energies of the groups are deduced.

If the magnetic field is uniform, any one experiment gives the relative $H\rho$'s of the lines on the plate independently of any knowledge of the magnetic field. Hence by a stepwise comparison method it is possible to obtain the relative $H\rho$'s even of lines which differ in energy too greatly to be obtained on the same plate. I have determined in this way the $H\rho$'s of most of the radium C groups relative to the $H\rho$'s of the strong radium B groups whose absolute values are already known. No attempt was made to search for new weak groups since this would have required a great deal of work and all my experiments tended to show the thoroughness of the original measurements of Rutherford and Robinson.

The results are given in Table I which also includes for convenience the lower energy groups measured by Ellis and Skinner†. I actually measured all these lines except some of the very faint ones (intensity 1) in the region $H\rho$ 6000 onwards. These were obtained by interpolation to the new scale from Rutherford and Robinson's results.

It is difficult to estimate the accuracy of these measurements. The most important factor is the uniformity of the field. This was tested directly for the lower fields during the course of the previous experiments and found to be uniform to 1 in 1000. For the higher fields used in this work no direct test was carried out, but the ratio of the ρ's of two groups kept the same value to 1 in 1000 for different positions on the plate. It appears reasonable to take the relative values of two neighbouring lines as correct to 1 in 1000, and the absolute values of the energies as correct to 1 in 300.

The great number of lines in this spectrum makes it necessary to distinguish the stronger lines in some manner, and this has been done in Table I by writing the $H\rho$'s of the stronger lines in a separate column and similarly for the energies. More figures are given in the values for the energies than are justified by the accuracy, but this is convenient in calculation. In the first column is the name of the line and in the second the intensity. These intensities represent roughly the ordering of the line relative to its neighbours for about ten lines on either side.

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* Ellis and Skinner, *Roy. Soc. Proc.*, A., vol. 105, p. 60 (1924).
† *Ibid.*, p. 165 (1924).

Table I. *β-Ray Groups of* RaC.

Name	Intensity	$H\rho$		Energy, volts × 10^{-5}	
E 1	1		1379		1·470
E 2	1		1438		1·582
E 3	4	1557		1·819	
E 4	2		1586		1·882
E 5	2		1834		2·406
E 6	2		1912		2·580
E 7	6	2085		2·975	
E 8	1		2156		3·142
E 9	3		2256		3·379
E 10	1		2390		3·704
E 11	1		2550		4·100
E 12	2		2720		4·529
E 13	1		2840		4·846
E 14	1		2890		4·965
E 15	30	2980		5·199	
E 16	5		3145		5·632
E 17	5		3203		5·785
E 18	15	3271		5·966	
E 19	5		3307		6·062
E 20	5		3326		6·113
E 21	5		3584		6·807
E 22	5		3824		7·462
E 23	7	4196		8·489	
E 24	5		4404		9·070
E 25	13	4866		10·370	
E 26	3		4991		10·747
E 27	3		5136		11·134
E 28	3		5178		11·256
E 29	7	5281		11·550	
E 30	2		5428		11·970
E 31	3		5552		12·327
E 32	2		5708		12·755
E 34	16	5904		13·340	
E 35	2		5948		13·467
E 36	2		6030		13·704
E 37	8	6161		14·082	
E 38	4		6212		14·234
E 39	1		6350		14·629
E 40	1		6523		15·131
E 41	1		6656		15·518
E 42	1		6800		15·937
E 43	2		6932		16·321
E 44	2		6998		16·515
E 45	8	7109		16·838	
E 46	1		7240		17·222
E 47	3	7380		17·631	
E 48	1		7530		18·069
E 49	1		7690		18·537
E 50	1		7974		19·370
E 51	1		8090		19·710
E 52	1		8313		20·365
E 53	3	8617		21·259	
E 54	1		8885		22·047
E 55	1		9165		22·874
E 56	1		9425		23·641
E 57	1		9655		24·320
E 58	1		10020		25·390

The energies are calculated to three figures behind the decimal point for convenience in calculation, although this exceeds the accuracy of the measurements.

Analysis of the Radium C β-ray Spectrum.

In this section I give the formal analysis of the β-ray lines. The point I wish to establish is that the greater portion of this spectrum appears to be due to the conversion according to the quantum relation of certain high frequency monochromatic γ-rays in the *K*, *L*, *M*, *N* ... levels of the radium C atom.

To show this one searches for a group of four lines which satisfy the following conditions. Firstly, the energies must be such that when we add to them respectively the *K*, *L*, *M*, *N* absorption energies of the radium C atom we obtain a constant quantity representing the *hν* of the γ-ray. Secondly, the intensities must follow a regular sequence, the line due to conversion in the *K* level being the strongest, and the others in a descending order. Table II shows that an analysis in this way is possible. The β-ray lines due

Table II. *Analysis of the* RaC *β-ray Spectrum* (explanation in text).

$_{4}$E 3...*K*	1·819 + 0·900	2·719	2·717...γE 3
$_{2}$E 6...*L* III	2·580 + 0·134	2·714	
$_{2}$E 5...*K*	2·406 + 0·900	3·306	3·29...γE 5
$_{1}$E 8...*L* III	3·142 + 0·134	3·276	
$_{6}$E 7...*K*	2·975 + 0·900	3·875	3·86...γE 7
$_{1}$E 10...*L* III	3·704 + 0·134	3·838	
$_{3}$E 9...*K*	3·379 + 0·900	4·279	4·26...γE 9
$_{1}$E 11...*L* III	4·100 + 0·134	4·234	
$_{30}$E 15...*K*	5·199 + 0·900	6·099	6·10...γE 15
$_{15}$E 18...*L* III	5·966 + 0·134	6·100	
$_{5}$E 19...*M*	6·062 + 0·03	6·092	
$_{5}$E 20...*N*	6·113 + 0·005	6·118	
$_{13}$E 25...*K*	10·370 + 0·900	11·270	11·27...γE 25
$_{3}$E 27...*L* III	11·134 + 0·134	11·268	
$_{3}$E 28...*M*	11·256 + 0·03	11·286	
$_{7}$E 29...*K*	11·550 + 0·900	12·450	12·45...γE 29
$_{3}$E 31...*L* III	12·327 + 0·134	12·461	
$_{16}$E 34...*K*	13·340 + 0·900	14·240	14·23...γE 34
$_{8}$E 37...*L* III	14·082 + 0·134	14·216	
$_{4}$E 38...*M*	14·234 + 0·03	14·264	
$_{8}$E 45...*K*	16·838 + 0·900	17·738	17·75...γE 45
$_{3}$E 47...*L* III	17·631 + 0·134	17·765	
$_{3}$E 53...*K*	21·259 + 0·900	22·159	22·16...γE 53
$_{1}$E 54...*L* III	22·047 + 0·134	22·181	

Absorption Voltages for Atomic Number 83

K ...	0·900 × 10^5 volts.
L III ...	0·134
M (mean)	0·03
N (mean)	0·005

to one γ-ray are arranged together. In the first column is put the name of the β-ray line with an initial subscript showing the intensity and followed by the name of the electronic level from which it is supposed to originate. Thus $_{13}$E 25 ... *K* means that the β-ray line E 25 of intensity 13 originates in the *K* level. Next follows the energy of the β-ray line plus the absorption energy of the level concerned. In the example quoted this is 10·370 + 0·900, these figures representing units of 10^5 volts. The next column shows the sum of these two quantities and should be the energy of the γ-ray. If the analysis is correct the last quantity should be constant within a group, for example, in the group containing E 25 the figures obtained are 11·270, 11·268, and 11·286. The last column shows the best value of the γ-ray and also its name which is taken from the *K* line it gives.

It can be seen from the table that remarkably good agreement is obtained, furnishing strong support for the analysis, but in this connection there are several more points which must be mentioned.

The absorption levels used are those for a body of atomic number 83 (radium C) since the β-ray lines are usually assumed to come from radium C and from analogy with the radium B E group it would appear reasonable to use these values. It is possible, however, that the correct values might be those of atomic number 84 for the β-rays might come from the following product radium C′. It has only been shown that these β-ray lines come from radium C or from radium C′, no separation of these two bodies is possible owing to the period of radium C′. No certain decision is possible on purely numerical grounds since sufficient, although distinctly worse, agreement is obtained by using the *K* and *L* I levels of atomic number 84. This latter solution would be distasteful since it would involve still another inversion in the absorbing powers of the *L* levels, and therefore on general grounds I decide provisionally for atomic number 83 which gives excellent agreement.

The values used are those given by Bohr and Coster*. The energies used for the *M* and *N* levels are mean values, since the separation of the different levels is unimportant.

Having decided to use the absorption levels of atomic number 83, the measurements lead definitely to the conclusion that the *L* lines originate in the *L* III sub-group, a result which is consistent with previous experience.

Considering the groups now in greater detail I shall pass over the first four up to E 9 since measurement is difficult in this region owing to general faintness, and these lines have been already treated by Ellis and Skinner. The remaining six groups of lines show a quite surprising accuracy, the γ-ray energies agreeing to 1 in 1000. This was not to be expected, since although with the large apparatus

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* Bohr and Coster, *Zeit. f. Physik*, vol. 12, p. 350 (1923).

I used it is possible to obtain the relative values of two close lines to this accuracy, the relevant quantity here is the absolute accuracy since we are combining these results with X-ray evidence. I anticipated an absolute accuracy of about 1 in 300 in the energy, but it appears, provided the decision about the appropriate atomic number be correct, that by a fortunate chance the absolute energies are considerably more accurate than this.

The general analysis is greatly strengthened by some previous work where I showed directly that the γ-ray E 15, energy $6{\cdot}10 \times 10^5$ volts, really exists*. This was done by the "excited spectrum" method by measuring the energies of the groups of electrons ejected from platinum and uranium by this γ-ray. Repeated attempts to confirm the existence of the other γ-rays by this method led to no conclusive results, due probably to the low absorption of these hard radiations.

The origin of these high energy β-ray lines cannot be considered to be settled until some further confirmation is obtained, but at the moment the numerical agreement shown in Table II is convincing and affords strong evidence for the correctness of this point of view.

It might be thought that owing to the great number of lines it would always be possible to find companions to the strong lines having the required separation. Rutherford has pointed out that the lines are relatively evenly spaced with a separation of about 40,000 volts. Two such intervals do give approximately the *K–L* III separation but fortunately precisely with the two strongest γ-rays we find *M* lines which could not be given in this way. This is a strong argument.

For these reasons I shall take this analysis as correct and discuss the bearing of the results in the following paragraphs. The

Table III. γ-*Rays of* RaC.

Name	Intensity	Energy (volts)	Wave-length in *XU*
E 3	4	$2{\cdot}72 \times 10^5$	45·3
E 5	2	3·29	37·5
E 7	6	3·86	32·0
E 9	3	4·26	29·0
E 15	30	6·10	20·23
E 23	7	9·39	13·15
E 25	13	11·27	10·95
E 29	7	12·45	9·91
E 34	16	14·23	8·67
E 45	8	17·75	6·95
E 53	3	22·16	5·57

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* Ellis, *Roy. Soc. Proc.*, A., vol. 101, p. 5 (1922).

analysis is as complete as could be expected, all the strong lines being explained with the one exception of E 23. This is probably a K line but no corresponding L line was measured. This is not significant as it is easy to miss faint lines.

The γ-rays are collected in Table III, which shows their energies in volts and corresponding wave-lengths. Under the heading "Intensity" is given the intensity of the photographic impression of the β-ray groups they eject from the K level.

Origin of the γ-rays.

The most striking feature of the analysis of the radium C β-ray spectrum is that γ-rays of one and a half million volts and probably also two million volts show no fresh absorption phenomena. The only observed effect is conversion in the K, L, M ... electronic levels as in the case of the lower energy γ-rays of radium B, no indication being found of any conversion in a level deeper in the atom than the K level. This is an important result, the energies of these γ-rays are very great, far greater than those of any X-rays that can be produced in the laboratory and yet they are absorbed in the familiar way. The conversion is certainly internal, that is in the same atom that emits the γ-rays, but the general evidence indicates that this would not make a great difference. This result may be stated in another way by saying that there is no evidence of any nuclear conversion of radiation for radium C. It has been apparent for some time from the β-ray results that this was true up to γ-rays of six hundred thousand volts, and it was thought possible that still higher frequencies might show some such effect.

The fact that nuclear conversion does not happen to any measurable extent brings into evidence for the first time from the purely γ-ray side a difference between the nucleus and the electronic structure. The various γ-ray level systems did not differ from the X-ray levels in anything but complication and magnitude of the energy changes, but it appears now that there must be a real essential difference shown by the fact that there is no appreciable absorption.

It is not possible to put forward a definite hypothesis to explain this behaviour because so little is known about the origin of the γ-rays. In fact it is precisely this point which seems to be the fundamental problem in this subject and therefore it appears relevant to summarize what evidence there is at the moment.

It appears very probable that the γ-rays are emitted by transitions between definite stationary states in the nucleus, and the energy differences of these states are known quite accurately. Contrasted with this detailed knowledge we meet a disappointing lack of evidence bearing on the nature of the particle executing the transitions and its connection with the disintegration process. It is most natural to think of an electron emitting these γ-rays, but it is, of course, possible that it may be an α-particle. We find here a complication which is not present in the electronic system owing to there being only one type of particle, but in the nucleus there are at least both α-particles and electrons and before the detailed γ-ray evidence can be used some decision must be made as to which particle emits the γ-rays.

It is helpful, and also probably physically correct, to distinguish at least two separate systems in the nucleus, the electronic system which emits β-rays during a β-ray disintegration and the corresponding α-particle system which emits α-particles. The simplest assumption is that the γ-rays are emitted by one of these two systems. If we consider a β-ray type of disintegration like that of radium B we find that, even supposing the γ-rays do come from the nuclear electronic system, the actual disintegration electron is very unlikely to be actually responsible for the γ-ray emission. For instance, the evidence suggests that one of the γ-rays is emitted after the disintegration electron has left the nucleus and further that the departure of the disintegration electron does not greatly alter the energy differences of the stationary states. There is also the well-known case of radium E which shows a β-ray type of disintegration but emits only an extremely small amount of γ-radiation. This seems to be more easily understandable if we consider in general that the disintegration electron is not directly connected with the emission of γ-rays. Further evidence on this point is provided by the interesting observations of Meitner who showed that the two α-particle bodies, radium and radiothorium, both emit γ-rays of definite frequency indicating that the γ-ray particle can execute transitions and emit γ-rays also during an α-ray disintegration*. This behaviour appears to be the exception rather than the rule with α-particle bodies and even in these two cases the γ-ray emission is small.

Thus it appears that a γ-ray system can exist in both nuclei showing the β-ray type and the α-ray type of disintegration, but that actual emission of γ-rays is far more usual and more intense in the former case than in the latter. Provisionally we might consider the seat of the γ-ray emission to be in the nuclear electronic system but distinct from the disintegration electron, and while neither the disintegration electron nor the α-particle is actually concerned in the γ-ray emission this view does suggest in a general way that strong γ-ray emission is more likely to be followed by instability of the electronic system than of the α-particle system.

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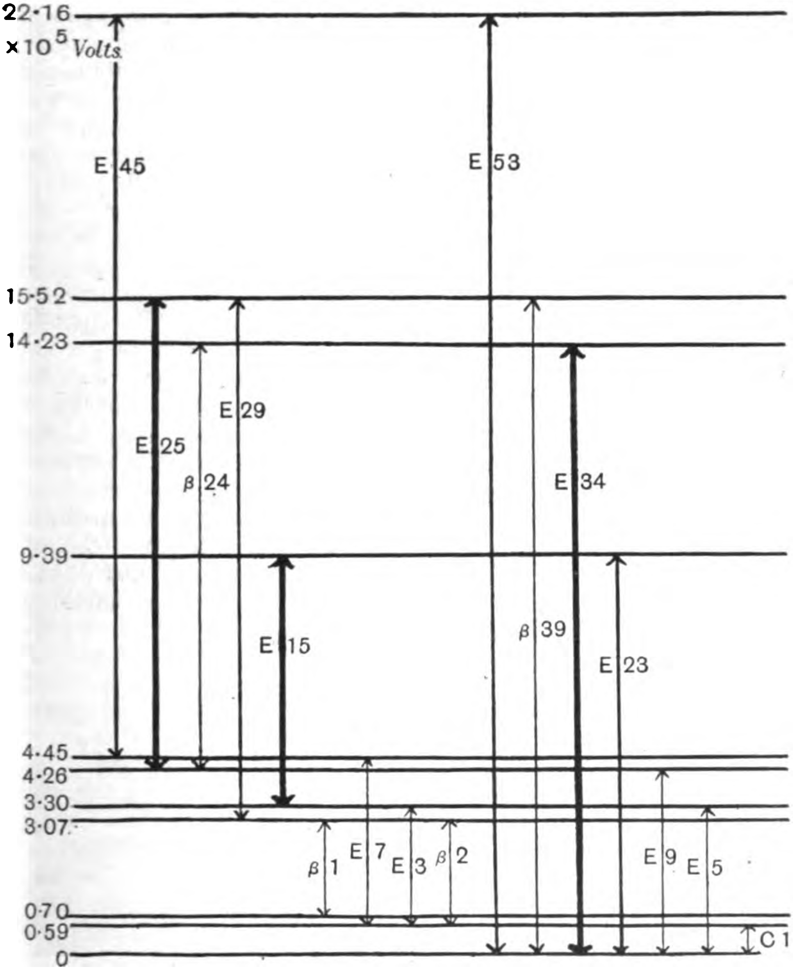
* Rosseland (*Zeit. f. Physik*, vol. 14, p. 173 (1923)) has pointed out the importance of these results in this connection.

The Nuclear Level System of Radium C.

A system of levels capable of giving the lower energy γ-rays has been already given by Mr Skinner and the author. We were guided largely by a close analogy which seemed to exist between

Energy Levels of the Ra C Nucleus



the γ-rays of radium B and those of radium C, and this we interpreted as showing a similarity in the γ-ray level systems of these bodies. It is natural to try to extend this system to include the new higher energy γ-rays, and an attempt at this is shown in the diagram. The transitions are marked with the name of the γ-ray, and further symbols like β 24 are used to show that the γ-ray corresponding to this transition if converted in the K level would give the β-ray line number 24. The β-ray lines referred to in this way are very faint and it may reasonably be supposed that the corresponding β-ray line due to conversion in the L level was too faint to be noticed. We may consider the prediction of these γ-rays to be a certain support for the level system, but yet the evidence is not sufficient to determine the level system uniquely.

The γ-rays undoubtedly show simple additive combinations with each other, indicating that they are due to transitions between stationary states, but these combinations can be represented in more than one way by a set of levels; still it is convenient to show these combinations by a diagram. The set of levels shown here does give a better numerical agreement than any other I could find, but the general accuracy is not sufficient to justify a decision on these grounds alone, and evidence of a more general nature will be needed. The only conclusion one can draw at present is that all the γ-rays emitted by radium C appear to be due to transitions between one connected set of levels.

I wish to express my thanks to Sir Ernest Rutherford for his continued help and for supplying the radioactive material, and to Mr G. R. Crowe for the preparation of the active sources used. I also thank the Government Grant Committee of the Royal Society for a grant which defrayed part of the cost of the apparatus.

Note added in proof. Conclusions of the same general nature as those given above have been reached by Thibaud (*Comptes Rendus*, No. 21, T. 178, 1924) [May 19]. He was not able to proceed to a detailed analysis of the spectrum as sufficiently accurate measurements were not at his disposal.

The relative absorbing powers of the L-levels for radiation of varying wave-length. By H. W. B. SKINNER, Coutts-Trotter Student of Trinity College.

[*Read* 19 May 1924.]

§ 1. *Introduction.*

The experiments to be described were undertaken in consequence of some results obtained from the analysis of magnetic β-ray spectra. It has been shown* that the relative intensities of the lines due to electrons which are ejected from the L_{I}, L_{II}, L_{III} levels of the RaB atom by a certain γ-ray of RaB are in the inverse order from that which would be expected from measurements of the relative strengths of X-ray absorption in the L-levels which had then been made. These, however, as we shall see, refer to the absorption of wave-lengths only just below the critical L-absorption wave-lengths of the element concerned, whereas of course the wave-length of the γ-ray referred to was several times smaller than those of the L-absorption edges of RaB. The present experiment was started† by Mr L. L. Whyte at the suggestion of Mr Ellis, with a view to extending our knowledge of the relative strengths of X-ray absorption by the various L-levels to the case where the wave-length of the absorbed X-ray is also considerably below those of the L-absorption edges. He was, however, unable to complete the work, and so I took it over.

In the present paper, therefore, we shall obtain some evidence on the absorption of X-rays of varying wave-length by the various L-levels of an atom and thus attempt to connect up with the evidence obtained from β-ray spectra.

In this way we are also led to consider how that part of the absorption coefficient of radiation of a given wave-length by an element which applies to *an individual L-level* varies with the wave-length of the absorbed radiation and to compare the result with the known fact that the coefficient of absorption by L-levels, taken as a whole, obeys approximately the λ^3 law.

Previous to the start of this work, the only evidence on the relative absorbing powers of the L-levels for X-radiation had as stated above been confined to radiation of wave-length only just less than the critical L-absorption wave-length of the element concerned. General radiation had been fixed on to an absorbing screen and the transmitted rays analysed with a crystal and ionisation chamber*. Owing to the fact that the absorption coefficient of a substance decreases approximately as the cube of the wave-length absorbed, the effective wave-lengths in these experiments were those only just shorter than the wave-lengths of the L-absorption edges.

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* C. D. Ellis, *Proc. Camb. Phil. Soc.*, vol. 21, p. 121 (1922). (See also further Ellis and Skinner, *Proc. Roy. Soc.*, A., vol. 105, p. 185 (1924)).

† I am informed by Dr Ellis that Dr D. Coster also had the idea, about the same time, of comparing the results obtained from β-ray spectra with those of experiments on X-rays to be tried.

But during the course of the experiments, a paper by H. Robinson† appeared. He carried out an analysis in a magnetic field of the electrons ejected photoelectrically by monochromatic X-rays from a target made up in succession from various elements. In this way evidence on the relative absorbing powers of the L-levels under different conditions was obtained.

In the present paper, the problem is attacked by an entirely distinct method. It involves the excitation of fluorescent X-ray spectra by radiation of varying wave-length. There are, in the L-spectrum, lines corresponding to the return of electrons into all three of the L-levels. If we take, in a fluorescent L-spectrum, two of the lines which are due to the removal of an electron from different L-levels, for example, β_3 ($M_{\text{III}} \rightarrow L_{\text{I}}$) and β_2 ($M_{\text{V}} \rightarrow L_{\text{III}}$), the relative intensities of these lines will depend on two factors: (1) the relative strengths of the absorption of the incident radiation in the L_{I} and L_{III} levels, and (2) the relative probabilities of the electron switches $M_{\text{III}} \rightarrow L_{\text{I}}$ and $N_{\text{V}} \rightarrow L_{\text{III}}$. The latter of these two factors, involving as it does simply the reorganisation of the atom after ionisation, may be expected to be independent of the wave-length of the radiation which has caused the ionisation.

We thus see that if we observe the relative intensities of the lines $L\beta_3$ and $L\beta_2$ in a fluorescent L-spectrum, excited by incident radiation of wave-lengths λ_1 and λ_2, we have a means of determining the change of the relative number of electrons ejected from the L_{I} and L_{III} levels of the atom by radiation of wave-lengths λ_1 and λ_2 which may exist. In this way therefore changes in the relative absorbing powers of the various L-levels for radiation of various wave-lengths can be observed.

It will be clear from the above that changes in the relative intensities of certain lines (e.g. β_3 and β_2) in a fluorescent L-spectrum, excited by radiation of various wave-lengths, were the effects looked for in the present method of finding any changes in relative absorbing powers of the L-levels for radiation of varying wave-length.

The next paragraph will be concerned entirely with the experimental details. § 3 will give the results, and in § 4 the bearing of the available X-ray evidence on the problem of the absorption of radiation by an individual L-level will be discussed. In § 5 the results obtained from β-ray spectra and their relation to the X-ray results are mentioned.

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* See e.g. Duane and Patterson, *Proc. Nat. Acad.*, vol. 6, p. 477 (1920); A. Dauvillier, *Comptes Rendus*, t. 178, p. 477 (1924).

† H. Robinson, *Proc. Roy. Soc.*, A., vol. 104, p. 445 (1923).

§ 2. *Experimental Method.*

The X-rays were produced by a gas filled tube of the type used by Müller*, the anticathode of which could be easily changed and which is very close to the thin Al window. The power was supplied from an induction coil with mercury break and about $\frac{1}{10}$ k.w. was put into the tube. In this way various characteristic K-radiations could, with suitable filtration, be fixed on to the fluorescent radiator, for which cerium oxide was always employed. The K-lines of Cu have energy only about $1\frac{1}{2}$ times the energy of the L-levels of Ce, while the K-lines of silver have an energy of about four times that of these levels. The method of taking the fluorescent spectra was to photograph them using a small Seeman spectrograph with a radius of about 6·5 cm. This method dispenses with slits and rotation of the crystal, since it uses a divergent beam of X-rays which is limited only by a wedge which is placed against the crystal in such a way as to form a narrow slit, and seems to be very

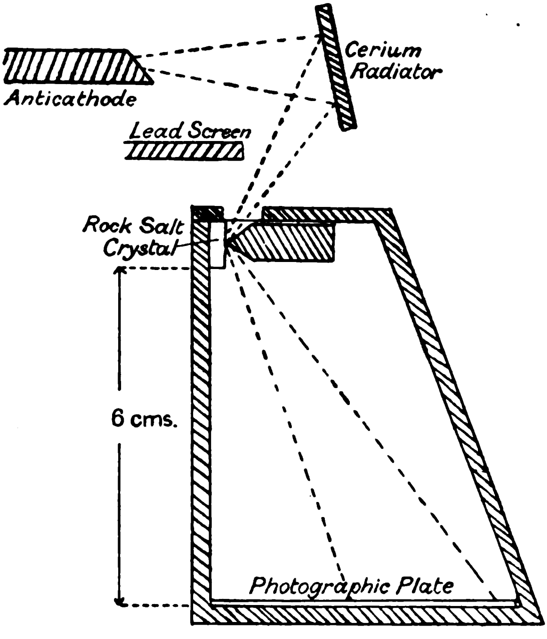

Diagram I

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[*] *Phil. Mag.*, vol. 42, p. 419 (1921).

suitable where accurate wave-length measurements are not required, and where the X-rays can come from a considerable area of the surface emitting them. The crystal was rock salt. The diagram illustrates the way in which the apparatus was set up in order to take the photograph.

It will be observed that the different rays which go to build up the lines of the *L*-spectrum on the plate come from different parts of the radiator. For this reason it is important to have the radiator evenly illuminated. This could only be tested by placing a fluorescent screen in the place of the radiator and judging the evenness of the fluorescence by eye. But the self consistency of sets of plates taken with the same exciting radiation but with the apparatus reset every time seems to prove that this was sufficient and also in fact we shall only be dealing with lines comparatively close together such as are found by rays coming from a strip of the radiator only a few mm. in breadth.

Table I.

	Wave-length Å.U.
AgK_α	0·57
MoK_α	0·71
CuK_α	1·54
Ce L_I absorption edge	1·89
Ce L_{II} " "	2·01
Ce L_{III} " "	2·16

It is thus seen that the apparatus used was very simple; but its efficiency may be judged from the fact that it was possible to photograph strongly a fluorescent *K*-spectrum, under the most efficient conditions, with 10 mins. exposure while even in the much more difficult case of fluorescent *L*-spectra, the exposures were by no means prohibitive.

The exciting radiations used were the *K*-radiations of Cu, Mo, and Ag together with a certain amount of unavoidable general radiation. The wave-lengths of all these radiations (which are sufficiently long not to excite the *K*-radiation of Ce) are given in Table I. In the case of Cu, the tube was run fairly soft (at 30–40 kilo-volts) and the amount of general radiation did not appear, judging from direct photographs of the beam coming from the anticathode, to be very considerable. Since the wave-length of the Cu *K*-radiation (1·54 Å.) is not very much less than those of the Ce absorption edges (about 2·0 Å.) and the λ^3 law of absorption holds approximately, it did not seem to be necessary to take any precaution to cut out the general radiation, particularly since the Cu radiation was used to produce the fluorescent *L*-spectrum of a substance excited by wave-lengths only just less than its characteristic L-absorption wave-lengths. But in the other cases filtration was essential.

In the case of the Mo K-radiation (0·7 Å.), it was very difficult to cut out the general radiation of longer wave-length without also cutting out the K-radiation excessively. In filtering out the long wave-lengths, one has to depend on the λ^3 law, and 0·36 mm. of Al was used. But since the efficiency of excitation decreases rapidly as the wave-length increases, and since the general radiation was quite obvious, under the conditions at which the bulb was run, the effective average wave-length was probably considerably above 0·9 Å., even with this filtration.

In the case of Ag K-radiation (0·56 Å.) it is possible to ensure that the effective average wave-length is considerably less than the wave-length of Cu K-radiation. Al filters of thickness 0·6 and 1·2 mm. were used in two different cases. Calculation shows that in the case of a filter 0·6 mm. in thickness, Ag K-radiation is cut down to two-thirds while a wave-length of 1·0 Å. is cut down to about one-eleventh so that the effective average wave-length is certainly considerably less than this. In the case of the 1·2 mm. filter, Ag K-radiation is cut down to 45 per cent., while a wave-length of 0·7 Å. is cut down to one-fifth. Assuming that these wave-lengths were equally represented in the spectrum coming from the anticathode (which is not the case as the direct photograph shows the Ag K-lines standing clearly from the background) these would be roughly equally effective in exciting fluorescent radiation and so the effective average wave-length certainly lies somewhere between 0·6 and 0·7 Å. Direct photographs also showed that this filter practically wiped out the whole background of wave-lengths above that of the Ag K-lines.

§ 3. *Results.*

The Ce L-spectrum excited by the radiation from the Cu anticathode was obtained clearly with an exposure of 2½ hours. Nine lines were visible, three pairs of which are however inseparable with the spectrometer used. These are the lines $(\alpha_1\alpha_2)$, $(\beta_1\beta_4)$, $\beta_3\beta_2\gamma_1$ $(\gamma_2\gamma_3)$, the brackets indicating the inseparable doublets. Since however consideration of all these lines is unnecessary for the present purpose, we shall confine attention to the lines β_3 and β_2 which are situated close together. Table II shows their origins.

Table II.

Line	Origin
β_2	$N_V \rightarrow L_{III}$
β_3	$M_{III} \rightarrow L_I$

The excitation of the fluorescent Ce *L*-spectra by the Mo and Ag radiation (filtered as above) was much more difficult since the efficiency of excitation decreases rapidly as the exciting wave-length is diminished. Twenty hours' exposure was given and then the lines were not very strong but all the above lines were visible. The presence of a background which could not be avoided made the effectiveness of longer exposure very doubtful.

For convenience of reference we shall call for instance a plate of the Ce *L*-spectrum excited by the radiation from a silver anticathode "an Ag → Ce Plate."

In the Mo → Ce plate, the relative intensities of the lines β_3 and β_2 appeared to be as far as one could see the same as in the Cu → Ce plates.

In the Ag → Ce plates, however, the line β_3 appeared to be distinctly stronger relative to β_2 than in the Cu → Ce plates; but on account of the very faint nature of the lines it appeared to be highly desirable to test this point more precisely by means of a photometer, and I am greatly indebted to Mr G. M. B. Dobson* for allowing me to use the photometer he described in a recent paper. Unfortunately, however, it was not constructed to suit the measurement of faint lines on a plate on which considerable background exists owing to the fact that the background cuts down the brightness of the light, and so the sensitivity, considerably. The Cu → Ce plates were, however, easily measurable and the densities of the lines $(\beta_1\beta_4) : \beta_3 : \beta_2$ were in the ratio 0·49 : 0·12 : 0·26 with an error of about 0·02 in each figure. The Ag → Ce plates, on the other hand, could only be measured with difficulty since the passage of the line over the illuminated slit only gave a motion of the electrometer needle of 1 or 2 scale divisions. From observations on the deflection of the needle, however, a ratio of 1 : 2 for the densities of the lines $\beta_3 : \beta_2$ was definitely excluded, the ratio being slightly less than 1 : 1 in the case of filtration of the exciting radiation through 0·6 mm. of Al and about 1 : 1 when the filtration was through 1·2 mm. of Al.

Thus the photometer measurement supports the visual observation and the agreement of the two methods renders it certain that the line β_3 is distinctly stronger relative to β_2 in the Ag → Ce than in the Cu → Ce photographs.

It is seen from this result that in the case of Mo → Ce, if we bear in mind the fact that the effective average wave-length was greater than 0·9 Å.U., a noticeable change in the intensities of the lines from those found in the Cu → Ce case could scarcely be expected.

It is unfortunately impossible to try excitation by harder X-rays as the exposure required to photograph, for example, the

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* *Proc. Roy. Soc.*, A., vol. 104, p. 248 (1923).

Ce L-spectrum excited by Tungsten K-radiation would be altogether prohibitive, since it can be shown to rise approximately inversely as the cube of the wave-length of the exciting radiation where the latter is sufficiently below the Ce excitation wave-length.

We therefore arrive at the main experimental result of this paper, namely, the conclusion that when the Ce L-spectrum is excited by Ag K-radiation the line β_3 is distinctly stronger relative to β_2 than when the excitation is by Cu K-radiation. It would be interesting to know by how much the ratio is increased. However, one cannot lay any stress on the result obtained for the absolute magnitude of the increase because, in order to find it, one would have to know the density-X-ray intensity curve of the photographic plate. But as we shall see, the mere fact that an increase in the ratio exists is sufficient to enable us to deduce some interesting results.

The relative intensities of the lines in a fluorescent L-spectrum are therefore not constant but depend on the method by which the spectrum is excited.

Of the lines $L\beta_3$ and $L\beta_2$ the former involves a transition to the L_{I}-level and the latter to the L_{III}. It appears therefore that if there were any change in the relative absorbing powers of these levels with the wave-length of the exciting radiation, a corresponding change in the intensity of the lines in the L-spectrum would be observed. To reverse the argument, however, by stating that the observed change of the relative intensities of the lines in the L-spectrum indicates a change in the relative absorbing powers of the L-levels one is compelled to make an assumption. This is that the probability of a given transition by an excited atom is independent of the radiation density of some particular wave-lengths existing round the atom when the transition takes place. If, following Einstein*, we assume that the probability of the transition is $A + B\rho$, where ρ is the radiation density of the emitted frequency which exists in the general radiation from the anticathode, and the term $B\rho$ represents "negative absorption," then since the conditions of production of the exciting radiation in the cases of a Cu and an Ag anticathode are different, the term $B\rho$ might not be the same in the two cases. However, a calculation of the $B\rho$ term shows that its effect would be very insignificant and we are thus fully justified in neglecting it. That the probability of the transition is in fact approximately a constant is supported by the observation that the lines $L\alpha_1$ ($M \rightarrow L_{\mathrm{III}}$) and $L\beta_2$ ($N \rightarrow L_{\mathrm{III}}$) do not apparently change in relative intensity as the wave-length of the exciting radiation is changed from Cu K to Ag K.

Thus we can deduce from the fact that the lines β_3 and β_2 change in relative intensity with the wave-length of the exciting radiation the result that the relative absorbing powers of the L_{I}- and L_{III}-levels vary in a corresponding way with the wave-length of the absorbed radiation.

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* Einstein, *Phys. Zeit.*, vol. 18, p. 121 (1917).

§ 4. *Discussion of Results.*

In § 1 we have seen that there exist several sets of data on the absorbing powers of the L-levels for various wave-lengths. The discussion of the β-ray evidence will, because of its complexity, be deferred until the next paragraph. In the present one, we shall simply be concerned with linking up, and attempting to interpret the various groups of evidence obtained on the X-ray absorption.

In the first place, the results of the pure absorption experiments of the type mentioned in § 1 must be given. We saw that these experiments really refer to the absorption of radiation whose wave-length is only just below that of the levels, by the L-levels. The conclusion in this case is that the L_{III}-level is much the most powerful absorber, and that the L_{II}-, L_{I}-levels follow on in importance in that order.

Our results have shown that when the wave-length of the absorbed radiation is (in the case of a Ce absorber) about one-quarter the wave-lengths of the L-absorption edges, the importance of the L_{I}-level as an absorber is appreciably increased relative to its value when the wave-length of the absorbed radiation is only just below those of the L-absorption edges. We shall now show that this result alone is sufficient to lead to an interesting conclusion.

It is perhaps clearest to express it in terms of an absorption coefficient. Let μ be the atomic absorption coefficient of the material. Then we may divide μ into parts, $\mu_K, \mu_L, \mu_M \ldots$, such that

$$\mu = \mu_K + \mu_L + \mu_M + \ldots,$$

where $\mu_K, \mu_L \ldots$ are the absorption coefficients for the various sets of levels of the atom considered alone. Similarly, we may divide μ_L writing

$$\mu_L = \mu_{L_{\mathrm{I}}} + \mu_{L_{\mathrm{II}}} + \mu_{L_{\mathrm{III}}},$$

where $\mu_{L_{\mathrm{I}}}, \mu_{L_{\mathrm{II}}}, \mu_{L_{\mathrm{III}}}$ are the absorption coefficients of the single L-levels considered above.

What our experiments have done is to provide a rough, qualitative measure of the relative magnitudes of $\mu_{L_{\mathrm{I}}}, \mu_{L_{\mathrm{II}}}, \mu_{L_{\mathrm{III}}}$. In fact we have mentioned that for radiation of wave-length only just below the wave-length corresponding to the L-absorption edge

$$\mu_{L_{\mathrm{III}}} > \mu_{L_{\mathrm{II}}} > \mu_{L_{\mathrm{I}}},$$

while for radiation considerably above the wave-length corresponding to the edge (e.g. Ag K-radiation falling on the Ce L-levels) our experiments have shown that the inequality is less marked.

It is known that the λ^3 law of absorption holds approximately for μ_L. But the present experiments show that it cannot hold for μ_{L_I}, $\mu_{L_{II}}$, $\mu_{L_{III}}$. For if $\mu_{L_I} = C_I\lambda^3$ and $\mu_{L_{III}} = C_{III}\lambda^3$ obviously the ratio of $\mu_{L_I}/\mu_{L_{III}}$ remains constant for all values of λ. We have seen that this is by no means the case and therefore the λ^3 law is merely an average law for the three *L*-levels, and can have no significance when applied to a single level.

This result confirms in an interesting way the experiments of Robinson mentioned in §1. These experiments dealt with "the corpuscular spectrum" of the electrons ejected from various targets by X-rays. Robinson kept his absorbed radiation constant and varied the absorbing element. He records, amongst other things, the estimated relative strength of the lines in the corpuscular spectrum which are due to electrons which have come respectively from the L_I-, L_{II}- and L_{III}-levels. Table III gives his results.

Table III.

At. no.	Target	Mean *L*-absorption wave-length Å.U.	Estimated Rel. intensities of lines		
			L_I	L_{II}	L_{III}
56	Ba	2·2	1	1	6
53	I	2·6	2	3	6
50	Sn	3·0	2	3	5
47	Ag	3·4	3	4	5
42	Mo	4·1	3	4	5
38	Sr	5·7	5	5	5
29	Cu	13	6	5*	5*

* L_{II}, L_{III} unseparated in this case.
Absorbed radiation CuK$_\alpha$ 1·54 Å.U.

It is clear that they show the same tendency as was observed in our experiments, indeed in more detail than the experimental difficulties in our case allowed. In particular they provide strong evidence for our deduction that the λ^3 law is inapplicable to the case of absorption by an individual *L*-level.

As to the detailed agreement on the magnitude of the change in the value of the ratio $\mu_{L_I}/\mu_{L_{III}}$ it is difficult to decide with definiteness. The absorbing elements, and the wave-length of the radiation used in the two cases, were not the same. It is, however, seen from Table III that the wave-length for which equality in the absorption by the L_I-, L_{II}- and L_{III}-levels is obtained in the case of a Sr absorber is 1·54 Å.U., namely a wave-length of about one-fourth the wave-lengths of the absorption edges.

We have already stated in § 3 that in our case it is difficult to obtain a precise idea of the magnitude of the change, but we have seen that a distinct change in the ratio $\mu_{L_I}/\mu_{L_{III}}$ is obtained in the case of a Ce absorber when the wave-length of the absorbed radiation is about one-third or one-fourth of the wave-lengths of the Ce L-absorption edges. While we do not suggest that the ratio of the wave-length of the radiation which is absorbed equally by all three L-levels to the corresponding absorption edge is necessarily a constant, independent of the absorbing element, it will be clear that there is at any rate no antagonism between Robinson's results and ours numerically. However, the important fact is that, generally, the two sets of results obtained by entirely different methods agree.

§ 5. *Relation to evidence obtained from β-ray spectra.*

The only evidence we have on which to rely when the wave-length of the absorbed radiation (compared with the wave-lengths of the absorption edges concerned) is still shorter is derived from β-ray spectra. It is of considerable interest to compare these results with the results obtained from X-ray evidence described above.

It has been shown* that the electrons which form many of the groups of β-rays of homogeneous velocity in such spectra can be considered as ejected from the K, L_I-, L_{II}-, L_{III}-, etc. levels of the atom concerned by the γ-rays. In a recent paper, C. D. Ellis and the present writer† extending the work of Ellis mentioned in § 1 have drawn attention to the fact that the relative intensities of the β-ray lines, due to electrons which come from the L_I-, L_{II}-, L_{III}-levels of the RaB atom, show a series of curious changes with the wave-length of the γ-ray which is ejecting them. This indicates that the three L-levels in a radioactive atom differ markedly in their relative capacity for absorbing a γ-ray according to its wave-length. At the same time it should be pointed out that this conclusion is possibly not a necessary one‡, and in any case the absorption of a γ-ray must be regarded as peculiar. The absorption, in fact, must take place in the same atom from which the radiation itself is emitted.

The results mentioned thus supply data on the absorbing powers of the L-levels in the region of very short wave-lengths. But it is not certain that the evidence is comparable with the X-ray evidence. For we have stated that the absorption of a γ-ray takes place in the same atom from which the radiation itself comes, whereas of course the absorptions of X-rays we have dealt with take place in a different atom. We may call the former type of absorption "internal" absorption. Obviously the probability of the ejection of an electron by a quantum of a γ-ray coming from the inside of an atom will not be the same as the probability of its ejection by a quantum of the same radiation, coming from the outside, even if internal and ordinary absorption are identical processes, on account of the different conditions under which the atom is struck in the two cases. A comparison between them cannot be made without assuming on what character of the radiation the probability of the ejection of an electron depends. Such assumptions (e.g. that it is proportional to the radiation density existing in the neighbourhood of the absorbing level) are easily made and the comparison has been performed*. It was shown that the most obvious assumptions lead to difficulties. Thus the problem of the identity of "internal" and ordinary absorption is still a doubtful one.

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* See e.g. Ellis and Skinner, *Proc. Roy. Soc.*, A., vol. 105, p. 165 (1924).

† Ellis and Skinner, *Proc. Roy. Soc.*, A., vol. 105, p. 185 (1924).

‡ See the last quoted paper, p. 185, where the possibility of the "radiationless" ejection of an electron, from the radioactive atom, as suggested by Rosseland, is discussed.

On the other hand, for our present comparison of the relative absorbing powers of the *L*-levels in the two cases, we are concerned only with the relative probabilities of the ejection of an electron, and so the special probability assumption just referred to does not come in. In this way it is possible that we are touching more closely on the function of the levels themselves in the absorption process. In this connection it is extremely suggestive to note that the relative number of electrons ejected by a γ-ray from the *K*, *L* and *M* levels of the atom, taken as a whole, is roughly independent of the wave-length of the γ-ray throughout the range of the spectra and agrees at least approximately with what one would expect from the corresponding data of X-ray absorption. This, though not in itself conclusive of the validity of a comparison between the β-ray evidence and that obtained from X-rays, is suggestive that it will be interesting to attempt such a comparison.

We therefore propose to consider the facts known about the ejection of electrons from the *L*-levels of a radioactive atom by the γ-rays.

In the case of the ejection of electrons from the RaB atom by the softest γ-rays dealt with, it was found that the L_{I}-level is the most important absorber and that the L_{II}, L_{III} follow on in a sharply descending scale. Since the wave-length of this γ-ray is 0·23 Å.U. and the wave-lengths of the *L*-absorption edges of Pb (with which RaB is isotropic) are about 0·8 Å.U., it appears that this result would fit in well with the X-ray results given above, though perhaps the complete change over in the order of importance of the *L*-levels occurs for a somewhat longer wave-length than might have been expected from the X-ray evidence. This may, however, be due to the higher atomic number of the element involved. On the other hand with γ-rays of smaller wave-length this order of importance for the L-levels does not persist. In Table IV the full data available are given; letters in round brackets imply that no line corresponding to them was visible.

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* See the last quoted paper of Ellis and Skinner, p. 187.

It is seen that the relative absorbing powers of the L-levels must suffer violent fluctuations when the absorbed radiation has a wave-length less than 0·2 Å.U. In particular, when the wave-length of the γ-ray is 0·05 Å.U. or less, the $L_{\rm III}$-level is the most important absorber, just as it is when the wave-length of the absorbed radiation is only just below those of the absorption edges. It is interesting to note that this condition, when once it has set in, appears to be a permanent one. When once the wave-length of the γ-ray has fallen below 0·05 Å.U., a further decrease in the wave-length brings with it no change in the relative importance of the L-levels. The region of violent fluctuation appears to be confined to wave-lengths between 0·2 and 0·05 Å.U.

Table IV.

Wave-length of γ-ray Å.U.	Order of Intensity of lines due to the action of the ray on the three L-levels
0·23	$L_{\rm I}, L_{\rm II}, L_{\rm III}$
0·16	$\left\{\begin{matrix}L_{\rm I}\\ L_{\rm III}\end{matrix}\right\}(L_{\rm II})$
0·05–0·03	$L_{\rm III}\ (L_{\rm II}, L_{\rm I})$
0·02–0·006*	$L_{\rm III}\ (L_{\rm II}, L_{\rm I})$

As we have stated, we do not know with certainty that the absorption in the two cases is of the same type, but we have seen that in the case of the softest γ-ray of RaB dealt with, the relative absorbing powers of the L-levels are about what one would expect from an extrapolation of the X-ray results and this fact makes it worth while to consider in a little more detail the relation of the two sets of results. We shall make use of illustrative curves, and it is most convenient to draw $\frac{\mu}{\lambda^3}, \lambda$ curves in the region of short wave-lengths. It is simplest in the case of the β-ray evidence to treat μ simply as the probability of ejection of an electron from the appropriate level by the γ-ray. Since we are dealing only with relative values of the absorptions, this will raise no difficulties.

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* I owe to the kindness of Mr C. D. Ellis the permission to insert the result for wave-lengths between 0·02 Å.U. and 0·006 Å.U. He has arrived at it from a hitherto unpublished reinvestigation of the γ-rays of high energy which come from RaC. The fact that it applies to RaC (At. no. 83) and not to RaB (82) is not likely to be of any significance in the present connection.

In Diagram II are drawn the $\frac{\mu}{\lambda^3}$, λ curves*. The small circles represent the values of λ (see Table IV) at which the β-ray evidence is available.

The precise ratios of the absorptions at these wave-lengths are however somewhat arbitrarily chosen. The solid curves join up the points so obtained and the dotted curves show a rough attempt to link up the β-ray evidence with the X-ray evidence. It is seen that at any rate the assumption that the two types of absorption are strictly comparable leads to no inconsistency and this suggests that they may actually be the same. It is an interesting question involving as it does the nature of "internal" absorption and is one on which further work on the relative absorbing powers of the L-levels for X-rays in the region of short waves may be expected to throw light.

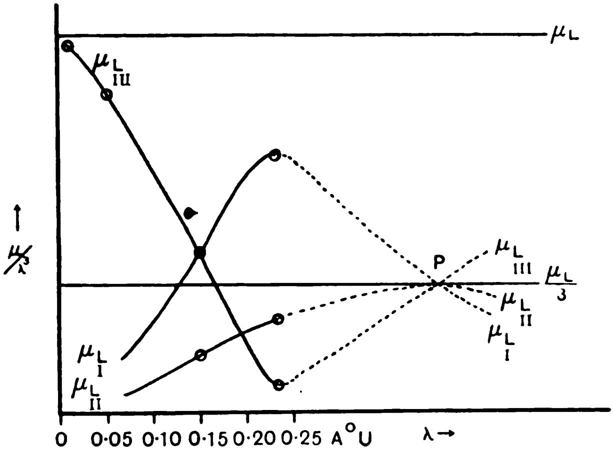

Diagram II

If the β-ray evidence is valid for extending our knowledge of the breakdown of the λ^3 law as applied to the individual L-levels to the case of short wave-lengths, it is seen that the result is to make the relation between μ and λ extremely complex in the region of short wave-lengths. However, the fact that the λ^3 law is inapplicable to the single L-levels is a deduction which can rest entirely on the X-ray evidence brought forward in this paper.

In conclusion I should like to express my thanks to Professor Sir E. Rutherford for his constant interest and advice during the course of the work.

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* It should be stated that for the purpose of drawing the diagram we have assumed, without any direct evidence in the region of such short wave-lengths, that the λ^3 law still holds for the L-levels taken as a whole.

SUMMARY.

1. Changes in the relative intensities of the lines in the fluorescent *L*-spectrum of Cerium excited by radiation of various wave-lengths have been observed.

2. These results imply a change in the relative absorbing powers of the three *L*-levels as the wave-length of the absorbed radiation diminishes from a value just below the absorption wave-length of the *L*-levels to a value considerably below. The absorbing power of the L_{I}-level becomes increased relative to the absorbing powers of the other *L*-levels as the wave-length diminishes. The results agree with those published by H. Robinson in a recent paper.

3. These results imply a breakdown of the law that μ/λ^3 is a constant (where μ is the absorption coefficient of X-rays of wave-length λ) as applied to the *individual* *L*-levels of an element.

4. A comparison is made between the above results, and some results on the relative absorbing process of the *L*-levels obtained by Ellis and Skinner from β-ray spectra.

Some electrical properties of liquid sulphur. By D. H. BLACK, M.SC., Emmanuel College. (Communicated by Dr F. W. ASTON.)

[*Read* 19 May 1924.]

As is well known, the viscosity of molten sulphur tends to a minimum value between 150° C. and 160° C., and then increases suddenly to a maximum at about 200° C.; from that point it decreases again. Following up some work of Farr and Macleod on viscosity measurements, experiments were undertaken to find out whether the electrical conductivity of sulphur underwent similar changes.

Apparatus. Threlfall* found that in dealing with sulphur aluminium was the best metal to use as it is not attacked to any appreciable extent by the sulphur. Another important point is the fact that it is extremely difficult to obtain sulphur in a reasonably pure state, and when obtained to keep it so. The chief difficulty throughout the work was caused through the sulphur films becoming contaminated.

At first an attempt was made to obtain a film of sulphur between two polished aluminium plates, separated by quartz fibres; but this proved unsatisfactory. Finally, the sulphur film was obtained between two cones as shown in Fig. 1. The inner cone was supported on a cylinder of hard glass *D*, supported in a movable sleeve *C*, the height of which could be adjusted by distance pieces when necessary. Contact was made with the inner cone through a mica insulated spark-plug as shown; and the top bolted down as tightly as possible.

The cones were immersed in an oil bath and kept at constant temperatures by means of a thermostat and stirrer in the usual way. Conductivity was measured by direct reading on a high resistance galvanometer (8000 ohms); standard deflections being obtained by means of a standard resistance and shunt. The battery consisted of eight carefully insulated accumulators. Particular care had to be taken with the insulation of the connecting wires, as the resistances being measured were of a fairly high order†. For the problem in hand it was not necessary to work out the specific resistance of the sulphur in each case, relative values only were required.

The samples of sulphur used were prepared by carefully distilling ordinary flowers of sulphur three times in hard glass retorts previously washed with boiling nitric acid, followed by distilled water. The distilled sulphur was kept in closed vessels until required.

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* *Phil. Trans.* (1896).

† At 163° C. the specific resistance was found to be $7 \cdot 5 \times 10^{10}$ ohms (approx.).

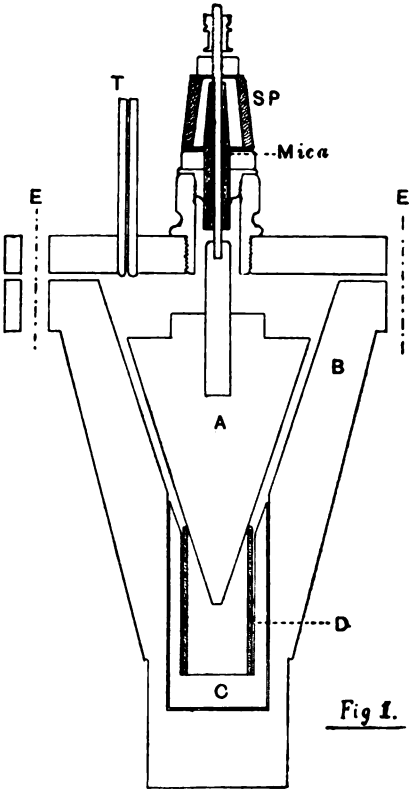

Fig 1.

Variation of conductivity with temperature. A large number of measurements were taken with various films of sulphur. Many of these had to be discarded as it was found that the sulphur had become dirty, rendering the readings of no value. Consequently the whole of the work occupied a considerable time. Of the measurements that are reliable one set has been chosen as being typical.

The procedure was as follows: the sulphur was first of all brought to the highest temperature at which it was desired to take observations, approx. 200° C. It was kept at this temperature for at least one hour and the deflections of the galvanometer read with the current in both directions. For reasons which will be described later it was necessary to take the galvanometer readings at definite intervals from the time at which the current was first switched on. Also it was necessary to pass the current in one direction first without taking a reading; and then to reverse the current twice, taking readings in each case. The temperature was then allowed to fall to any required point and the process repeated. The results are plotted in Fig. 2, where relative conductivity is plotted against temperature. It will be seen that the conductivity follows a similar type of change to the viscosity. The portion of the curve at maximum conductivity was determined more accurately in subsequent measurements and was found to follow the graph more or less as indicated.

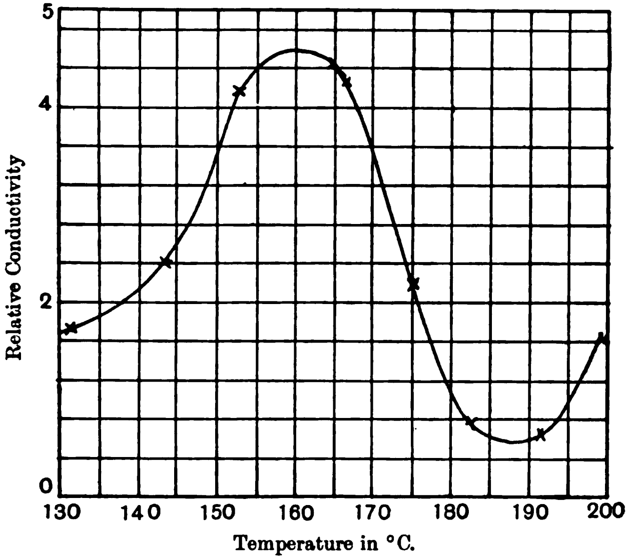

Fig. 2

An attempt was also made to see whether the curve was the same if the temperature were increased from point to point instead of being decreased. Such evidence as was obtained tended to show that this was not the case; but that with rising temperatures the points lay practically on a straight line and followed the given curve when falling. However too much reliance cannot be placed on this point as it was subsequently found that impurities had reached the film, but it was unknown at what stage of the experiment this had taken place.

Variation of conductivity with time of flow of current. From the first observations that were made it was found that the deflection of the galvanometer altered with the time from which the current was switched on. By observing more closely it was found that on reversing the direction of the current the galvanometer deflection rapidly rose to a maximum and then fell off gradually. For this reason the observations on the variation of conductivity with temperature were taken at fixed intervals from the time of reversing the current. Observations were also taken to find out how the deflections, i.e. the conductivity, varied with the time. A set of such observations at different temperatures is shown in Fig. 3. It will be noticed that the curves are all of the same type, though some are flatter than others as would be expected, as the final conductivities vary with the temperature.

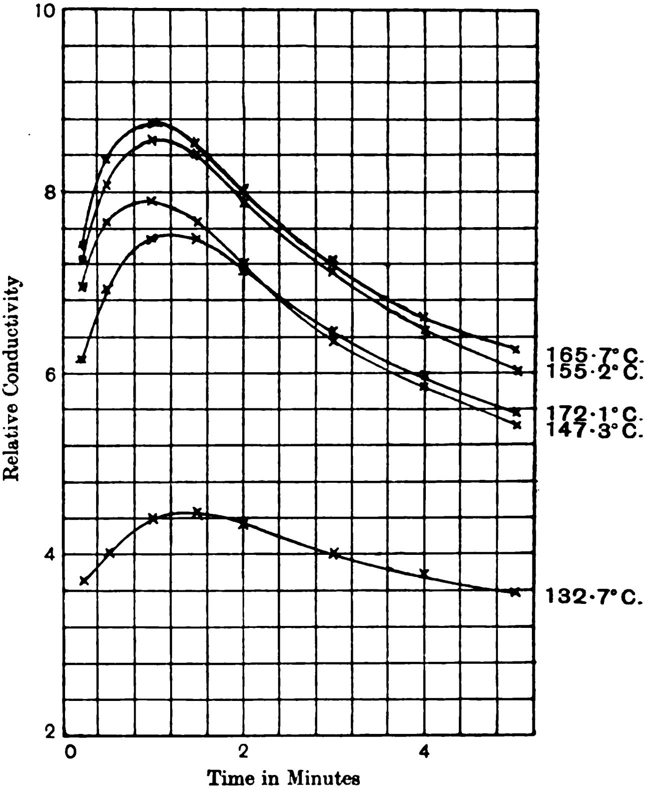

Fig. 3

Polarisation. By studying the conductivity-time curves I came to the conclusion that the conduction of liquid sulphur might be to some extent electrolytic in character. Later I found that Threlfall and Monckman had both come to the same conclusion.

Threlfall mentioned some attempts to measure polarisation effects, but, it seems, without success. On the other hand, one would obviously expect an E.M.F. of polarisation if the conduction is electrolytic. To measure such potentials, if they existed, some form of electrometer must be used; for the resistances involved would be much too high for galvanometer measurements.

A quadrant electrometer with a phosphor bronze suspension was set up; and a standard deflection was obtained from a cadmium cell. Paraffin wax switches were arranged so that the aluminium cones could be connected directly across the electrometer. A current was passed through the sulphur and the leads disconnected. The electrodes were then short-circuited to eliminate any capacity effects that might have been present; and finally the difference of potential read from the electrometer. The process was then repeated with the current in the reverse direction. It was found that the electrometer deflections were in the same direction in each case, but of different values. This was almost certainly due to thermoelectric effects; but the true deflection for the polarising potential was obviously half the difference of the two readings.

Several observations were taken on two different films at various temperatures, and in every case an E.M.F. was observed. Sufficient care was not taken with the heating of the sulphur to enable any relation between the E.M.F. and the temperature to be observed; the experiments were only to prove the existence of the E.M.F. In both cases the measured potentials ranged from ·03 to ·08 of a volt according to the temperature.

The work was discontinued at this point and no further opportunity has arisen to clear up many of the points that arose. In any future work there are several points which would have to be considered. First of all it is very important that oil fumes, etc. must be kept from the sulphur. Although the metal cover was bolted down tightly oil worked its way along the threads of the screws and thus into the sulphur. Either some other method of heating would have to be used or a containing vessel of a different shape used. Secondly, it seems that the manner in which the sulphur is heated, whether slowly or quickly, is very important, and therefore it should receive exactly the same treatment in every case.

Discussion. It seems to be the general opinion that liquid sulphur can exist in two forms S_λ and S_μ which are in a state of equilibrium, thus

$$S_\mu \rightleftharpoons S_\lambda .$$

The proportions in which the two kinds are present depend on the temperature; the thin yellow S_λ predominates up to 160° C., the point of minimum viscosity, and the proportion of the thick brown S_μ increases up to the point of maximum viscosity between

190° C. and 200° C. A very simple explanation of the variation of the conductivity with the temperature is obtained if the resistance of S_μ is higher than that of S_λ. In that case we should expect the conductivity to vary in a similar manner to the viscosity. The experiments bear out this result very well, the conductivity rising to a maximum at about 160° C., and falling to a minimum between 190° C. and 200° C., the respective points of minimum and maximum viscosity.

As previously stated, the study of the falling off of the conductivity with the time the current had been flowing, led me to the conclusion that the conduction of electricity in liquid sulphur might be electrolytic in character. If this is so, it is reasonable to suppose that the "ions" consist of S_λ and S_μ, or something akin to those states. Consequently when a current of electricity passes through the sulphur, a film of S_λ would be formed on one surface and a film of S_μ on the other. This film of S_μ would increase the total resistance in much the same way as hydrogen tends to increase the resistance of a simple cell. That the falling off of the galvanometer deflection is not due to the E.M.F. of polarisation is absolutely certain as in that case the polarising E.M.F. would have to be over one volt.

Assuming, then, that the greater part of the resistance is due to the S_μ present, the resistance R after a time T may be approximately determined. Let ρ be the resistance per unit thickness of the film of S_μ and x the thickness of the film at time T, then

$$x = \frac{R}{\rho}.$$

But x depends on the amount of S_μ deposited, and this depends on the total quantity of electricity, Q, which has passed through the sulphur.

$$\therefore\ x = KQ \quad [K = \text{a constant}],$$

but

$$Q = \int_0^T I\,dt \text{ and } I = \frac{E}{R},$$

$$\therefore\ x = \frac{R}{\rho} = K\int_0^T \frac{E}{R}\,dt,$$

i.e.

$$R^2 = 2\rho KT + A^2,$$

where A^2 is the constant of integration, A representing the resistance when $T = 0$. The time would be measured from the point of maximum conductivity, as the above holds only for the formation of a film of S_μ; whereas when the current is reversed a previously formed film has to be broken up and the conductivity rises.

If the above formula is correct the conductivity should be inversely proportional to the square root of the time. In Fig. 4 the logs of the times are graphed against the logs of the deflections.

It will be seen that the points all lie on approximately parallel straight lines of gradient $-\frac{1}{2}$, thus bearing out my conclusions.

If the decrease in conductivity is due to the formation of a film of S_μ, one would expect that the difference between the maximum conductivity and the conductivity after a certain length of time at any particular temperature, would depend on the proportion of S_μ originally present. This fact is borne out by the curves, for when the average conductivity is low, the curves are much flatter.

If the conductivity is electrolytic in character, one would expect to find some E.M.F. of polarisation. The fact that such an E.M.F. does exist bears out the conclusions.

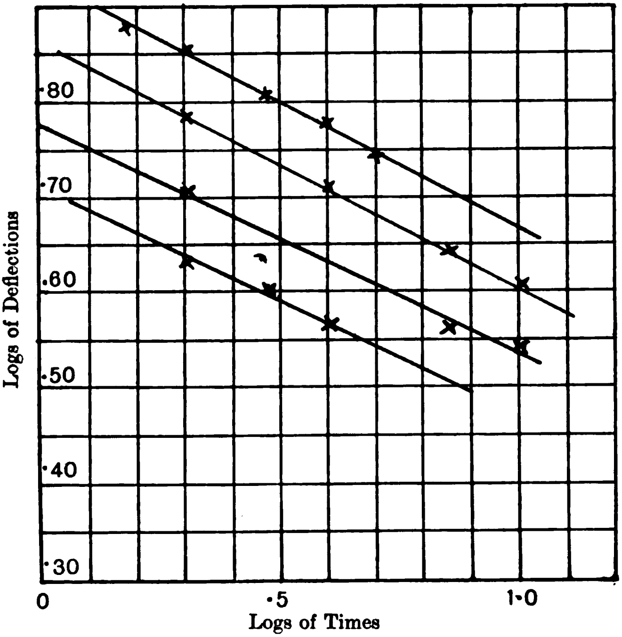

Fig. 4

SUMMARY.

1. The conductivity of liquid sulphur is found to vary with the temperature in a similar manner to that in which the viscosity varies.

2. The conduction is electrolytic in character.

3. There is a definite E.M.F. of polarisation.

I wish to thank Dr C. Coleridge Farr of Canterbury College, New Zealand, for his continued interest in the experiments and for his kindness in giving me every facility in the construction of the apparatus involved.

The number of β-particles from Radium E. By K. G. Emeléus, B.A., Hutchinson Student of St John's College. (Communicated by Professor Sir E. Rutherford, F.R.S.)

[*Read* 19 May 1924.]

When a radioactive element disintegrates with expulsion of β-rays, and has no other mode of disintegration, it is to be expected that at least one β-particle should be emitted for each atom breaking up: one particle arising from each nucleus and possibly others by absorption of accompanying γ-radiation in the orbital electron system of the atom. In the cases of Radium B and Radium C, Moseley[*], by measuring the charge acquired by a chamber exposed to the β-radiation from known quantities of the substances, and also from the rate at which the sources themselves charged up positively when insulated, concluded that about 1·1 β-particles came from each disintegrating atom. By a similar method, Danysz and Duane[†] found that Radium B and Radium C together gave at least three β-particles. Moseley[‡] found however that Radium E appeared to emit little more than half a β-particle, and pointed out that this peculiar result seemed in agreement with some ionization measurements by Geiger and Kovarik[§]. In order to explain this he supposed that Radium E emitted soft rays, not detected in his experiments. On the other hand, Hahn and Meitner[‖] photographed its magnetic spectrum, and found a broad continuous band between $H\rho = 5000$ and $H\rho = 1600$, most intense at 2100: a result recently confirmed by unpublished work of Bothe, employing photographic registration, and of Madgwick, using an electrical method; and even if, as appears probable, the band is continued to smaller values of $H\rho$ than originally found, its intensity is feeble, and Moseley's explanation improbable.

Moseley experienced considerable difficulty in the case of Radium E on account of the small current he had to measure, and probably his result was much less accurate than he estimated it to be. In the present investigation the particles have been recorded individually by an electrical counter[¶], a far more sensitive method, and it has been shown that about one β-particle is ejected from each atom breaking up.

Two methods suggest themselves. In the first a strong source of pure Radium E is taken, and its β-ray emission found; it is then allowed to decay, and from the α-ray activity of the resulting Radium F, and the known transformation constants, the quantity of Radium E initially present can be deduced. Unfortunately it was not found possible to reduce the initial quantity of Radium F present sufficiently to permit of an accurate determination of its subsequent increase. Radium F disintegrates at only one-thirtieth the rate of Radium E, and so, for example, if only 3 per cent. of the Radium F in equilibrium with the Radium E on the source were present, it would give almost the same number of α-particles as would result from the Radium F produced by decay of this Radium E.

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[*] *Proc. Roy. Soc.*, A., 87, 230 (1912).

[†] J. Danysz and W. Duane, *C.R.*, 155, 500 (1912); *Sill. J.*, (4), 35, 295 (1913); J. Danysz, *Thèses, Univ. Paris* (1913).

[‡] *Loc. cit.*

[§] *Phil. Mag.*, (6), 22, 604 (1911).

[‖] *Phys. Z.*, 9, 321, 697 (1908).

[¶] H. Geiger, *Phys. Z.*, 14, 1129 (1913).

In the second, which was adopted, the two substances are taken in equilibrium. Then the rates at which they are breaking up are equal, and the ratio of the number of β-particles to the number of α-particles gives the number of β-particles per disintegrating atom of Radium E, on the assumption that one atom of Radium F gives one α-particle.

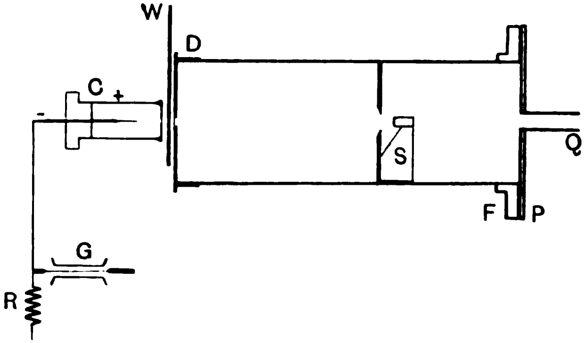

Fig. 1

The apparatus is shown in Fig. 1. The counter (*C*) consisted of a brass tube, 3½ cm. long and 1½ cm. in diameter, polished internally. The particles entered through a mica window, equivalent in stopping power for α-particles to 8 mm. of air at N.T.P., and covering a hole ½ mm. in diameter in a sheet of platinum 0·8 mm. thick, which was waxed on to the annular brass cap of the case with the hole central. The needle was of platinum, with its point accurately centred 1 cm. from the mica, and was soldered in a brass rod insulated from the case by an ebonite stopper; it was in connection with the fibre of a string electrometer (*G*), and earthed through an enclosed alcohol xylol resistance (*R*) of about 10^3 megohms. The case was connected to the positive terminal of a motor-generator capable of giving up to 3000 volts. Entry of an α- or β-particle caused a discharge between the point and the case of the counter, and the resulting momentary change in potential was indicated by a kick of the fibre of the electrometer.

The source (*S*) was supported on the axis of a glass tube, 14 cm. long and 5 cm. in diameter, resting between the pole-pieces of an electromagnet capable of giving a field of up to 500 gauss. The end opposite to the counter was closed by a brass cap (*D*) with a central hole of 4 mm. diameter covered with mica of 1·2 cm. stopping power: the other end carried a brass flange (*F*) and could be closed by bringing against its ground face a flat brass plate (*P*) with an exit tube (*Q*). In front of the source, and soldered to its support, was a brass diaphragm, whose aperture was sufficiently small to prevent β-rays passing out to the counter after a single reflection in the tube. To minimize reflection, the edges of the holes in the cap (*D*) and diaphragm were tapered, with a flat face towards the source. A brass frame bolted on to *D* carried the counter and was insulated from it. Between the counter and *D* was a brass wheel (*W*), by rotation of which 2 mm. of the metal, or mica of various thicknesses, could be put in the path of the particles. The wheel and *D* were earthed and the front of the counter was as close as possible to the wheel without sparks passing, giving a path in the air of 0·45 cm.

The source was part of a 3-year old glass emanation tube, broken into small pieces, and mounted on a piece of brass rod on the brass base (*S*). This was slipped into the tube, which was then closed, and exhausted through *Q*, first by an oil-pump, and finally by a charcoal-liquid air tube. The active material of this source consisted of Radium D + E + F, the quantities of the last two being to within 2 per cent. in their equilibrium ratio*. In addition to the β-radiation from Radium E there is β-radiation from Radium D, but Meitner† has shown that it is very soft, with an absorption coefficient of $\mu = 5500$ cm.$^{-1}$ in aluminium, although her results do not preclude the existence of a small amount of more penetrating radiation. There is also a little γ-radiation from all three bodies, chiefly soft, but negligible for the present purpose, for which the source can be regarded as consisting of Radium E and Radium F in equilibrium.

It was found by preliminary experiments that the number of β-particles recorded by the counter increased rapidly when the potential was raised from 1200 to 1550 volts, but by only a further 3 per cent. between 1580 and 1740 volts; above 1800 volts the point discharged frequently without apparent external stimulus. 1740 volts were used, and at frequent intervals during the course of the counting the number of spurious discharges was found by bringing the metal part of the wheel in front of the counter; these were mainly due to γ-rays and X-rays from other parts of the building, and were reduced to one or two per minute by lead screens.

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* St. Meyer and E. v. Schweidler, *Radioaktivität*, p. 365.

† *Phys. Z.*, 16, 272 (1915).

A count was first made with an air gap in the wheel, giving the total number of α-particles from the Radium F and β-particles from the Radium E entering the counter in a given time. A field of 350 gauss was then applied to deflect the β-particles. The α-particles were not appreciably affected and their number was found by a second count. The field in the neighbourhood of the counter was less than 12 gauss, and this was shown by a subsidiary experiment not to affect the number of α-particles detected. The complete deflection of the β-particles was verified by putting a 1·6 cm. air-equivalent sheet of mica in front of the counter, when nothing was recorded beyond the usual spurious effect. Finally, a set of counts was taken with various thicknesses of mica inserted and without the magnetic field, to allow for absorption of β-rays in the material unavoidably present between the source and interior of the counter.

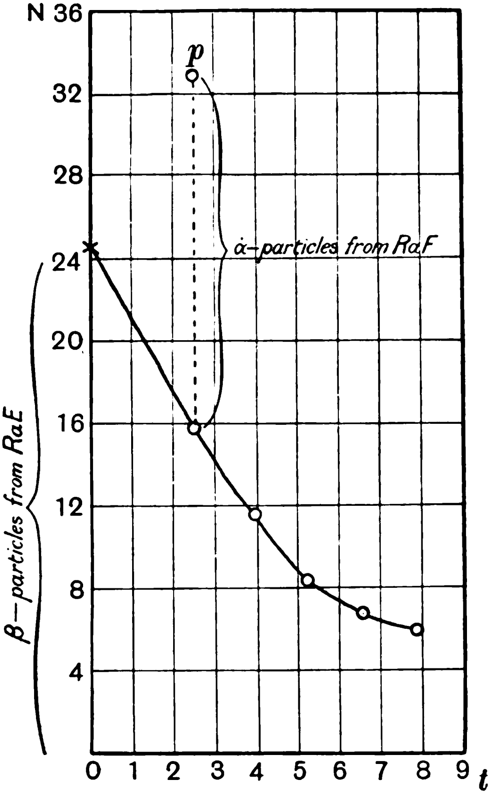

Fig. 2

The results are shown in Fig. 2, where the number of β-particles (N) recorded per minute, corrected for spurious effects, is plotted against the total quantity of absorbing material (t) in their path, expressed in cm. air-equivalent for α-particles. The observed data are given by circles; the cross gives the number of β-particles for zero absorption, found by logarithmic extrapolation; and p gives the result of the first count, with the air gap in the wheel and no magnetic field.

To every 17·1 α-particles from the source there correspond 24·5 β-particles. Not all of the latter have been initially projected towards the counter; some have been emitted in the opposite direction, and have had their direction of motion reversed by collisions in the glass source. This effect has been investigated by Kovarik*, and although his results refer in the first instance to the resulting ionisation, yet it was shown later by the use of a counter that this is proportional to the actual number of reflected particles†. From his data it appears that in the present experiment about 23 per cent. of the β-particles recorded have arisen in this way. Applying this correction, it follows that for each α-particle emitted from Radium F, 1·10 β-particles are emitted from Radium E, and since the two substances are practically in equilibrium this gives the number of β-particles per disintegrating atom. Great accuracy cannot be claimed for this result, because of the uncertainty as to the exact correction to be applied for reflection at the source, and because of the necessary extrapolation to allow for absorption in the air gap and the mica windows, but it should be correct to within 10 per cent.

SUMMARY.

Using an electrical counter, the number of α- and β-particles from a source of Radium D, Radium E and Radium F in equilibrium has been measured: it has been found that after correction for reflection at the source, their numbers were about equal. According to previous work on their absorption, the β-particles from Radium D would not be recorded under the conditions of these experiments. On this assumption the observed β-radiation was due to disintegration of Radium E, and since this was in equilibrium with Radium F, it follows that about one β-particle is emitted per disintegrating atom of Radium E.

In conclusion, the author wishes to express his thanks to Sir E. Rutherford for his interest in the research and to Dr Chadwick for his continual help. He is indebted for partial maintenance to a training grant from the Department of Scientific and Industrial Research.

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* *Phil. Mag.*, 20, 849 (1910).

† Kovarik and McKeehan, *Phys. Z.*, 15, 434 (1914).

On the measurement of the critical potentials of gases. By Mr E. G. Dymond, St John's College.

[*Read* 19 May 1924.]

In all methods of measuring excitation and ionisation potentials which are in common use the current-voltage characteristic curves of the apparatus are plotted. In the absence of gas these curves are smooth, but on the admission of gas at a suitable pressure sharp upward bends occur at definite voltages, which are interpreted as the critical potentials of the gas. In a variation by Franck and Hertz*, the so-called inelastic impact method, these bends are downward, the critical points being represented by maxima.

Difficulties arise in the exact fixation of these points, because they are not, due to a variety of causes, sudden but rather gradual changes in the current, and it is customary to extrapolate the upper and lower portions of the curve (which are to a first approximation linear) until they meet and call this the critical point. This proceeding is a little arbitrary and it seemed desirable in order to heighten the accuracy of measurement to increase the sharpness of the bends. A possible method of doing this is to plot the differential of the current-voltage curve instead of the curve itself, preferably arranging the apparatus to record this curve directly.

This may be done simply by the following device. Superimpose on the potential V a small alternating potential dV, so that the applied voltage oscillates between the values $V + dV$ and $V - dV$. The current will accordingly oscillate between the values $I + dI$ and $I - dI$. Now let this current be passed through a commutator rotating with the same frequency as the alternating potential dV. The current will now oscillate between the limits $dI + I$ and $dI - I$, i.e. it consists of an alternating part I and an unidirectional part dI. Any galvanometer with sufficiently long period compared with the frequency of alternation will give a steady reading proportional to dI, so that if dV is kept constant while V is varied, the galvanometer will record very approximately the differential of the current-voltage curve when dV is sufficiently small.

It will be seen that in this scheme unusually large alternating currents will be circulating in the galvanometer, to its possible harm. An alternative procedure would be, first to transform the current and then to rectify on the commutator, or to balance out the large current by a bridge method before rectifying. In practice, however, with a frequency of 25 per sec. and a long period galvanometer it was found that it would take an alternating current a hundred times larger than that corresponding to a full scale deflection without sensible vibration, and the more complicated alternative methods were discarded.

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* Franck and Hertz, *Verh. d. D. Phys. Ges.*, 16, p. 457 (1914).

The gas chosen to test the method was helium, both on account of its ease of purification and of the positive nature of the results hitherto obtained. It was purified in the usual manner before use by standing over charcoal cooled in liquid air.

The electron tube (Fig. 1) was of familiar type, and consisted of a short tungsten filament *F*, a cylinder *G* of nickel, closed at either end with nickel gauze, and a collecting plate *P*, also of nickel. The electrons emitted from the filament are accelerated by a potential V_1 between *F* and *G*, and suffer collisions with the gas molecules in the field-free space inside the cylinder *G*. In order to measure ionisation, a retarding potential V_2, which is numerically greater than V_1, is applied between *P* and *G* so that only positive ions can be collected on *P*. In order to measure excitation, V_2 is made small, of the order of 1 volt, which serves to stop only those electrons which have given all their energy up to atoms in raising them to higher quantum states. The filament was mounted on a ground glass sleeve to facilitate easy removal. The sleeve was used without lubricant, and when waxed on the outside was quite tight.

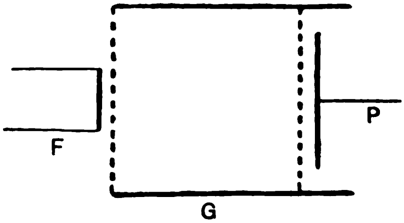

Fig. 1

To remove as much occluded gas as possible from the metal parts and glass walls of the apparatus it was heated, except for the ground glass joint, in a furnace at 360° C. for several hours, while the cylinder and plate were bombarded by an intense electron stream of 1000 volts velocity. The source of alternating current for the differential method was a potentiometer and a commutator mounted on the same shaft as the rectifier.

The results obtained are essentially preliminary and indicate the capabilities of the method, rather than contributing any fresh data on the critical potentials of helium. Immediately on starting work it was seen that the method promised greatly increased sensitivity, for irregularities of the characteristic curves due to traces of impurities were plainly indicated whereas they would have passed quite unnoticed on the normal curves. The extent of this increase of sensitivity may be gauged from Fig. 2, which shows the curve due to helium, both normal (A) and differentiated (B).

These curves were taken with the "inelastic impact" arrangement of potentials. The gas pressures were, for the curve A 1·38 mm., and for B ·98 mm. (This difference of pressure favours the normal curve in the comparison.) The retarding potential V_2 in each case was ·5 volt, while the alternating potential was ·05 volt. The critical points in A are the maxima, and in B the point where the curve crosses the axis. The distinction between the peaks due to the ortho and par-helium levels is just indicated in A, but becomes strongly marked in B.

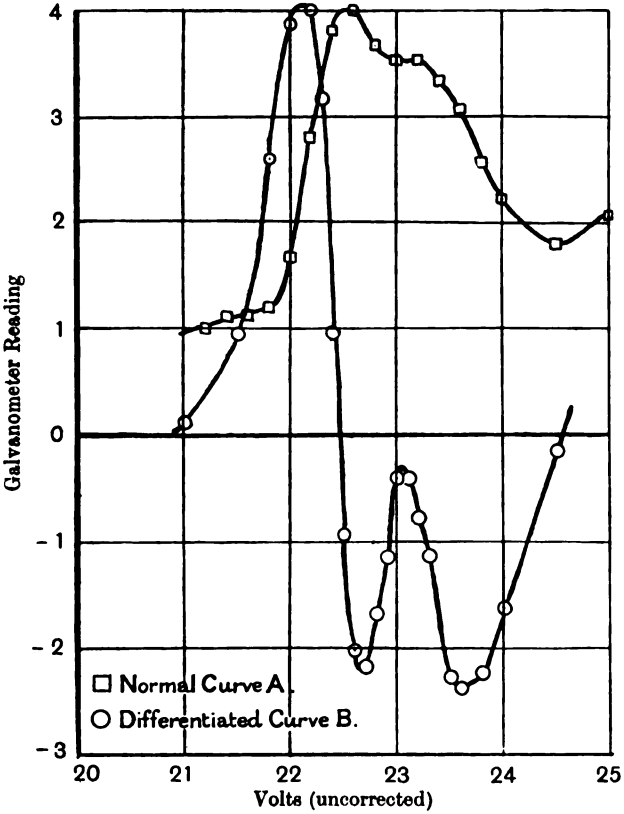

Fig. 2

It would appear at first sight that this null method would yield exceedingly accurate results, as the zero point can be fixed definitely to ·02 volt, or to one part in a thousand. It is however impossible to interpret this point as the required critical point, to the same accuracy, due to the lack of uniformity of the velocity of the electron stream and to the finite magnitude of the alternating potential. (The first order corrections due to initial velocities and contact potentials can be taken care of by the well-known "double impact" method.) This distribution of velocities expands the region of sharp curvature of the characteristics, and the position of the maximum in the normal curves is a function of the type of distribution and of the "efficiency of inelastic collision[*]." The problem of determining the exact value of the excitation potential and of forming an estimate of the form of the probability function of inelastic collision, taking this velocity distribution into account, is at present under investigation.

In conclusion I wish to express my gratitude to Sir Ernest Rutherford for suggesting this research and for his constant interest and advice.

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[*] Hertz, *Proc. Amst. Acad.*, 25, p. 90 (1922); Blackett, *Proc. Camb. Phil. Soc.*, 22, p. 56 (1924).

The Spectra of some Lithium-like and Sodium-like Atoms. By Mr D. R. HARTREE, St John's College.

[*Read* 19 May 1924.]

§ 1. *Introduction.*

In a recent paper, Millikan and Bowen[*] have tabulated a large number of lines appearing in the hot spark spectra of various elements; these spectra, obtained with very violent conditions of excitation, must generally include lines of the optical spectra of atoms in different states of ionisation, and probably lines of X-ray spectra as well[†]. The subject of this paper is the identification of lines of some particular series spectra from among many lines observed.

The method of identification depends on some quantitative relations between terms of the optical spectra of different atoms ionised to such an extent that they possess the same electron structure. The theoretical deduction of these formulae, and a comparison with the results of observation, will be given in another paper; here they will be used simply as extrapolation or interpolation formulae, for obtaining estimates of terms of spectra so far unknown from those of known spectra of atoms with the same electron structure. From the estimated terms, lines can be calculated, and actually the accuracy of extrapolation is in many cases good enough to enable the calculated lines to be correlated with definite observed lines with some confidence.

It is convenient to specify the type of electron structure by the name of the neutral atom in which such a structure is the normal one. Thus the atoms Li, Be^+, B^{++}, C^{+++},... will be called lithium-like. This nomenclature is consistent with the already adopted term "hydrogen-like" for atoms which have lost all electrons except one[‡].

This paper is confined to the spectra of lithium-like and sodium-like atoms. For the lithium type the theoretical relations, which are only approximate, are probably most nearly correct, and they are also particularly simple; on the other hand, spectra of four atoms of the sodium type are already known, so that extrapolation can be carried out more accurately for this type of electron structure than for most others, since for no other single type are spectra of more than two atoms known*.

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[*] R. A. Millikan and I. C. Bowen, *Phys. Rev.*, vol. 22, p. 1 (1924).

[†] In any atom, ionised or not, the most loosely bound electrons may be called the *series electrons*, and the spectrum arising from their transitions between various orbits will be called the *optical spectrum* even if it lies partly in the "X-ray" region. The rest of the atom or ion will be referred to as the *core* and the term *X-ray spectra* must be taken to mean spectra arising from transitions involving core orbits.

[‡] Millikan (*loc. cit.*), however, uses the term "hydrogen-like" for any atom "with one valence electron."

If an atom is ionised to such an extent that the core charge is C (which is greater by one than the degree of ionisation), the series constant of its spectrum is C^2 times the Rydberg constant. Following the notation of Fowler and Paschen in recent papers†, the spectrum of such an atom will be named by writing the core charge in Roman numerals after the symbol of the element. Thus the spectrum of doubly ionised aluminium is named Al III, and has a series constant nine times the Rydberg constant.

§ 2. *General Outline of Method.*

The two relations of which use will be made connect the values of terms, corresponding to orbits of given quantum number, in the optical spectra of different atoms with the same electron structure. If the wave number of a term is ν, and R is the Rydberg constant, it is convenient to work with ν/R rather than ν itself, and ν/R will be called the *term value*.

For a hydrogen-like atom of core charge C the term values are given by $\nu/R = C^2/n^2$; for other atoms the relation of the term value for the orbit of quantum numbers n_k to the series of hydrogen-like term values may be expressed by writing

$$\nu/R = C^2/n^2 + \alpha, \qquad \text{......(2·1)}$$

or

$$\nu/R = C^2/(n-q)^2. \qquad \text{......(2·2)}$$

The quantity α will be called the *term excess* and q is already known as the *quantum defect*. Both the quantities are functions of the two quantum numbers n and k, but in many cases the quantum defect q is nearly independent of n.

The orbits corresponding to terms of the optical spectrum of an atom may be divided into two classes, consisting of those which remain always outside the core and those which penetrate into it. The relation between the values of a term for given quantum numbers, in different atoms of the same electron structure, depends on the class to which it belongs. The relations are:

(A) For orbits which remain well outside the core for the whole time, $(C^4/\alpha)^{\frac{1}{3}}$ should be approximately linear in the atomic number N. That is

$$(C^4/\alpha)^{\frac{1}{3}} = a\,(N-\gamma). \qquad \text{......(2·3)}$$

(B) For orbits which penetrate the core, $1/q$ should be approximately linear in N, at any rate in the case of lithium-like and sodium-like atoms. That is,

$$1/q = b\,(N-\gamma'). \qquad \text{......(2·4)}$$

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* Note added in Proof. From Prof. Fowler's work on the spectra of silicon (Bakerian Lecture, 1924) the spectra of three Mg-like atoms are now known.

† A. Fowler, *Proc. Roy. Soc.*, vol. 103, p. 413 (1923); vol. 105, p. 299 (1924): F. Paschen, *Ann. der Phys.*, vol. 71, pp. 142, 537 (1923).

There is no simple relation between the coefficients a and b, but for lithium-like atoms, γ and γ' should be approximately the same.

Though the two relations are approximate only, they do indicate quantities which may be expected to vary in a nearly linear way with atomic number N, and which therefore will probably be easy to extrapolate fairly accurately.

The general procedure to be followed is to use the quantities $(C^4/\alpha)^{\frac{1}{3}}$ or $1/q$ (according to the class to which the orbit belongs) to extrapolate term values of unknown from those of known spectra. Lines are then calculated by combinations of terms and correlated with observed lines where possible. From the observed lines so identified, revised term values are calculated, and these can then be taken as part of the known data in extrapolating to the spectrum of the next element in the periodic table (ionised, of course, to one higher stage, since the electron structure must remain the same).

In no case considered in this paper is the number of lines of any one series large enough for term values to be found in the usual way, by fitting a formula and determining series limits. One or more term values have to be assumed in order to calculate others from the combinations which give the observed lines. Usually, however, it is possible to arrange that the term values which have to be assumed are so nearly hydrogen-like that they can be estimated fairly certainly to the accuracy required.

§ 3. *The Spectra of Lithium-like Atoms.*

For Li-like atoms all orbits except those for $k = 1$ (corresponding to s terms) lie well outside the core and relation (A) must be used for them; Bohr* leaves the question open whether the s orbits do actually penetrate into the core or not, but at any rate they must approach very near it. From other work it seemed necessary to treat them as if they do penetrate, and it will be assumed that relation (B) is applicable to them.

The values of relevant quantities for Li I are given in Table I.

Table I. *Term Values for* Li I.

n_k	ν/R	q	n_k	ν/R	a	n_k	ν/R	a
2_1	·39628	·412	2_2	·26046	·01046			
3_1	·14836	·405	3_2	·11446	·00335	3_3	·11121	·00010
4_1	·07723	·402	4_2	·06395	·00145	4_3	·06254	·00004

2_1, $1/q = 2·43$. 2_2, $(C^4/a)^{\frac{1}{3}} = 4·56$

The only spectrum of a lithium-like atom so far known is the spectrum Li I itself; the first principal pair of Be II has been recognised[*], but no other lines. Millikan and Bowen[†] have identified the corresponding pair of B III (the lines suggested by these authors as corresponding lines of C IV, N V and O VI are probably incorrectly identified, as will appear later).

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* N. Bohr, *Ann. der Phys.*, vol. 71, p. 263.

§ 3·1. *Boron.*

The spectrum of boron was first investigated and it is convenient to consider it first. The value of γ has not so far been specified; from other work it was estimated that it probably lay between 0·5 and 1, and the value $\gamma = 0{\cdot}8$ was taken; this leads to the term values given in Table II.

Table II. *Estimated Term Values for* B III.

n_k	ν/R	n_k	ν/R	n_k	ν/R
2_1	2·83	2_2	2·34		
3_1	1·16	3_2	1·05	3_3	1·00
4_1	0·63	4_2	0·58	4_3	0·56

Table III gives all Millikan's observed lines, and the calculated values of ν/R for those lines of B III which would be expected to be at all strong and which lie in the region observed.

Table III.

Observed lines			Calc. lines B III
Intensity	ν	ν/R	
5	147748	1·346	$2_2{-}3_3 = 1{\cdot}39$
0	136771	1·246	
3	131832	1·201	$2_2{-}3_1 = 1{\cdot}23$
2	113336	1·033	
2	92450	·824	
1	81301	·741	$3_3{-}6_2 = 0{\cdot}75$???
5	73383	·669	$3_2{-}5_1 = 0{\cdot}66$??? *
8	61560	·561	$3_1{-}4_2 = 0{\cdot}58$? †
0	54767	·499	
0	54650	·498	
2	54262	·494	$3_2{-}4_3 = 0{\cdot}49$?
Very strong	48445 48403 $\Delta\nu = 42$	·441	$2_1{-}2_2 = 0{\cdot}44$ ‡

* Observed line very strong for this transition, probably not B III, but it may be obscuring the B III line.

† Observed line stronger than would be expected for this transition.

‡ Not tabulated by Millikan but mentioned in text as $\lambda\lambda = 2066{\cdot}0,\ 2064{\cdot}2$, strongest lines on plate; and identified by him as first principal pair.

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[*] Paschen-Götze, *Seriengesetze der Linienspektren*, p. 71.

[†] Millikan and Bowen, *loc. cit.*

The calculated values are written against the observed line to which they are supposed to correspond, with one or more queries according to the degree of doubt about the identification. From the lines, the revised term values given in Table IV are determined as follows:

The identification is at first most certain in the case of the first lines of the principal and diffuse series, $2_1 - 2_2$ and $2_2 - 3_3$. Now the 3_3 orbit is so nearly hydrogen-like that its term value can be estimated to three decimal places with some confidence; then from the two lines already mentioned the term values 2_1 and 2_2 can be obtained. The 3_1 term value can be estimated afresh, assuming q to be the same as for the 2_1 term, and the line $2_1 - 3_1$ calculated; the result is $\nu/R = 1{\cdot}195$ and it is now clear that the observed line $\nu/R = 1{\cdot}201$ must correspond to this transition, and not the line $\nu/R = 1{\cdot}243$.

Table IV. *Revised Term Values* for* B III.

n_k	ν/R	q	n_k	ν/R	a	n_k	ν/R
2_1	2·789	·205	2_2	2·348	·098		
3_1	1·147	·199	3_2	(1·057)	(·057)	3_3	*1·002*
4_1	[0·622]		4_2	(0·586)	(·023)	4_3	*0·563*
5_1	(0·388)	(·184)					

2_1, $1/q = 4{\cdot}88$. 2_2, $(C^4/a)^{\frac{1}{4}} = 9{\cdot}40$

The 4_3 term has also been estimated, as the line suggested as $3_2 - 4_3$ is the only one observed which depends on either of the terms. The ratio of values of a for the 3_2 and 2_2 orbits is a good deal larger than for Li, which throws doubt upon the identification of the line.

In tables of term values deduced by making use of observed lines, the following notation will be used:

Term values *in italics*, those which have been assumed, such are values of the 3_3 and 4_3 terms for B III.

Term values in ordinary type without brackets, those deduced from the observed lines of which the identification is fairly certain.

Term values enclosed in round brackets, those deduced from observed lines of which the identification is doubtful.

Term values enclosed in square brackets, those obtained by interpolation or extrapolation from terms deduced from the more certain lines.

It is clear from Table IV that investigation of the hot spark spectrum in the visible and near ultra-violet would be very helpful in checking the term values given, and determining them and others accurately (the same is true for the other spectra considered here). For instance, the measurement of the line $3_2 - 4_1$, which would be expected to lie in the neighbourhood of $\lambda = 2300$, would settle whether the value of the 3_2 term given is correct, and so whether the line from which it was obtained was correctly identified. Millikan quotes the first principal pair as being given as a very weak spark line in Eder and Valenta's tables, so there is not much likelihood that any of the other lines of B III in the near ultra-violet have been recorded.

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* See addendum for more accurate term values based on later work of Millikan and Bowen.

In the original work carbon was the next element investigated, but the extrapolation is made easier and more certain if we consider beryllium next.

§ 3·2. *Beryllium.*

Interpolating $1/q$ for the 2_1 orbit, and $(C^4/\alpha)^{\frac{1}{4}}$ for the 2_2 orbit, linearly between the values for Li I and B III, the following term values are estimated for Be II:

$$2_1 \quad 1{\cdot}341 \qquad 2_2 \quad 1{\cdot}047 \qquad 3_3 \quad 0{\cdot}445$$

The only lines observed are:

(1) A very strong doublet $\nu/R = 0{\cdot}291$, $\Delta\nu = 6{\cdot}6$ already recognised* as the first principal pair of Be II. The calculated value of ν/R for the line is 0·294, in good agreement with observation. (This line is not tabulated by Millikan but mentioned in the text.)

(2) A line, intensity 3, $\nu/R = 0{\cdot}602$ which agrees exactly with the line calculated for the transition $2_2 - 3_3$.

These are the only lines given by Millikan for Be, but he mentions that further experiments are in progress and in this connection Table V of approximate term values may be given.

Table V. *Approximate Term Values for* Be II.

n_k	ν/R	q	n_k	ν/R	α	n_k	ν/R	n_k	ν/R
2_1	1·338	·271	2_2	1·047	·047				
3_1	[0·535]		3_2	[0·460]		3_3	*0·445*		
4_1	[0·287]		4_2	[0·259]		4_3	*0·251*	4_4	*0·250*

$$2_1, \quad 1/q = 3{\cdot}69. \qquad 2_2, \quad (C^4/a)^{\frac{1}{4}} = 6{\cdot}99$$

§ 3·3. *Carbon.*

We have now obtained some data for the first three Li-like atoms, and it appears that the linear relations (A) and (B) hold very closely indeed.

The values of γ deduced from the 2_1 and 2_2 orbits are certainly not exactly the same, but this is not surprising in view of the approximations involved in deducing the relations. For extrapolation purposes it seems best to treat the relations for the p and s terms as independent. For the d terms we have to estimate α and for want of better means it will be assumed that the ratio of the values of α for the 2_2 and the 3_3 orbits is the same as for Li. Only the 2_1, 3_1 and 2_2 term values can be extrapolated directly from the data available so far, since the 3_2 and 4_2 values for B III are doubtful. It is assumed in deducing values for carbon that for each n_2 term the ratio of the value of α to that for the 2_2 term is the same as for Li. For 4_1 and higher s terms it is assumed that q is the same as for 3_1. The term values given in Table VI are thereby obtained, those for 3_2, 4_2 and 5_2 being the least certain.

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* Paschen-Götze, *loc. cit.*

Table VI. *Extrapolated Term Values for* C IV.

n_k	ν/R	n_k	ν/R	n_k	ν/R	n_k	ν/R
2_1	4·740	2_2	4·155				
3_1	1·982	3_2	1·826	3_3	1·780		
4_1	1·084	4_2	1·022	4_3	1·001	4_4	1·000
5_1	0·682	5_2	0·651	5_3	0·641	5_4	0·640

Table VII.

Observed lines			Calc. lines C IV
Int.	ν	ν/R	
4	260166	2·371	$2_2-3_3=2{\cdot}375$
1	238197	2·170	$2_2-3_1=2{\cdot}173$
1	146929	1·339	$3_3-6_4=1{\cdot}336$?? * $3_1-5_2=1{\cdot}331$??
1	127141	1·159	$3_2-5_1=1{\cdot}144$???
5	125016	1·139	
6	123957	1·130	$3_3-5_2=1{\cdot}129$??? †
5	123464	1·125	‡
4	105752	0·964	†
0	104775	0·955	$3_1-4_2=0{\cdot}960$?
3	91522	0·834	$3_2-4_3=0{\cdot}825$
3	87920	0·801	
4	87608	0·798	
15	85062	0·775	$3_3-4_4=0{\cdot}780$??? †
3	83743	0·763	$3_3-4_2=0{\cdot}758$??? §
2	81289	0·741	$3_2-4_1=0{\cdot}742$?
15	74909	0·683	
4 3	64566 64481 } $\Delta\nu=85$	0·588	$2_1-2_2=0{\cdot}585$ ‖
5	64048	0·584	†

* Not observed by Millikan, given by Simeon.

† Observed in spectrum of 30 volt arc.

‡ Observed in spectrum of 40 volt arc.

§ Given by Millikan as probably Si. Doublet $\Delta\nu=84$ according to Simeon.

‖ Given by Simeon, as $\nu\nu$ 64587, 64483 ($\Delta\nu=104$).

The lines observed by Millikan are so numerous that a complete table would take up too much space, so Table VII only includes some of them. In some cases there is no doubt about the observed line to be correlated with a calculated one; where there might be a doubt, all the observed lines which lie in the neighbourhood of the calculated one and which might be concerned are tabulated. Those which have been observed* in the spectrum of a 30 volt arc are indicated; they are presumably not due to C IV, but there is always the possibility that C IV lines may be hidden by them.

Of the first two lines there is very little doubt, and the doublet $\nu/R = 0{\cdot}588$ is most probably the first principal pair, though the doublet separation $\Delta\nu = 85$ is rather smaller than would be expected from the separations 0·34, 6·6, 42 for Li I, Be II, B III respectively. (According to Simeon's measurements $\Delta\nu = 104$ for the doublet in question.) On the other hand, the agreement between observed and calculated ν/R for the pair is quite good and it is the only fairly strong pair in this region with about the right separation†.

Assuming term values for 3_2 and 4_2, we get the revised term values given in Table VIII.

Table VIII. *Revised Term Values for* C IV.

n_k	ν/R	q	n_k	ν/R	a	n_k	ν/R
2_1	4·739	·1625	2_2	4·151	·151		
3_1	1·981	·158	3_2	1·835	·057	3_3	*1·780*
4_1	(1·094)	(·176)	4_2	(1·026)	(·026)	4_3	*1·001*
5_1	(0·676)	(·13)	5_2			5_3	*0·641*

2_1, $1/q = 6{\cdot}15$. 2_2, $(C^4/a)^{\frac{1}{3}} = 11{\cdot}92$

In this table, the values for 3_2, 4_1 and 5_1 are derived from the estimated 4_3 term using observed lines, and the values for 2_1, 2_2. 3_1 from the estimated 3_3 term. This is rather important, since it is possible that values of a for the diffuse ($k = 3$) terms have been underestimated, and if so the correction to be added to 3_3 and the terms deduced from it will be greater than that for the 4_3 term and those depending on it. There are no possible tests in the region covered by the lines observed by Millikan, but in the visible region there is a doublet λλ 5812, 5801 ($\nu/R = 0{\cdot}157$, $\Delta\nu = 31$), observed in

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* See F. Simeon, *Proc. Roy. Soc.*, vol. 104, p. 368 (1923). Of the lines which are considered as probably due to C IV, Simeon (*loc. cit.* and *Proc. Roy. Soc.*, vol. 102, p. 488 (1922)) only gives the first principal pair, and a line which may be the same as Millikan's line $\nu/R = 2{\cdot}170$, though the two observers' wave lengths differ a good deal more than for most other lines.

† The strong line $\nu/R = 0{\cdot}683$ ($\lambda = 1335{\cdot}0$) is suggested by Millikan for the first line of the principal series of C IV, but as it appears in the 30 volt arc this is unlikely. Also it does not fit in at all with the regular progression of the corresponding lines of the earlier elements (cf. § 3·6).

Wolf-Rayet stars and found by Merton* using a very heavy discharge between graphite electrodes in hydrogen, which has been suggested by Fowler† as possibly due to C IV. The separation is about what might be expected for the 3_2 term, and the doublet may possibly correspond to the transition $3_1 - 3_2$ ($\nu/R = \cdot 146$ from term values in Table XI).

Assuming the lines $2_2 - 3_1$, $2_2 - 3_3$ and $3_2 - 4_3$ have been correctly identified (of this there seems little doubt, as each of the lines is well separated from the next observed line), then in order to fit in the doublet $\nu/R = 0{\cdot}157$, it is necessary to increase the difference between the estimated 3_3 and 4_3 term values by ·011; to do this and preserve a reasonable ratio of a for the two term values it is necessary to increase 3_3 and term values deduced from it by about ·020. The line $3_3 - 4_4$ would then be about $\nu/R = 0{\cdot}800$; this line would be expected to be fairly strong, but if the term values of Table XI are correct, it might have been hidden by the very strong line $\nu/R = 0{\cdot}775$. Actually there is a line at $\nu/R = 0{\cdot}798$, of about the same intensity as the other strong C IV lines. If provisionally we identify this as $3_3 - 4_4$ and assume $4_4 = 1{\cdot}002$, the term values given in Table IX are obtained. The term 4_2 in this table has been determined from $3_1 - 4_2 = \cdot 955$, and 4_1 from $3_2 - 4_1 = \cdot 741$.

Table IX. *Alternative Term Values for* C IV.

n_k	ν/R	q	n_k	ν/R	a	n_k	ν/R	a	n_k	ν/R
2_1	4·759	·166	2_2	4·171	·171					
3_1	2·001	·172	3_2	1·844	·067	3_3	1·800	·022		
4_1	(1·103)	(·191)	4_2	(1·046)	(·046)	4_3	1·010	·010	4_4	*1·002*

The objection to these term values is the way in which q varies with ν/R for the s terms; in almost all cases in which the orbit penetrates the core, $dq/d\,(\nu/R)$ is positive; it is so for the s terms of Li I and there is no obvious reason why the sign should change as N increases. In fact, an approximate theoretical relation for the variation of $dq/d\,(\nu/R)$ with atomic numbers can be found for orbits which penetrate the core; it shows firstly that $dq/d\,(\nu/R)$ must be positive for large atomic numbers at any rate, so that a negative value for C IV would involve two changes of sign, and secondly, that, considering two given terms of the same sequence, the difference of the values of q is roughly proportional to q^3 times the difference of values of ν/R. This relation is roughly true for sodium-like atoms and there is no reason to expect it to fail entirely for lithium-like atoms. Applying it to the difference between the values of q for the 2_1 and 3_1 terms of lithium-like atoms, we get from the known difference 0·007 for Li I, the value 0·005 for C IV, which agrees excellently with Table VIII.

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* Merton, *Proc. Roy. Soc.*, vol. 91, p. 498 (1915). A table of the chief unidentified Wolf-Rayet lines is given in this paper. One line and one pair observed by Merton are included in the spectrum C II given recently by A. Fowler, *Proc. Roy. Soc.*, vol. 105, p. 299 (1924).

† A. Fowler, *Report on Series in Line Spectra*, p. 164.

The values of $1/q$ and $(C^4/a)^{\frac{1}{4}}$ obtained from Table IX do not, of course, fit the linear relations deduced from the earlier elements, but these have been based on assumed values of a for the 3_3 terms of Be II and B III which would be underestimated if the value given in Table IX for carbon were correct.

There is some evidence from the spectra N V and O VI that the high value of the term excess a for the 3_3 term in Table IX is incorrect, and altogether the available data appear to favour the set of term values in Table VIII, which will therefore be used in preference to those in Table IX for extrapolation to other elements.

§ 3·4. *Nitrogen.*

For N V the term values given in Table X are obtained by extrapolation of the results for the previous elements.

Table X. *Calculated Term Values for* N V.

n_k	ν/R	n_k	ν/R	n_k	ν/R	n_k	ν/R
2_1	7·190	2_2	6·459				
3_1	3·037	3_2	2·866	3_3	2·780		
4_1	1·668	4_2	1·605	4_3	1·564	4_4	1·563

Of the observed lines, which do not cover at all a large range ($\nu/R = 1{\cdot}38$ to $0{\cdot}61$), the only two which it seems are possibly due to N V are:

Int. 1 $\nu = 133871$ $\nu/R = 1{\cdot}220$ Calc. $3_3 - 4_4 = 1{\cdot}217$.

Int. 0 $\nu = 80500$ $\nu/R = 0{\cdot}734$ Calc. $2_1 - 2_2 = 0{\cdot}731$.

Both the observed lines are well separated from those on either side of them and the only doubt about the identification of the latter is that it is recorded as a single line, while a doublet $\Delta\nu = 250$ (approx.) would be expected. The extrapolation of the 2_1 term seems more certain than that of the 2_2 term, so taking the former to be correct, we get $\nu/R = 6{\cdot}456$ for the 2_2 term.

The other line mentioned is important, since if the proposed identification is correct, we get $a = {\cdot}005$ for 3_3. This would appear definitely to eliminate the possibility that the large value of a for this term (·022) in Table IX is correct, but the identification of the nitrogen line in question can probably not be taken as certain, though the occurrence of a similar line for oxygen strengthens the evidence for it.

§ 3·5. *Oxygen.*

Extrapolated term values are given in Table XI; only a few terms can be extrapolated accurately enough to warrant a third place of decimals.

Table XI. *Calculated Term Values for* O VI.

n_k	ν/R	n_k	ν/R	n_k	ν/R	n_k	ν/R
2_1	10·140	2_2	9·261				
3_1	4·31	3_2	4·11	3_3	4·007		
4_1	2·39	4_2	2·31	4_3	2·254	4_4	2·251

The observed lines which might be due to O VI are:

Int. 0 $\nu = 192965$ $\nu/R = 1{\cdot}758$ Calc. $3_3 - 4_4 = 1{\cdot}756$.

Int. 0 $\nu = 96952$ $\nu/R = 0{\cdot}883$ Calc. $2_1 - 2_2 = 0{\cdot}879$.

As in nitrogen, both these lines are well separated from those on either side of them. It is worth noting that the term values 2_1 and 2_2 are about 10, so the third decimal in ν/R depends on an accuracy of 1 in 10,000 in the term value, and so for 2_1 the fourth decimal place in q is important. Assuming the calculated 2_1 term value is correct, the 2_2 term value is 9·257.

§ 3·6. *Collection of Results for the Lithium-like Atoms.*

The only line which has been identified for all elements considered is the first line of the principal series, $2_1 - 2_2$. In Table XII the values of ν/R for this line, and also the ionisation potentials of the different atoms, are given; the regular progression of the value of ν/R suggests very strongly that the identifications are really correct up to N V, but not for O VI. For fluorine, no line is tabulated by Millikan near the estimated position of the corresponding line of F VII. The detailed comparison of the observed term values with the theoretical relations will be given in a separate paper together with the deduction of these relations.

Table XII.

	Li I	Be II	B III	C IV	N V	O VI
2_1-2_2 ν/R	·1358	·2910	·4415	·5882	·7336	·8834
Difference		·1552	·1505	·1467	·1454	·1498
Ionisation potential (volts)	$5\frac{1}{2}$	18	38	64	97	137

The general way in which the intensities of the lines, suggested as due to the lithium-like atoms, vary from element to element is significant. Speaking very roughly, it would be expected that for given conditions of excitation the strongest optical spectrum would correspond to about the same degree of ionisation (more accurately, to the same ionisation potential) for all elements, so that the spectrum of the lithium-like atom should decrease in intensity with respect to the spectra due to lower ionised atoms as the atomic number increases. This is in fact what happens. For boron the first principal doublet of B III is quoted by Millikan as being the strongest line observed; for C IV the corresponding pair is only of intensity 4 compared to lines of strength 15 which are observed in the spectrum of a 30 volt arc and so are presumably due to lower ionised carbon atoms; it is of intensity 0 for N V and for O VI if it appears at all, and for F VII it does not appear.

§ 4. *Sodium-like Atoms.*

In the case of lithium-like atoms only one spectrum was initially known, so there were no data as to how nearly the approximate formulae (2·3) and (2·4) held, and the extrapolation process was at first rather tentative. For sodium-like atoms the case is different, as here four spectra are already known, and so extrapolation can be carried out with greater certainty. The doublet separations of the 3_2 and 4_2 terms are appreciable and extrapolation of these quantities may assist in the identification of lines.

The regular progression of the first principal doublet for the lithium-like atoms suggests that it might be easy to extrapolate lines directly for the sodium-like spectra, but except for the first principal pair and possibly a few other lines, the successive differences are not small enough and do not converge quickly enough for such extrapolation to be of much value (see § 4·3).

For these atoms, the *p* terms ($k = 2$) as well as the *s* terms ($k = 1$) correspond to orbits which penetrate the core; other orbits remain always well outside.

§ 4·1. *Phosphorus.*

For phosphorus quite a number of lines are observed, and those which lie in the neighbourhood of the lines calculated for P V are tabulated in Table XIII.

The estimated doublet separations are $3_2\,\Delta\nu = 770$, $4_2\,\Delta\nu = 280$. There seems to be no doubt about the first principal pair and very little about the first diffuse pair*; these have already been recognised by Millikan, by extrapolation from spectra of previous sodium-like atoms; the use here of $1/q$ and $(C^4/\alpha)^{\frac{1}{4}}$ has enabled the extrapolation to be performed more accurately. The first line given in the table seems fairly certain also. One or other of the two pairs tabulated in the region $\nu/R = 0{\cdot}910$ probably represents the transition $3_3 - 4_2$; for Si IV the corresponding separation is $\Delta\nu = 162$, so for P V the pair $\Delta\nu = 341$ is more likely than the other, though the separation is rather large.

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* There is just a possibility that the lines $\nu/R = 1{\cdot}053$, $1{\cdot}046$ form the first diffuse pair; the difference of intensity is rather large but it is in the right direction and the separation is about correct.

Table XIII.

Observed lines				Calc. lines P V
Intensity	ν	$\Delta\nu$	ν/R	
1	117984		1·075	$4_1-5_2=1{\cdot}077$
0	115541	770	1·053	
2	114771	803	1·046	$3_2-3_3=1{\cdot}043$ (mean)
2	113968		1·038	
3	103676		·945	$4_3-6_4=0{\cdot}938$??? *
1	100211	149	·913	
1	100062	341	·912	$3_3-4_2=0{\cdot}914$?
1	99721		·909	
5	89457	787	·815	$3_1-3_2=0{\cdot}822$ (mean)
3	88670		·808	

* If the observed line did correspond to this transition the line 4_3-5_4 ($\nu/R = {\cdot}631$ calc.) would be expected to be stronger, but no line is observed near the calculated position.

As for lithium-like atoms, the value of some term has to be assumed in order to calculate other terms from observed lines. For sodium-like atoms, the f terms ($k = 4$) are the best for the purpose, as the term excess α is only a few units in the third decimal place. But for P V no lines involving f terms are observed; in particular the $3_3 - 4_4$ line (ν/R calc. $= 1{\cdot}345$) lies outside the range covered by the observations. It is therefore necessary to depend on an assumed value for the 3_3 term, so it is as well to make an estimate of α for this term more carefully than is required in extrapolation for identifying lines. By using the results for S VI the required value of α can be obtained by interpolation instead of extrapolation. The extrapolation to S VI is sufficiently accurate to make it possible to identify the line $3_3 - 4_4$ without much doubt, and then from an assumed value of α for the 4_4 term, that for the 3_3 term can be obtained for S VI and then interpolated for P V.

Neither of the terms concerned in the first line given ($4_1 - 5_2$) appears elsewhere, so one has to be estimated. The extrapolation of the 4_1 term seems to be the less uncertain of the two.

Table XIV gives the term values obtained; for the doublet levels, the value of ν/R for the p_1 term (the smaller of the two) and the doublet separation $\Delta\nu$ are given.

Table XIV. *Revised Term Values for* P V.

n_k	ν/R	n_k	ν/R	$\Delta\nu$	n_k	ν/R	n_k	ν/R
3_1	4·772	3_2	3·957	795	3_3	*2·919*		
4_1	*2·291*	4_2	(2·007)	(341)	4_3	[1·633]	4_4	*1·565*
5_1	[1·345]	5_2	1·216		5_3	[1·029]	5_4	*1·002*

§ 4·2. *Sulphur.*

There are a large number of lines observed for sulphur and the identification of lines of S VI is in some cases rather doubtful. As in some of the other tables, only a selection of the observed lines is given in Table XV. The estimated doublet separations are $3_2\,\Delta\nu = 1230$, $4_2\,\Delta\nu = 700$ if 4_2 for P V is correct; $\Delta\nu = 450$ if not. The P V terms have been used in extrapolating the S VI terms for $k = 1$ and 2, but as already explained, the value of a for the 3_3 term has to be extrapolated without the use of the P V value. There is, however, very little doubt about the identification of the line $3_3 - 4_4$ for S VI and the line $4_3 - 5_4$ is almost as certain.

Table XV.

Observed lines				Calc. lines S VI
Intensity	ν	$\Delta\nu$	ν/R	
0	214943		1·959	$3_3 - 4_4 = 1·954$
0	159243		1·451	$4_1 - 5_2 = 1·444$?
4	153172	877	1·396	$3_3 - 4_2 = 1·404$???
4	152295		1·388	
1	149330		1·361	$4_3 - 6_4 = 1·358$??
1	144502	867	1·318	
2	143635	616	1·310	$4_2 - 5_3 = 1·310$ (mean) ?
4	143019		1·304	
4	142576	1305	1·299	
3	141271		1·287	$3_2 - 3_3 = 1·276$ (mean) ???
4	140639	1155	1·281	
4	139484		1·271	
4	137887		1·256	$4_2 - 6_3 = 1·256$??
1	107105	1279	0·976	$3_1 - 3_2 = 0·979$ (mean)
1	105826		0·964	
1	100373		0·915	$4_3 - 5_4 = 0·916$

There is also little doubt about the first principal pair; both separation and position agree approximately with the calculated values (it might be mentioned that direct extrapolation for the line confirms the identification). The first diffuse pair does not seem to be observed; there are a number of observed lines in the region of the calculated pair, only some of which are given in the table, but all are of intensity 3 or 4, and as the first principal pair is only of intensity 1, it seems unlikely that any of the observed lines belong to the diffuse pair in question, though of course they may be obscuring one or both members of it; two pairs of such lines with about the right separation are tabulated.

The estimation of the doublet separation of the 4_2 term is very doubtful, on account of the uncertainty about the corresponding P V term; two pairs with separation about $\Delta\nu = 870$ are given, which are observed in the region of the calculated lines involving the 4_2 term. Both components of the pair near $\nu/R = 1{\cdot}39$ are of intensity 4, so the identification is very doubtful; it is more probable that the other pair ($\nu/R = 1{\cdot}318, 1{\cdot}310$) is correct, though the doublet separation is rather large.

Revised term values are given in Table XVI; there are no observed lines that can with certainty be correlated with the transition $3_2 - 3_3$, so the terms 3_1 and 3_2 both occur in the first principal pair only, and the value of one of them has to be assumed. The extrapolation seems to be slightly better for 3_1 than for 3_2, so the calculated value for the former is taken. As for P V the p_1 terms and $\Delta\nu$ are given for $k = 2$.

Table XVI. *Revised Term Values for* S VI.

n_k	ν/R	n_k	ν/R	$\Delta\nu$	n_k	ν/R	n_k	ν/R
3_1	*6·456*	3_2	5·480	1279	3_3	4·215		
4_1	*3·164*	4_2	(2·812)	(867)	4_3	2·359	4_4	*2·256*
		5_2	(1·713)		5_3	*1·502*	5_4	*1·444*

§ 4·3. *Collection of Results for Sodium-like Atoms.*

The values of ν/R for the stronger line of the first principal pair and the corresponding line of the first diffuse pair of sodium-like atoms are given in Table XVII, also the doublet separations of the first two p terms; the ionisation potentials are also given. The comparison of actual term values with the theoretical relations will be treated in the paper in which those relations are established.

As for lithium-like atoms, the progression of the first principal pair into the ultra-violet rapidly becomes nearly uniform when the positions of the lines are expressed in wave numbers; the progression of the first diffuse pair is not so uniform, though of course it is quite "smooth."

For the first four sodium-like atoms the ratio of the doublet separations for the 3_2 and 4_2 terms is very nearly 3 to 1; the large divergences from this ratio for P V and S VI cast doubt on the identification of the lines involving the 4_2 term. No simple formula for the doublet separation of the 3_2 term has been found; the empirical formula proposed by Fowler* for the spectra Na I to Si IV gives values for P V and S VI which are too small.

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* A. Fowler, *Proc. Roy. Soc.*, vol. 103, p. 426 (1923).

Table XVII.

	Na I		Mg II		Al III		Si IV		P V		S VI
3_1-3_2 ν/R	0·1547		0·3259		0·4913		0·6537		0·8152		0·9761
Difference		1712		1654		1624		1615		1609	
3_2-3_3 ν/R	0·1112		0·3256		0·5653		0·8080		1·0383		
Difference		2144		2397		2427		2305			
3_2 $\Delta\nu$	17·2		91·6		238		460		795		1279
4_2 $\Delta\nu$	5·5		30·5		80		162		341?		870?
Ionisation potential (volts)	5		15		28		45		64½		87½

As the atomic number increases the spectrum of the sodium-like atom decreases in intensity, roughly in the same way as for the lithium-like atoms. But the correspondence is not exact in detail; for instance, the spectrum Si IV is more prominent than the spectrum C IV, P V than N V, S VI than O VI. This is what would be expected from the different ionisation potentials of the corresponding atoms; for instance the ionisation potential of C IV is 64 volts, that of Si IV 45 volts. In stellar spectra a similar difference is observed; the silicon spectrum corresponding to each degree of ionisation appears lower down the temperature scale than the corresponding carbon spectrum.

§ 5. *Summary.*

It is possible to obtain approximate theoretical relations between the spectra of different atoms ionised to such an extent that they possess the same electron structure. In this paper these relations are used to extrapolate values of the terms of the spectra of various lithium-like and sodium-like atoms.

From the terms so estimated lines can be calculated, and in many cases correlated with lines observed by Millikan and Bowen in the hot spark spectrum of the corresponding element. Revised term values can then be calculated from the lines so identified.

The spectra of lithium-like atoms treated in this way include those from Be II up to O VI, and those of sodium-like atoms, the spectra P V and S VI.

ADDENDUM. (Added in proof.)

Since this paper was written, further observations by Millikan and Bowen on the hot spark spectrum of boron have been published (*Proc. Nat. Acad. Sci.*, vol. 10, p. 199 (1924)). The first principal pair of B III has been remeasured, and its doublet separation is given as $\Delta\nu = 36$; the lines $\nu/R = 1{\cdot}346$, $1{\cdot}201$ have been observed as doublets with about the same separation, confirming the identifications given in this paper. The lines $3_3 - 4_4$ and $4_4 - 5_5$ have been observed, so it is now possible to find all the terms from an assumed value of a for the 5_5 orbit, which must be very small. The values given by Millikan, divided by R to be comparable with Table IV, are given in Table XVIII.

Table XVIII. *Term Values for* B III.

n_k	ν/R	q	n_k	ν/R	n_k	ν/R	n_k	ν/R
2_1	2·7881	·2036	2_2	2·3471	—	—	—	—
3_1	1·1460	·1979	3_2	—	3_3	1·0016	4_4	·5630

We can now extrapolate the value of a for the 3_3 term of C IV to four decimal places with some confidence, and obtain the term values given in Table XIX, confirming the values given in Table VIII as opposed to those given in Table IX.

Table XIX. *Term Values for* C IV.

n_k	ν/R	q	n_k	ν/R	n_k	ν/R
2_1	4·7396	·1627	2_2	4·1512	3_3	*1·7805*
3_1	1·9808	·1579	3_2	1·8355	4_3	*1·0014*

It is clear that if all the hot spark lines concerned are correctly identified, the Wolf-Rayet pair $\nu/R = {\cdot}1570$ cannot be $3_1 - 3_2$ of C IV. But possibly the identification of the line $\nu/R = {\cdot}8340$ as $3_2 - 4_3$ is incorrect (the observed line is perhaps rather strong for this transition); if the identification of the Wolf-Rayet pair is accepted in preference, the term value for the 3_2 term becomes $\nu/R = 1{\cdot}8238$. The present evidence is not enough for a definite decision between these two alternatives to be given. The 3_1 term is almost certain to be correct, as the difference of q between the 2_1 and 3_1 terms given is quite consistent with the values for Li I and B III.

Improvements in the Apparatus for Measuring the Velocity of very rapid Chemical Reactions. By H. HARTRIDGE, M.D., Sc.D., Fellow of King's College, Cambridge and Mr F. J. W. ROUGHTON, Fellow of Trinity College, Cambridge.

[*Read* 19 May 1924.]

INTRODUCTION.

During the last six months we have succeeded in achieving two of the improvements which were foreshadowed in a former paper on this subject*.

These were (*a*) to make the apparatus available for the investigation of more rapid chemical reactions, (*b*) to employ a thermal method for determining the stages reached by the reaction at different instants of time from its commencement.

SECTION I.

With regard to (*a*), we have shown previously that (i) the time taken for the complete mixing of the two reagents and (ii) the time taken by the analysis of the fluid, at different selected stages of the reaction, must both be short in comparison with the time taken by the reaction to proceed to half completion.

No improvement has so far been necessary in the case of (ii): for even in the fastest reactions which have been investigated with the improved methods, the physical analysis of the moving fluid has only involved the examination of a width of the fluid which is small in comparison with the length of the observation tube "occupied by half the reaction." The competency of the method to deal with fast reactions, or as we may perhaps call it "the resolving power of the apparatus," depends at present upon its ability to reduce the time of mixing T to as small a value as possible. For consideration will show that T represents the maximum time interval which can separate the arrival, at a given cross-section of the observation tube, of two parts of the fluid which commenced to react at the same instant of time.

The mixing tests, which we have previously described, only enable us to decide whether the fluid *emerging from the orifice of the mixing chamber* is or is not sufficiently well mixed. If that is the case, then the time of mixing "T" is less than the time taken by the fluid to pass through the mixing chamber, the latter in its turn being equal to the quotient of the volume of the mixing chamber divided by the volume of fluid flowing through it per second. Our aim has therefore been to diminish the volume of the mixing chamber as far as is possible without at the same time diminishing either the total volume of fluid flowing through the apparatus per second (owing to an increase in the total resistance) or the perfection of mixture in the fluid escaping from the orifice of the chamber. In order to satisfy these antagonistic conditions we have (1) reduced the resistance of the other parts of the system to a minimum by use of (*a*) pipes of large bore to convey the fluid to the jets, (*b*) eight jets delivering into the mixing chamber instead of a smaller number, (*c*) short observation tube, (*d*) a conical tube of gradually increasing diameter to receive the fluid from the observation tube (so that the kinetic energy of the rapidly moving fluid is not wasted) and (2) have employed a very small conical mixing chamber fed by sharply tapering rectangular section jets milled out from the solid metal. (By this means a very high velocity is obtained in the mixing chamber, and correspondingly a very rapid rate of mixture.)

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* *Roy. Soc. Proc.* A, 104, p. 376, 1923.

The improvement resulting from these modifications may be readily appreciated by inspection of Table I, in which are contained the values of the time interval "*T*" and the values of the various data from which they are calculated both for the two apparatus which had been constructed at the time of our previous paper and for the four improved apparatus which have been designed since.

Table I.

N.	B.	V.	D.	T.	R.
I	1·28	90	·29	·0032	1/100
II	·63	400	·74	·0019	1/200
IIa	·63	370	·41	·0011	1/300 Obs. I
III	·50	450	·31	·0007	1/450
IV	·50	400	·31	·0008	1/400
V	·50	345	·12	·00035	1/1000 Obs. II
VI	·28	1000	·45	·00045	1/700

N. = No. of apparatus in order of construction.
B. = Bore of observation tube in cms.
V. = Velocity of flow of fluid down observation tube in cms. per second at 50 cms. Hg pressure.
D. = Greatest distance between any two parts of the fluid passing down observation tube which commenced to react at the same instant.
T. = Difference in time of arrival of any two such parts at a given cross-section of the observation tube.
R. = Time of half completion of the fastest chemical reaction which can be dealt with by means of an apparatus.
Obs. I = Observed by experiments with mono-molecular reactions and found reliable for reactions half over in 1/300 sec.
Obs. II = Observed by experiments with bi-molecular reactions and found reliable for reactions half over in 1/700 sec.

It will be seen that the values of *T* have fallen from ·0011 for apparatus No. IIa to ·00035 for apparatus No. V. The latter apparatus is therefore competent to deal with reactions rather more than three times as fast as those which could be dealt with by the former.

In the last column of the table it will be seen that whereas apparatus No. IIa was found by experiment capable of dealing with mono-molecular reactions half completed in 1/300th sec., apparatus No. V should be capable of dealing with reactions half completed in 1/1000th sec. This apparatus was tested experimentally on bi-molecular reactions and was found to give reliable results for reactions half completed in $\frac{1}{700}$ sec.

These two experimental tests are shown in diagrams I and II.

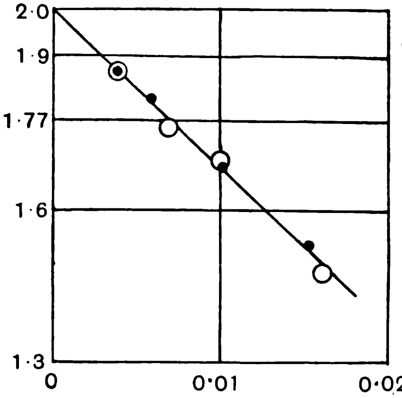

Diagram I

Mono-molecular Reaction.
Vertical = Log of Concentration.
Horizontal = Time in seconds.
Rate of flow through observation tube = for circles 100 cms. per second and for dots 300 cms. per second.

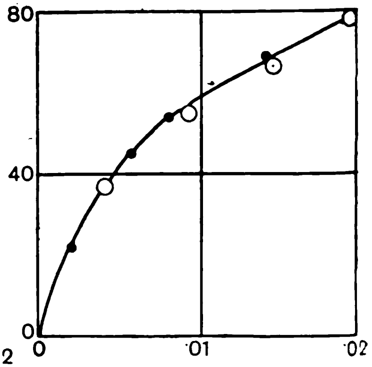

Diagram II

Bi-molecular Reaction.
Vertical = Concentration.
Horizontal = Time in seconds.
Rate of flow through observation tube = for circles 100 cms. per second and for dots 350 cms. per second.

A similar comparison of apparatus No. I with apparatus No. IV shows that the latter has a resolving power roughly four times that of the former, yet both are applicable to the same type of reaction, since the former has an observation tube of about 1·28 cms. bore while the latter has two tubes parallel with one another each of ·5 cm. bore. The beam of light which illuminates the optical apparatus used for the measurement first passes through the one tube and then through the other, the combined effect of the two tubes of apparatus No. IV being practically the same as the single tube of No. I. But each tube of apparatus No. IV has its own mixing chamber, etc., so that substantially the same rate of flow can be obtained as with a single tube apparatus of similar dimensions. It is in this manner that the increased efficiency was obtained which makes apparatus No. IV superior to No. I.

SECTION II.

The Thermal Method.

Description of the Method. Most chemical reactions are accompanied either by an evolution or by an absorption of heat, the amount of which is in a given case proportional to the extent to which the reaction has proceeded. The principle of the thermal method consists simply in measuring (by thermo-electric or other methods) the temperature of the mixed fluid at different cross-sections of the observation tube, and in calculating from these temperature measurements the amount of heat evolved (or absorbed) and thence, in turn, the amount of chemical change which has occurred when such cross-sections are respectively reached.

The assembling of the apparatus. The glass observation tube used for previous optical measurements has in this case been replaced by an ebonite tube. Through the bore of the latter the fluid passes with uniform velocity in the usual way from the mixing chamber to the outlet. At required intervals along this tube radial conical holes are drilled from the outside right through to the bore of the tube. Into these holes are forced conical plugs of soft wood impregnated with paraffin-resin mixture. Passing in a groove round the ends of each of these plugs is a single thermo-electric element composed of an iron and a constantan wire. These two metals are silver soldered together, the junction corresponding in position with the small end of the wood plug. The smooth end of the drill used to bore the ebonite tube is put inside the bore before the soft wood plugs bearing the thermo-elements are forced into position. By this technique the thermo-junction is just brought flush with the bore of the tube without forming any obstruction to the flow of the liquid down it. The "cold" junctions, of which there were a pair, were placed in the observation tube at a point so far removed from the mixing chamber that the reaction must here have reached completion. These could be connected one at a time with any one of the six "hot" junctions and with a galvanometer (giving a deflection of 75 mm. per 1° C. difference).

The elimination of temperature changes due to other causes. There are two factors which usually operate together to produce a change of temperature on the fluid passing down the bore of the observation tube. One is the evolution or absorption of heat which accompanies the chemical reaction under investigation, the other is the evolution of heat due to the friction of the fluid, partly on the walls of the observation tube, and partly owing to turbulent motion within itself, since the rate of flow of the liquid is greatly above the critical velocity for stream line flow given by Osborne Reynolds' formula. In order therefore that the temperature change due to the reaction may be studied it is necessary to devise means of balancing off the heat due to friction. Another factor to be eliminated was the small E.M.F. between each of the "hot" junctions and the "cold" junctions even when both are at the same temperature. Experiment showed that it was easily possible to eliminate both these factors simultaneously by the same procedure. It is only necessary to subtract from one another corresponding pairs of observations. One member of the pair is taken when the two fluids under observation are together flowing through the apparatus, whereas the other member of the pair is taken a moment later at the same rate of flow, and with the same thermo-junction, the two reacting fluids being however both replaced by mixed fluid which had been previously passed through the apparatus and had therefore reached chemical equilibrium. The galvanometer deflection would in the first case be due to (*a*) friction, (*b*) excess E.M.F. of hot junction, (*c*) chemical change, and in the second case to (*a*) friction, (*b*) excess E.M.F. The difference between the two deflections is therefore that due to the temperature change produced by the chemical reaction alone.

Dimensions, rate of flow, etc. The first apparatus (using thermal methods) which we have made has been designed for the study of very fast reactions. With 50 cm. Hg pressure on the fluids, a rate of flow of one thousand cms. per second is obtained. The radial holes, in which the thermo-elements were placed, are 0·15 cm. apart; the time taken by the fluid to pass from one junction to the next is therefore one seven-thousandth part of a second. Altogether there are six such junctions, the first being 0·2 cm. from the bottom of the chamber in which the fluids commence to mix. Thus it is possible to make the first observation of the temperature of the fluid one five-thousandth of a second after the commencement of mixture and to make five subsequent determinations at intervals of one seven-thousandth part of a second. The other constants of the apparatus are shown in Table I (apparatus No. VI).

A Practical Application of the Method. With the aid of this thermal method, a further and very convincing "mixture test" was obtained. For this purpose the two reacting fluids were $\frac{N}{10}$ HCl and $\frac{N}{10}$ NaOH. The neutralisation on mixture is in this case accompanied by a rise of temperature of about 0·68° C. Any delay in reaching the full theoretical rise of temperature would be attributable to the definite time interval required for complete neutralisation and mixture. The first hot thermo-junction to register the full theoretical rise must therefore have indicated the earliest point in the observation tube at which both neutralisation and mixture had been completed.

Diagram III shows the results of such a test, and it will be seen that thermo-junction No. 3 is the first one to register the theoretical rise of temperature. But the distance of this junction from the base of the mixing chamber (which in this apparatus is of the same cross-section as the observation tube) is 0·5 cm. and this distance with a rate of flow of 1000 cms. per second is travelled in one two-thousandth part of a second. *A fortiori* mixture alone must be completed in this short length of time, and this conclusion confirms those already arrived at in our previous paper, and summarised in Table I of the present paper.

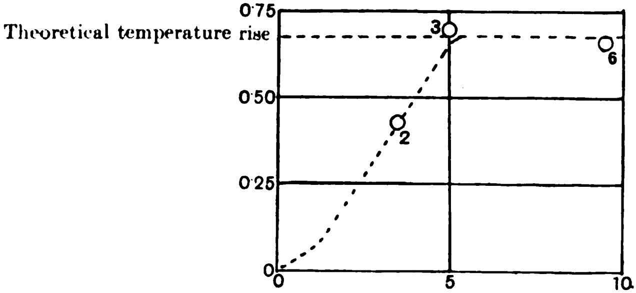

Diagram III.

Vertical = temperature rise in degrees Centigrade.
Horizontal = time in ten thousandths of a second.
Rate of flow through observation tube = 1000 cms. per second.
Numerals attached to observation points denote number of thermo-junction employed.

Our thanks are due once more to the Medical Research Council for defraying the costs of this research. We should also like to thank the Haemoglobin committee and in particular Professors Barcroft and A. V. Hill for the warm encouragement which they have given us.

Apparatus No. V will shortly be available to order from Messrs Baird and Tatlock.

Note on the Doppler principle and Bohr's frequency condition. By P. A. M. Dirac, St John's College. (Communicated by Mr R. H. Fowler).

[*Received* 19 May. *Read* 14 July 1924.]

When an atomic system emits or absorbs radiation, the frequency of the radiation is connected with the change of energy of the system by Bohr's frequency condition $\delta E = h\nu$. This equation is usually applied to the transition of an atom from one stationary state to another, the frequency of the radiation being measured in a frame of reference in which the atom as a whole is at rest, both before and after the transition. It has been pointed out by Bohr* that difficulties arise when no natural frame of reference presents itself in which to measure the frequency, which is the case for non-isolated systems, or even for isolated systems when the process of radiation is connected with a change of momentum.

The question has been considered by Schrödinger † on the basis of Einstein's hypothesis that an emitted quantum of energy $h\nu$ always carries off an amount of momentum $\frac{h\nu}{c}$. Schrödinger assumes that the frequency condition holds for all frames of reference, and deduces the following equation for the frequency in any frame of reference, which he calls a generalisation of the Doppler principle:

$$\nu_0 = \nu \left\{ \frac{c - v_1 \cos\theta_1}{\sqrt{c^2 - v_1^2}} \cdot \frac{c - v_2 \cos\theta_2}{\sqrt{c^2 - v_2^2}} \right\}^{\frac{1}{2}}.$$

Here ν is the frequency in a frame of reference in which the velocities of the emitting atom before and after the process are v_1 and v_2 in directions that make angles θ_1 and θ_2 with that of the radiation, and ν_0 is a constant. If, however, we eliminate v_2 and θ_2 by means of the relations connecting them with v_1 and θ_1 given by Schrödinger, then the following equation is obtained:

$$\nu' = \nu \frac{c - v_1 \cos\theta_1}{\sqrt{c^2 - v_1^2}},$$

where ν' is a constant. This shows that the law of transformation of ν is the same as that given by the ordinary Doppler principle for a source moving with velocity v_1 (and therefore the same as for a source moving with any velocity). Hence, provided the

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[*] *Camb. Phil. Soc. Proc.* (Supplement), "On the Applications of the Quantum Theory to Atomic Structure," pp. 28, 32.

[†] *Phys. Zeits.* 23, p. 301 (1922).

emitted radiation is entirely directed, consistent results will be obtained whichever frame of reference is used for applying the frequency condition.

This conclusion may be proved more directly by showing that the frequency condition can be put in the form of a vector equation in four-dimensional space-time. The momentum of the system multiplied by c and the energy form the space and time components of a vector E^i $(i = 1 \dots 4)$. The change in this vector during the transition process is another vector δE^i whose space and time components are equal, so that its direction in space-time is the same as that of the radiation. We now introduce the "frequency vector" for plane waves defined by

$$\nu_i = \frac{c}{2\pi}\frac{\partial\phi}{\partial x_i},$$

where ϕ is the phase angle of the wave at any point of space-time. This vector has its space and time components both equal to ν, and its space component is in the reverse direction to that of the radiation, so that the associated contravariant vector

$$\nu^i = \frac{c}{2\pi} g^{ik} \frac{\partial\phi}{\partial x_k}$$

lies in the direction of the radiation. Hence if

$$\delta E^4 = h\nu^4,$$

then

$$\delta E^i = h\nu^i$$

which is the required vector equation.

Some Experiments with an α-Particle Counter. By Mr E. V. Appleton, K. G. Emeléus, B.A., and M. A. F. Barnett.

[*Read* 14 July 1924.]

§ 1. *Introduction.* The α-particle counter, the prototype of which was used by Rutherford and Geiger*, is an ionization chamber in which one of the electrodes is a fine point or small sphere, the other being either a plane or the case of the chamber itself. The ionization produced by a swiftly moving electrified particle entering the chamber is magnified by ionization by collision caused by the intense field near the smaller electrode, the electrical disturbance in the circuit being detected by means of a string electrometer. The circuit normally used with the counter is illustrated diagrammatically in Fig. 1, where C and A are the point and case of the ionization chamber, R a large resistance, S and PP the string and plates of the electrometer.

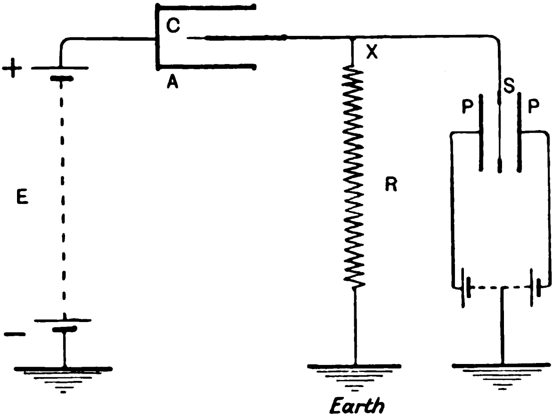

Fig. 1

It will be seen that the electrometer records the transient potential changes between the point X and earth, and, since the electrometer is insulated and of very small capacity, these potential changes indicate the variations of current through the non-inductive resistance R.

Although this method of counting electrified particles has been frequently used in radioactive experiments since its first description by Rutherford and Geiger, we know of no detailed explanation of its action. The experiments to be described were carried out with a view to elucidating the action of the counter, for there were many points about it which seemed to be a little obscure. For example, it is not at first sight clear what factors influence the nature and magnitude of the impulse produced by an α-particle, nor is it clear why the discharge produced by the trigger action of the α-particle should cease, the system being self-restoring. As a first method of attack it was therefore decided to examine the "wave-form" of the transient current impulses produced by the advent of electrified particles.

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* Rutherford and Geiger, *Proc. Roy. Soc.* A, 81, 141 (1908).

A method of delineating transient electrical impulses, involving the use of the cathode-ray oscillograph developed by the Western Electric Company, has recently been designed* for the examination of "atmospheric" wave-forms, and it was seen that the same method would be applicable here. We note that if the intensity, duration, and wave-form of the current impulse are known, the total quantity of electricity Q circulating round the system on the entrance of one α-particle in the chamber may be estimated. It may also be determined, as described below, by a separate experiment. From these determinations, since the voltage E of the system is known, the energy (i.e. QE) released by an α-particle may be estimated, and hence, the energy of an α-particle being known, the "energy trigger factor" of the system may be calculated.

In the first experiments the potential change across the resistance in the circuit was examined by means of a cathode-ray oscillograph. The circuit used is illustrated in Fig. 2, where the deflecting plates of the oscillograph (*mm* and *pp*) are also shown. The resistance tapped off between *a* and *b* was chosen so as to give a deflection

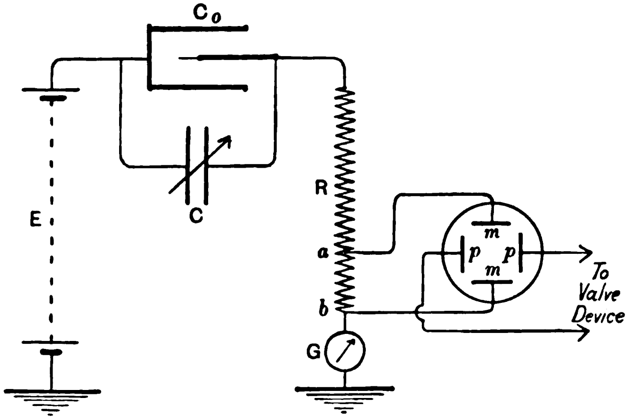

Fig. 2

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[*] Watt and Appleton, *Proc. Roy. Soc.* A, 103, 84 (1923).

corresponding to about 10 volts. In this way it was found that the current in the system momentarily reached a value as high as 5×10^{-5} amperes.

The ionization chamber of the counter was connected to drying tubes and a pumping system so that the pressure could be lowered. A fine capillary tube for allowing small quantities of air to enter could also be put in connection.

The behaviour of the counter as determined by pressure was first studied. As the pressure was lowered, keeping E constant at 700 volts, at first no effect was noticed when α-particles entered the chamber, but at about 20 mms. pressure small kicks of the cathode-ray spot (and of the string electrometer, which was also in connection) were noticed, which gradually increased as the pressure was further lowered.

On still further reducing the pressure spurious kicks occurred, which were found afterwards to consist of short trains of oscillations. Finally at a still lower pressure the system remained in a state of steady oscillation, generally of a super-audible frequency, which was detected by the valve heterodyne method.

The wave-form of these electrical oscillations was examined with the oscillograph as connected in Fig. 2, using a form of linear and unidirectional time base previously used by one of the authors in "atmospheric" wave-form experiments. By this valve-device, which is connected to the plates *pp* (see Fig. 2), the cathode-ray spot is caused to travel from left to right in a uniform manner and spring back almost instantaneously from right to left to begin the uniform movement again. By adjusting the time of the left to right transit of the spot to approximate equality with, say, twice the period of the oscillation to be examined, it was possible (due no doubt to the tendency of the oscillators to quasi-synchronize) to obtain a stationary pattern on the oscillograph screen. The pattern actually obtained is shown in Fig. 3*a* and obviously indicated two complete oscillations of a train such as is illustrated in Fig. 3*b*.

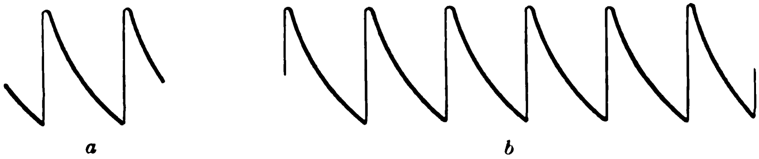

a *b*

Fig. 3

Each oscillation strongly suggests the current wave-form due to the charging of a condenser through a high resistance and small inductance, and our later experiments confirm this view. Our idea of the origin of these oscillations is therefore that the electrostatic capacity of the counter and the circuit resistance play a fundamental part in determining the time-period, and that the oscillation of Fig. 3 illustrates the current wave-form in the system due to the periodic charging of the condenser after the repeated discharge through the ionization chamber.

§ 2. *The relation between α-particle impulses and the oscillations.* Oscillograph determinations of the "wave-form" of the current impulse associated with the entry of one α-particle into the chamber showed that the impulse was almost identical with one unit of the oscillation shown in Fig. 3, and thus we must regard the current impulses produced in the circuit, when counting, as due to the recharging of the electrostatic capacity of the counter after a partial discharge has taken place between the point and the plane. It was therefore clear at the beginning of the experiments that, apart from the intrinsic interest of the oscillations themselves, the study of them would throw light on the factors determining the current impulses obtained with the instrument actually counting. The study of the oscillations therefore preceded the study of single current impulses produced by electrified particles.

§ 3. *Apparatus and Methods.* Two counters were used in the experiments, one of which is shown diagrammatically in Fig. 4. It consisted of a brass cylinder (*A*) having a metallic front with a mica window (*B*) of one millimetre diameter and 1·2 cms. stopping power for α-particles.

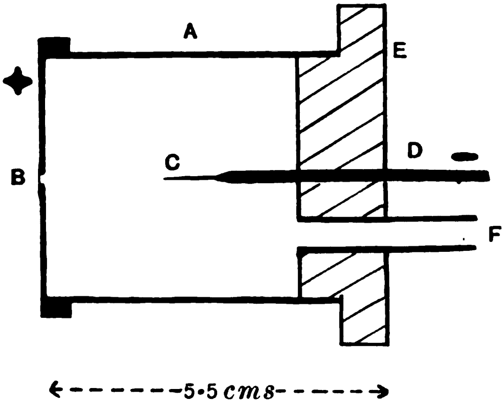


Fig. 4

An ebonite stopper (*E*) carried the inner electrode, a pointed piece of platinum wire (*C*) soldered into a brass holder (*D*). The chamber was connected to a pumping and drying system by a tube (*F*). The second counter was similar, except that the barrel was of glass, so that the discharge could be observed. Both counters gave essentially the same results. Neither would function as a "counter" with the voltages used, unless the point were made negative with respect to the case.

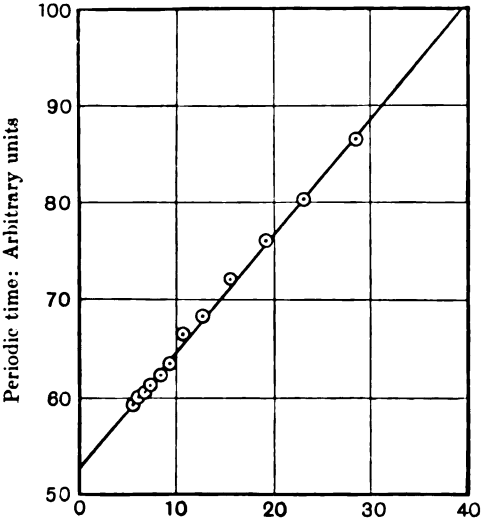
Various methods were used for determining the frequency of the oscillations and its variation with change of circuit constants and chamber pressure. In the first method a coil of small inductance was inserted in the earth lead and coupled to a calibrated oscillatory circuit with crystal-galvanometer detector. Resonance points were in this way sharply determined. In the second method a calibrated local oscillator was used, situated about 2 metres from the counter circuit, and adjustment to resonance with the fundamental, or harmonics, of the oscillations made by the heterodyne method. To test for resonance with the fundamental a Lissajous figure was obtained by connecting the counter circuit and the oscillating circuit to the two pairs of oscillograph plates. At the same time the wave-form of the counter oscillations was seen in terms of the sinusoidal base. This second method, which was more frequently used, was better than the first in that the counter circuit was not complicated by the addition of inductance.

To study the wave-form of the oscillations one pair of plates of the oscillograph was connected across all or part of the earthing resistance (R, Fig. 2), the other pair being in connection with the valve-maintained time-base previously mentioned, or a simple sinusoidal base. Generally only part of R was used, as the oscillograph contained gas and had a resistance between its plates of only about 2 megohms. By this method a record was obtained of the potential between the ends of R, and therefore of the current in the main circuit. The coupling between the time-base and the system under investigation was too weak to disturb the wave-form, but sufficiently strong to maintain synchronization, and a consequent stationary pattern on the fluorescent screen.

The wave-form was, as previously mentioned, such as to suggest that the oscillations were due to the periodic charging of the electrostatic capacity of the counter, following the periodic discharge through the chamber. If we assume for simplicity that the capacity C always discharges between the same limits of potential, and that the time of discharge is negligible compared with the time required for recharging, then it is clear that the period of the oscillation should be a linear function of the time-constant (CR) of the system. To test this point the variation of time-period with variation of C and R was studied, all other conditions being kept the same.

§ 4. *Variation of Frequency with Circuit-constants.* In studying the variation of period with capacity two experiments were made.

In the first the added capacity C (Fig. 2), a variable air condenser, was of the same order as the natural capacity C_0 of the system. In the second C was a mica condenser reading up to 1 microfarad. For the smaller capacities the relation between time-period and capacity is illustrated in Fig. 5: it is seen to be linear, indicating that the assumptions made above are valid for a fixed value of R. The line does not pass through the origin, since C and C_0 are comparable.



Capacity: cms.

$R = 4{\cdot}5\Omega$, $E = 600$ volts, $p = 3$ mms. Hg.

Fig. 5

From the point where the curve cuts the C axis C_0 was determined, and found to be 40 cms. With the large values of capacity the time-period (T) became of the order of seconds, and could be measured with a stop-watch, each discharge giving an audible click. The relation between T and C is shown in Fig. 6.

When C was very large, as in this case (of the order of 0·5 mfd.), the discharge was found to be of a totally different nature, an intense white spark passing in place of the purple glow previously obtained*. Discharge of the condenser was practically complete. The current in the main circuit was observed with a dead-beat galvanometer, and was found to rise very rapidly to a maximum value immediately after the passage of a spark, decreasing exponentially almost to zero before the next spark took place, the current wave-form being similar to that obtained at the higher frequencies with the oscillograph.

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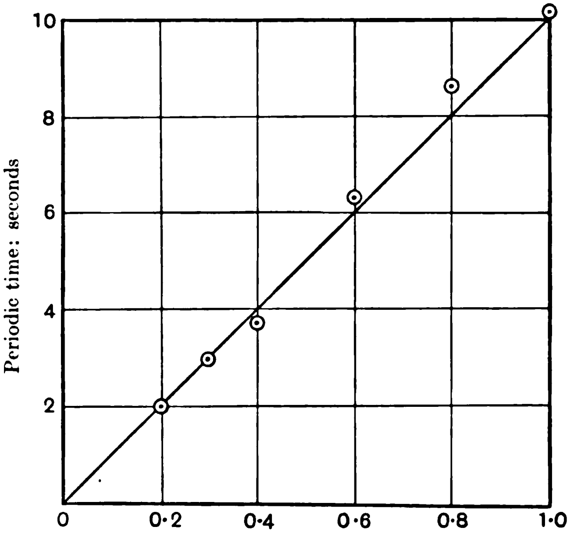
* Cf. Bispham, *Proc. Roy. Soc.* A, 81, 477 (1908); Milner, *Phil. Mag.* 24, 709 (1912).

Keeping the capacity constant, the period of the oscillations should, according to the interpretation above, vary with the resistance *R* (Fig. 2). With *R* sufficiently large, the discharge across the counter will last for a short time relative to that required to recharge the condenser, and the period of oscillation, being determined mainly by the time of recharge, should therefore be proportional to *R*. As *R* is diminished, the discharge occupies a greater relative time, and the period falls off less rapidly than if it were proportional to *R*. With further decrease of *R* the period increases again, since the condenser is now kept charged by the battery to a potential sufficient to maintain a current through the counter for a longer time, and ultimately a critical resistance is reached when the rate at which the condenser is being charged can keep pace with the rate at which it is discharging. With less resistance, the current ceases to be intermittent.



$R = 4{\cdot}5\Omega$, $E = 600$ volts, $p = 5$ mms. Hg.

Fig. 6

For large resistances, the period is found to be a linear function of the resistance; for smaller resistances it passes through a minimum value as shown in Fig. 7, and, extrapolating the curve, the critical resistance should be about 0·2 Ω. The analogy with the well-known flashing of the neon tube is thus fairly complete*.

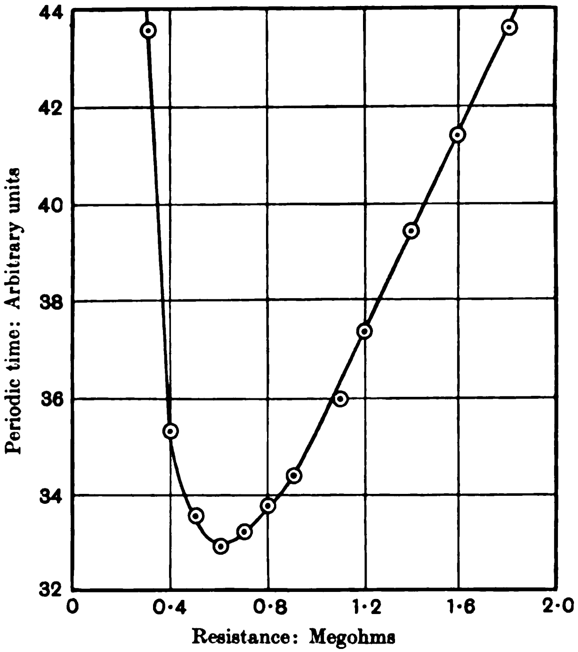


Fig. 7

§ 5. *Effect of Change of Pressure.* The variations in frequency and type of oscillation with pressure were studied by means of the wave-meter and cathode-ray oscillograph (Fig. 2). The relation between pressure and time-period for the metal counter is shown in Fig. 8.

It is seen that the period varies considerably and that there are two minima (β, δ) in the curve. As the critical pressure (p_c) was approached at which oscillations ceased their intensity diminished rapidly. For pressures immediately above p_c the counter gave frequent discharges of short duration without apparent external stimulus, but above about 14 mms. these had to be excited by α- or β-rays. It was noticed that for conditions represented by points just to the right of β, as well as along $\epsilon\delta$, the oscillations were somewhat irregular.

The effect of increase of voltage was to broaden the minimum at β and to displace the whole curve towards the region of greater pressures and shorter periods.

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* Pearson and Anson, *Proc. Phys. Soc.* 34, 204 (1922); Shaxby and Evans, *ibid.* 36, 253 (1924); Oschwald and Tarrant, *ibid.* 36, 262 (1924); Taylor and Clarkson, *ibid.* 36, 269 (1924).

Much the same phenomena were shown by the counter with the glass barrel. The relation between the time-period, the mean current as recorded by the galvanometer G (Fig. 2), and pressure is shown in Fig. 9, where parts presumed to correspond to those of the previous figure are indicated by the same lettering. The portion $\beta\gamma$ has now become flat. The irregularity previously mentioned for conditions represented by a point to the right of β in Fig. 8 seemed to have a parallel in the irregularity at β in Fig. 9, where a peculiar effect was obtained, the period and current proceeding along the wings η with increasing pressure, and then returning to the main curve discontinuously. With the cessation of oscillations at p_c a sudden drop occurred in the current.

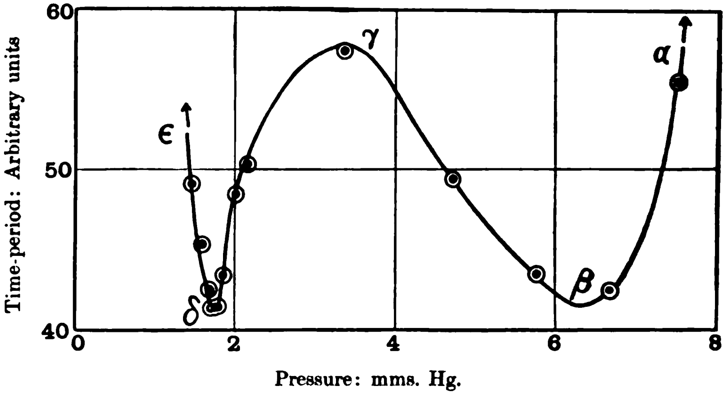

Fig. 8

No oscillations could be detected over the region δ (Fig. 9), and the frequency readings for $\delta\epsilon$ were difficult to obtain, the note being very impure. It is noticeable that at the lower pressures rapid changes in frequency accompany rapid changes in the current.

With this counter the luminosity in the tube could be watched. For pressures less than 0·06 mm. it was dark. Until oscillations of the main type started (at δ) the cathode glow extended up to the anode, no noticeable modification occurring along the range $\epsilon\delta$. When the main oscillations started an abrupt change occurred, the cathode glow shrinking on to the negative electrode, whilst the Faraday dark space appeared together with a trace of positive column. With increase of pressure a normal development* took place in the appearance of the discharge, three striae being present at γ, whilst the positive column was sensibly uniform beyond β.

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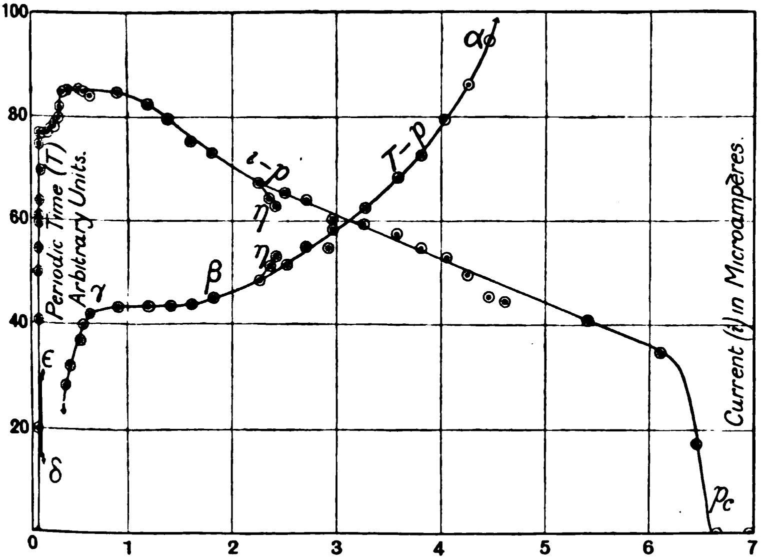
* Cf. Bispham, *loc. cit.*

After oscillations stopped (p_c) there was no permanent luminosity, but the bursts of oscillations at slightly greater pressures had a similar appearance to that seen just before the continued discharge stopped at p_c. About 3 mms. of the platinum needle were covered with a glow, and there was a dark space about 3 mms. long and 3 mms. broad*, and a positive cone of small angle proceeding to the anode. The luminosity became fainter as the pressure was further increased, and ultimately all that could be seen was a faint glow in the neighbourhood of the point, and this required an ionizing particle for excitation.



Pressure (p) mms. Hg.

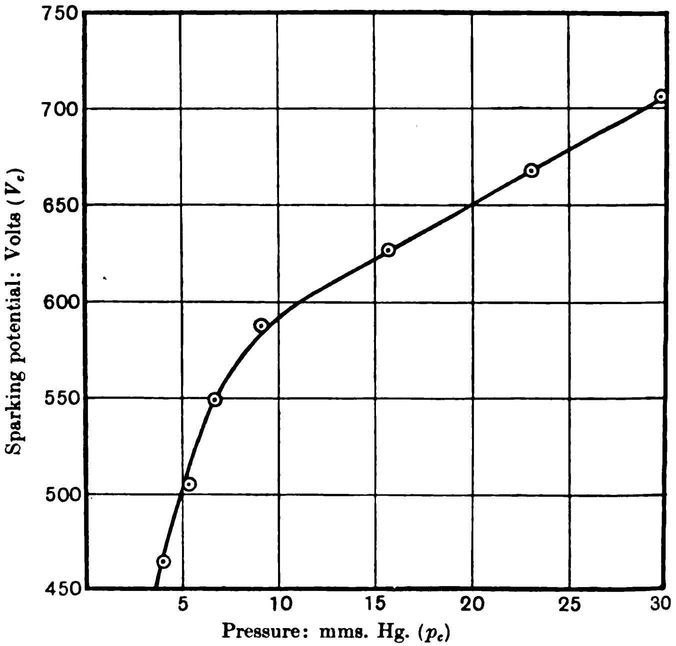
E=800 volts, R=4·5Ω, C=15 cms. T: 1 unit=2·4 × 10^{-6} sec. (approx.).

Fig. 9

Observations were also made on the relation between p_c and the critical voltage (V_c) across the counter at which oscillations begin. The results are shown in the curve of Fig. 10, which is very similar to one representing the relation between sparking potential and pressure for a point and plane discharge deduced from the data given by Villard and Abraham†. Such a figure illustrates the conditions under which counting is possible, for it is only with pressures higher and voltages lower than those represented by the curve that the system is stable.

* Cf. Townsend, *Electricity in Gases*, chap. XI, § 298.

† *C.R.* 150, 1286 (1910).



Metal-counter

Fig. 10

§ 6. *The study of Current Impulses produced by α-particles.* It has been mentioned previously that the oscillographic evidence suggested that the current impulses associated with single and isolated discharges initiated by α-particles were of the same "wave-form" as the units of the intermittent discharge constituting the permanent oscillation. Thus, if we assume, for simplicity, that the electrostatic capacity of the counter C_0 discharges very rapidly from a potential V_1 (equal in this case to that of the battery E) to a potential V_2 through the counter a quantity $C_0(V_1 - V_2)$ must be refurnished by the battery when the capacity is recharged, and it is this quantity of electricity, passing through the resistance R, which produces the string electrometer kick normally observed in "counting." According to this way of regarding matters it is seen that the quantity of electricity circulating round the system should increase with increase of electrostatic capacity of the counter. Experiments were therefore carried out using a capillary electrometer of fine bore, of the type devised by Mr C. T. R. Wilson[*], as an integrating coulombmeter to measure the small quantity circulating round the system at each discharge. It was not found

[*] *Proc. Roy. Soc.* A, 92, 555 (1916).

possible to measure each kick, but the integrated quantity of a number of kicks (recorded individually by the string electrometer, which was also in connection) could be measured. With

$$E = 760 \text{ volts},\ R = 10^3\Omega \text{ and } p = 15{\cdot}6 \text{ mm.}$$

it was found that, in the absence of added capacity in parallel with the counter, the quantity circulating for each electrometer kick was 10^{-10} coulombs, but this was increased very markedly by adding capacity in parallel with the counter. The increase of quantity per impulse was approximately proportional to the increase of capacity.

From the earlier experiments it was realized that the resistance R plays an essential part in the action of the counter other than that of supplying a potential difference for detection by the string electrometer. The variation of Q with R was therefore examined, and it was found that, when R was decreased, the value of Q rose so that with $R = 4{\cdot}5\Omega$ instead of $10^3\Omega$, Q was of the order of 10^{-6} coulombs and had to be measured with a less sensitive capillary electrometer.

It will be seen that, according to the simple view of the phenomenon outlined above, the quantity passing through the system for each impulse should not depend on R. This is probably true if the time of discharge is short, in which case very little current from the battery goes through the counter during a discharge. We may however treat the problem in general terms, and find an expression for the total quantity passing round the circuit during an impulse.

To simplify matters consider the duration of the impulse as being the sum of the time t_1 for the condenser C to discharge from the voltage E of the battery to the voltage V_c at which the discharge stops, and the time t_2 taken by the condenser to charge up again to the normal voltage E. Let Q_1 and Q_2 be the quantities given out by the battery in these two intervals. We see immediately that

$$Q_2 = C(E - V_c) \qquad \text{......(1).}$$

In calculating Q_1, if we take the voltage across the counter at any instant to be v, then the instantaneous current from the battery is given by

$$i = \frac{E - v}{R} \qquad \text{......(2).}$$

Also

$$i = C\frac{dv}{at} + i_1 \qquad \text{......(3),}$$

where i_1 is the discharging current through the counter.

Let us assume that, as in the case of the neon tube*,

$$i_1 = K(v - v_A) \qquad \text{......(4),}$$

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* Cf. Pearson and Anson, *loc. cit.*

where K and v_A are constants characteristic of the counter; then on substituting for i_1 in (3), we have

$$i = C\frac{dv}{dt} + K(v - v_A),$$

from which $Q_1 = \int i\,dt = C(v_c - E) - \int K v_A\,dt + \int K v\,dt$ ......(5).

But from (2) $$Q_1 = \frac{Et_1}{R} - \frac{1}{R}\int v\,dt \qquad \text{......(6).}$$

From (5) and (6) it follows that

$$Q_1 = \frac{(E - v_A)Kt_1 - C(E - v_c)}{1 + KR} \qquad \text{......(7).}$$

Therefore

$$\begin{aligned} Q &= Q_1 + Q_2 \\ &= \frac{C(E - Kv_c)R}{1 + KR} + \frac{(E - v_A)Kt_1}{1 + KR} \\ &= \frac{C(E - v_c)KR}{1 + KR} + \frac{(E - v_A)KCR}{(1 + KR)^2}\log\frac{E - \dfrac{E + KRv_A}{1 + KR}}{v_c - \dfrac{E + KRv_A}{1 + KR}} \qquad \text{...(8),} \end{aligned}$$

adopting the formula for t_1 calculated by Taylor and Clarkson*.

It is not possible to get a quantitative agreement between the observed and calculated values of Q, as the expression deduced involves so many quantities which are not known precisely for our experiments. But one or two points of qualitative agreement may be noted. The proportionality of Q and C for large values of R has been previously mentioned. The variation of Q with C when R is small is not so simple, as it was found experimentally that, although the quantity was at first reduced by decrease of capacity, a point was reached beyond which the quantity increased again with further diminution of capacity. This is possibly due to the fact that for small values of C the value of K is not really independent of C. It can also be shown from the formula that in cases where KR is always much greater than unity, Q does increase with decrease of R as described above.

From the experimental data mentioned above we may estimate the "energy trigger ratio" of the system, which is the ratio of the energy released in the system (i.e. QE) to that of the particle initiating the discharge. Assuming an α-particle to possess energy of 10^{-5} ergs the energy released was found to vary with resistance, pressure, and capacity from 1 erg to 10^4 ergs, indicating an "energy trigger ratio" of from 10^5 to 10^9.

We have now to consider in greater detail the action of the counter for normal working, and in particular with respect to its self-restoring property. It seems clear, *prima facie*, that hysteresis of the discharge, either statical or dynamical, plays some part in the phenomenon. Let us assume therefore that the relation between the current (i) through, and the potential (v) across the counter is represented by some such graph as is shown in Fig. 11 where *OA* represents the critical starting (sparking) potential and *OB* the minimum potential at which a discharge, once started, can be maintained.

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* Taylor and Clarkson, *loc. cit.*

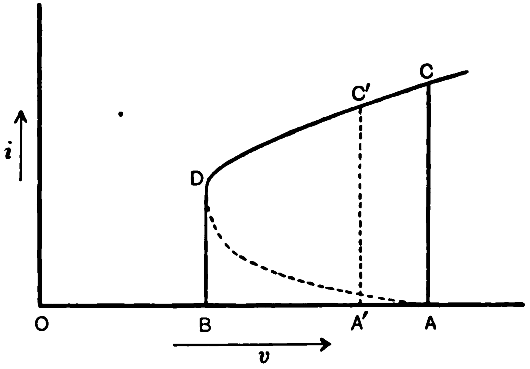

Fig. 11

Since *DC* and *BA* indicate stable values we may expect that there is a set of intermediate unstable values, such as is indicated by *DA*, between *DC* and *BA*. The values of current indicated by *DA* are the ionization currents which must be momentarily produced at the corresponding voltages in order that the discharge may be so increased by further ionization by collision that the current ultimately rises to the appropriate value on *DC*. We may conveniently term the values indicated by *DA* "threshold values," and it is probable, as assumed in drawing Fig. 11, that the threshold is smaller the nearer the applied voltage approaches the critical value represented by *OA*. Moreover, according to this point of view the magnitude of the current impulse should not be appreciably dependent on the initial value of the ionization current so long as the latter is greater than the threshold value.

For values of applied potential appreciably less than that represented by *OA*, we might expect to find a difference between the counting of α-particles and β-particles since their initial ionization effects are so dissimilar. This is found to be the case. Normally with potentials less than that required for complete "counting" the number of β-particles missed in counting is very much greater than the number of α-particles missed, the probable explanation being that many β-particles produce insufficient initial ionization to reach the threshold value of current. Nevertheless, if this is attained, the consequent discharge cannot be distinguished from that excited by an α-particle.

The function of the resistance of the circuit in producing stability may now be discussed. We have to ask ourselves the question: Why does the discharge not continue, when once started by an α-particle? The answer is most easily found by considering the conditions under which a discharge can be maintained. Although the potential difference across the counter is equal to the potential of the battery E before the discharge takes place it falls to $E - R$ when a current i flows through the resistance. If R is sufficiently small to make this resultant potential equal to the lowest value of potential sufficient to maintain ionization by collision, then a discharge will just be maintained. Thus in Fig. 11 if DB is represented by i_1 and OB by v_1 the critical value of R is given by

$$E - Ri_1 = v_1 \qquad \ldots\ldots(9).$$

If R is greater than this value no corresponding values of v and i can be found to fit the equation, except on the v axis, so that a system with voltage represented by OA' when "trigged" by an α-particle will follow the cycle represented by $A'C'DBA'$, and finally remain quiescent at A'.

It may be argued that there would be some value of the current, represented by a point on the line DB, for which a steady discharge could be maintained according to (9), but it may be remarked that, due to the residual inductance in the circuit, the condenser will always tend to over-discharge, so that the potential difference will be momentarily below the critical value represented by OB. The magnitude of the inductance necessary to produce this effect does not matter, so long as it is greater than a certain small value.

If the battery in the circuit has a voltage greater than that represented by OA, the system will not "count," but, if the resistance in the circuit is greater than the critical value necessary to prevent a steady discharge, a periodic charging and discharging of the condenser will take place, as previously described, and in a manner analogous to that obtained with a flashing neon lamp. It should, however, be emphasized that with such oscillations, as with the system "counting," a small amount of inductance is necessary to cause the capacity to over-discharge in each cycle, so that the potential is momentarily lowered below the value represented by OB.

We may further note that the above diagram suggests the possibility of counting without trigger action, but still involving the magnification of ionization currents by collision*. For example, if the voltage is maintained between the values represented by OB and OA (Fig. 11) the system will be stable so long as the momentary ionization currents initiated by ionizing particles do not reach the threshold. In such cases the magnitude of the effect would depend very largely on the initial ionization produced by the particle.

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* Probably this was the case in the original experiments of Rutherford and Geiger (*loc. cit.*), of Hess and Lawson (*Wien. Ber.* 127, 3 and 4 (1918)), and of Greinacher (*Zeit. f. Phys.* 23, 6, 361 (1924)).

In studying the variation of the quantity circulating round the system for each impulse, it was found that while the quantity increased with decrease of the resistance R, the deflection of the string electrometer diminished very markedly. This can be explained on the supposition that the string electrometer was acting ballistically, in which case its maximum deflection would be proportional to RQ, and even if Q increases with decrease of R the product RQ might well decrease. A consideration of the relation (8) for practical values suggests that the variation of Q with R is approximately of the type $Q = A + \frac{B}{R}$, where A and B are constants, so that the above explanation seems valid. But even if the action were not purely ballistic, the same result would follow on taking into account the inertia of the electrometer fibre, as has been shown by Hess and Lawson[*].

§ 7. *Further consideration of the oscillations.* Various additional points of interest concerning the experimental results may now be discussed. In the first place it is clear, from a survey of the literature dealing with glow discharges, that the type of sustained oscillations we have described has been previously noticed. For example, Zeleny[†] and Bispham[‡], using point and plane electrodes in a discharge tube at low pressure, noticed that the discharge was discontinuous. Since the frequency of the oscillation is largely controlled by the value of the circuit constants, this type of glow discharge oscillation must be distinguished from that previously described by Appleton and West[§], in which the frequency control was purely ionic. It would be an interesting problem to study the conditions (pressure, disposition of electrodes, etc.) under which these two types of oscillations are obtained. The type described in this paper is clearly analogous to the flashing of a neon tube, for the time of discharge was found by means of the oscillograph to last only a very short fraction of the complete time of oscillation[||]. When examined with a moving mirror this point was further verified.

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[*] *Wien. Ber.* 125, 285 (1916).

[†] Zeleny, *Phys. Rev.* 25, 305 (1907).

[‡] Bispham, *loc. cit.*; cf. also Valle, *N. Cimento*, 18, 155 and 205 (1919).

[§] *Phil. Mag.* 45, 879 (1923).

[||] At the lowest pressures worked with it was however noticed that the fraction of a time-period during which the counter conducted was greater than at higher pressures.

The change of frequency with pressure for constant applied voltage is complicated, and from Figs. 8 and 9 obviously depends on the disposition of the electrodes. We shall make no attempt to explain all the details, but one or two points may be mentioned. In the first place, since the time of oscillation is determined largely by the time of recharge of the condenser, and since the critical voltage at which the discharge starts falls at first with diminishing pressure, we should expect the time of charging to decrease at first with pressure, for a constant applied voltage E. On these grounds alone we should further expect there to be a minimum in the time-period pressure curve corresponding to the Paschen minimum for sparking potentials. But there are other factors which may be appreciable. The time of recharging up to the critical voltage after discharge will obviously depend on the stopping voltage as well as the starting voltage. We have, however, no information as to how the latter quantity varies with pressure*. Still, it is quite conceivable that it passes through a minimum, but at a different pressure from Paschen's, in which case both minima of Fig. 8 might be accounted for. Very little weight can, however, be attached to these speculations in absence of precise measurements.

The variation of mean current with pressure obviously depends on the same factors as the frequency variation (see Fig. 9). As the pressure is reduced to the critical starting value for the discharge the mean current suddenly rises. It can readily be shown that, if we again assume that the discharge across the counter occupies only a small fraction of the time-period, the mean current $\bar{i}$, as recorded by G, Fig. 2, is given by

$$\bar{i} = \frac{C\,(v_b - v_c)}{t_1} \qquad \ldots\ldots(10),$$

where

$$t_1 = CR \log \frac{E - v_b}{E - v_c},$$

and v_b is the starting voltage, v_c the stopping voltage. This formula shows that the mean current is independent of the capacity, as was verified experimentally over a large range of condenser values. The fact that $\bar{i}$ decreases with increase of pressure is probably accounted for mainly by the increase of time-period, as suggested by (10), the variation of $v_b - v_c$ not being large, but the matter is still further complicated by the fact that at the lower pressures the time of conduction of the counter is relatively greater.

It must be emphasized that the characteristic of a discharge tube is of very little significance, when it is determined by replacing the condenser of Fig. 2 by an electrostatic voltmeter, and measuring the corresponding current by the galvanometer G, if oscillations of the type which we have been investigating are taking place. G records the average current through the discharge tube, which is almost the same as the average charging current for the condenser, but, as the discharge actually consists of a series of pulses separated by relatively long intervals, the conductivity of the gas, during the short time when it is not acting as an insulator, is considerably greater than the value which would be deduced from the mean current.

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* Cf. however, Stead and Stoner, *Proc. Camb. Phil. Soc.* 21, 66 (1922), for the case of the low-voltage arc in mercury vapour.

It may be noted that the neon discharge tube, which has been previously mentioned as a source of oscillations very similar to those described in the present paper, can also act as a counter*. If it is set up in parallel with a capacity and in series with a resistance and battery (cf. Fig. 2), the applied voltage being just too small to produce oscillations, it will flash intermittently when a radioactive preparation is brought near. It is highly probable that other systems showing characteristic hysteresis could be made to act similarly, and further experiments on these lines are in progress.

§ 8. *The use of the counter.* Several important points arise from a consideration of the experiments described, in connection with the use of the counter for accurate enumeration of particles. If our interpretation of its action is correct there is a fundamental difference between counters like the original one of Rutherford and Geiger†, where the threshold value of current (cf. Fig. 11) is rarely if ever reached, and those where the threshold is passed and there occurs a much larger discharge of the condenser to a definite lower voltage. In the first type, since the change in potential across the capacity for each discharge is very small, the rate at which particles can be recorded will be determined, in general, by the inertia of the fibre of the electrometer. In the second type, where each particle causes a very much larger change in potential than in the former, the determining factor will be the time constant (CR) of the circuit; if two particles enter the ionization chamber in rapid succession, the second cannot initiate a discharge unless the battery has had time to recharge the condenser to a sufficiently high voltage. It is possible, however, to allow for particles missed on this account, by assuming that the number entering the chamber in a given time is governed by the laws of probability, as has been done in a slightly different connection by Lawson and Hess‡.

For this same reason, that the rate at which particles can be recorded is diminished, it would not be advisable to increase the magnitude of the deflections of the electrometer beyond a certain amount by increase of the capacity (C) or of the charging resistance (R). Moreover, it was found that the number of spurious discharges increased markedly with increase of capacity, and with the large mica condenser completely masked any effect resulting on deliberately introducing ionizing particles. A possible explanation of this is that the agents responsible for the spurious discharges, partly owing to their nature, partly to the place in the counter where they act, produce currents of about the threshold value. With a small capacity, the fall of potential consequent on the small initial current is sufficient to prevent the threshold ever being attained (cf. Fig. 11), but with a large capacity the same initial current now produces a correspondingly smaller change in potential; the current, further increased by ionization by collision, reaches the threshold, increases suddenly to a value on the line CD, and the condenser automatically discharges to the stopping voltage characteristic of the counter.

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* Baeyer and Kutzner, *Zeit. f. Phys.* 21, 1, 46 (1924).

† Rutherford and Geiger, *loc. cit.*

‡ Lawson and Hess, *Wien. Ber.* 127, 461 (1918); Hess and Lawson, *Phil. Mag.* 48, 200 (1924).

Another source of error may arise from the fact mentioned in § 5, that an α-particle can give rise under certain circumstances to a short train of oscillations, instead of to a single discharge. With the string electrometer the record obtained would be precisely the same as if a number of particles had entered, equal to the number of units of the oscillation actually occurring, and at a rate equal to the natural frequency of the circuit. The effect is the more marked the nearer the conditions approach to those where permanent oscillations set in, that is, with the usual disposition of apparatus, the higher the voltage. From an analysis of the registration by a counter of α-particles from polonium, Kutzner* came to the conclusion that there were fewer time-intervals in which the number of particles entering was considerably less than the average, than would be expected if the distribution were random; this may be a consequence of our observation†. His result, that there were, in addition, too few intervals in which there was a number of particles large compared with the mean, may be partly due to the impossibility of recording at more than a certain rate, determined by the time-constant of the circuit.

SUMMARY.

Experiments carried out with the aim of elucidating the action of the α-particle counter are described. A normal counting circuit including an ionization chamber with point and plane electrodes was used. The "wave-form" and intensity of the transient impulses

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* Kutzner, *Zeit. f. Phys.* 21, 5, 281 (1924).

† Cf. also Lawson, *Nature*, 114, 121 (1924).

resulting from discharges stimulated by single electrified particles have been determined using a cathode-ray oscillograph. It is found that the nature of the impulse is determined largely by the capacity and resistance of the system*, and not by the nature of the electrical stimulus. The potential impulse normally detected by string electrometer methods is shown to be proportional to the current responsible for the recharging of the electrostatic capacity of the counter after the rapid discharge of the capacity initiated by the electrified particle.

An explanation is given of the function of the high circuit resistance in causing the system to be self-restoring after a discharge.

When the pressure in the ionization chamber is made less than that at which self-restoring action is possible, persistent oscillations are found. The variation of the frequency of these oscillations with change of chamber pressure, and circuit capacity and resistance, is described. An oscillographic examination of the wave-form of the oscillations shows that they represent the periodic charging of the counter capacity after its periodic discharge between the point and the plane. The impulses normally produced by α-particles are thus represented by one unit of the sustained oscillation.

The total quantity of electricity circulating round the system when an α-particle enters the chamber has been measured using a sensitive form of capillary electrometer as an integrating coulombmeter. This is found in normal cases to be of the order of 10^{-10} coulombs, but in low resistance circuits may be made as high as 10^{-6} coulombs. According to the interpretation of the action of the counter given above the quantity impulse should normally increase when an external capacity is connected in parallel across the counter. This is found to be the case.

The "energy trigger ratio" of the counter for α-particles, which represents the ratio of the energy released in the system to that of the electrified particle acting as stimulus, is normally of the order of 10^5, but in certain circumstances may be made as high as 10^9.

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* Contrast this with Kutzner's statement (*Zeit. f. Phys.* 23, 128 (1924)) that additional capacity and resistance are not essential to the action of the point and plane counter.

A Helical Method of focussing β-Rays. By R. A. R. TRICKER, B.A., Trinity College. (Communicated by Professor Sir E. RUTHERFORD.)

[*Received* 2 August 1924.]

[With Plate III.]

Up to the present two methods of obtaining magnetic spectra of β-rays have been used; namely that employing a comparatively small deviation, used more particularly by Meitner, and that used by Rutherford and Robinson, C. D. Ellis, and others, with rays which have traversed a semi-circular trajectory. In the case of the latter, rays of a given velocity, from a line source, are brought to a fairly sharp focus, and darken a suitably placed photographic plate. While producing excellent spectra, both methods are subject to certain disadvantages, for example:

(1) It is difficult to reproduce accurately magnetic fields of the required area and intensity.

(2) The intensity is not sufficiently great for short period sources.

It was hoped to eliminate both these difficulties by means of a helical method of focussing, using an air coil, suggested by Dr Kapitza.

This method has now been tried, and although, owing to a difficulty of seeing faint lines, which will be referred to later, it does not appear to be so generally useful, or so simple for obtaining ordinary β-ray spectra, it possesses certain advantages which may render it useful for other purposes. Briefly these advantages are as follows:

(1) Large dispersion.

(2) Uniformity and ease of reproduction of the magnetic field.

(3) Good screening from γ-rays and secondary β-rays.

(4) Axial symmetry,—experiments are now in progress, with the β-particles subjected to an accelerating electric field, in order to test the relativity expression for $(H\rho)$.

(5) Greater intensity.

Kapitza's method is as follows. A short line source of β-rays is placed in a uniform magnetic field, parallel to the lines of force, and rays, projected at a given angle α to the field, are selected by means of an annular slit, whose axis contains the source. These rays are spread out into a magnetic spectrum, which focusses on a film lying along the axis of the slit, at a suitable distance from the source. The simple theory is as follows.

Consider a β-particle projected with velocity V (u, v, w) from O, in a magnetic field H, taken parallel to the axis of X.

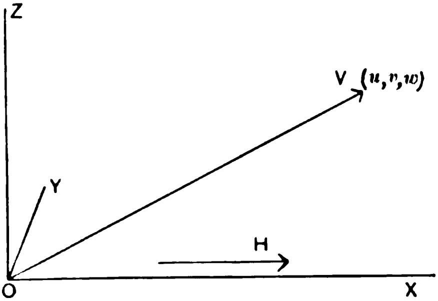


Fig. 1

The equations of motion are:

$$\left.\begin{aligned} m.\ddot{x} &= 0 \\ m.\ddot{y} &= eH\dot{z} \\ m.\ddot{z} &= -\,eH\dot{y} \end{aligned}\right\} \qquad \ldots\ldots(1).$$

Writing $p = \dfrac{eH}{m}$ and assuming that the particle is projected in the plane $y = 0$, equations (1) give

$$\left.\begin{aligned} y &= \frac{v}{p}.\sin pt \\ z &= \frac{v}{p}.(\cos pt - 1) \\ x &= ut \end{aligned}\right\} \qquad \ldots\ldots(2).$$

When $t = \dfrac{2\pi}{p}$ the particle recrosses the axis of x, $(y = z = 0)$. Thus the range R, along this axis, is

$$R = \frac{2\pi u}{p} \qquad \ldots\ldots(3).$$

Focussing. Let O, Fig. 2, be the source and S half the section of the annular slit, the axis of which is parallel to the axis of X, and the magnetic field, and contains the source O. Let x' be the distance from O to the slit. A displacement dx' in the position of the source, keeping S fixed, will alter the angle of projection of the particles, and consequently their range. Let dR be the corresponding displacement in the trace on the film, and du the change in the x component of the velocity.

Then $$dR = \frac{2\pi}{p}\,du - dx'.$$

We may write $$x' = f(u),$$

and therefore

$$dR = \frac{2\pi}{p}\,du - du\,.f'(u) - du^2.\frac{f''(u)}{2} - \dots \qquad \dots\dots(4)$$

by Taylor's Theorem.

The connection between u and x' is obtained from the constant radius of the slit, by means of equations (2). It is

$$r^2 = \frac{2.(V^2 - u^2)}{p^2}.\left(1 - \cos\frac{px'}{u}\right) \qquad \dots\dots(5).$$

where r is the radius of the slit.

We see from (4) that the condition of focus, using a long source and narrow slit, is that

$$\frac{2\pi du}{p} - f'(u).du = 0 = dR \qquad \dots\dots(6).$$

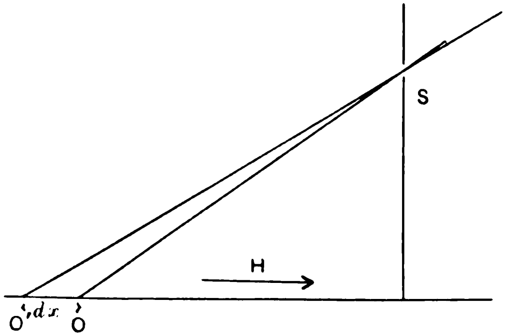


Fig. 2

Calculating the value of $f'(u) = \frac{dx'}{du}$ from (5), and inserting it in (6), the condition of focus becomes

$$\tan\frac{px'}{2u} = \frac{(V^2 - u^2)}{u^2}.\left(\pi - \frac{px'}{2u}\right) \qquad \dots\dots(7).$$

Equation (7) involves two ratios, $\frac{x'}{R} = \frac{px'}{2\pi u}$ and $\frac{u}{v}$, one of which may be fixed arbitrarily and the other obtained by means of the above relation. This was the procedure adopted in the design of the instrument. It is possible to go further however, and enquire the value of $f''(u)$, of equation (4). It is impossible to make this zero,—the best that can be done is to choose a minimum value. This optimum position was departed from slightly in the actual instrument in order that a somewhat smaller coil might be used. A theoretical increase in the thickness of β-ray lines of ¼ mm. was introduced thereby.

From equation (5) we find the value of $f''(u)$ to be

$$f''(u) = \frac{[3ur\,(V^2 - u^2) + 2u^3r]\cos^2\frac{px'}{2u} + u^3r}{(V^2 - u^2)^{\frac{3}{2}}\cos^3\frac{px'}{2u}} \quad \ldots\ldots(8).$$

Using the first focussing equation (7) to eliminate the ratio $\frac{V^2 - u^2}{u^2}$, and writing $a = \pi - \frac{px'}{2u}$, and adding the term $du^2 = \frac{p^2 dx'^2}{4a^2}$ we obtain

$$f''(u).du^2 = -\frac{\pi}{R}.\frac{1}{\sin a.\cos a}\left[1 + 2\cos^2 a - \frac{3\sin a.\cos a}{a}\right]dx'^2 \quad \ldots\ldots(9).$$

As stated before, there is no value of a which corresponds to a position of the slit between the source and the film, which makes the above expression zero. It possesses a minimum value at $a = 2{\cdot}13$ radians which fixes the optimum position of the slit. The angle of projection of the rays, and hence the diameter of the slit, is then determined from the first focussing relation (7).

Theoretical Breadth of Line. The object of deducing equation (9) was to determine the degree of focussing to be expected theoretically, which it limits. Inserting therein the value of a used in the apparatus, and the length of the sources used (5 mm.) for dx', we obtain by means of (4) the breadth of line to be expected as ·6 mm.

There is also to be considered in this connection variation in a, the angle of projection of the rays, due to the following causes.

(*a*) *Width of slit.* Considering a slit formed by a disc and a circular aperture of the same radius lying in parallel planes at a distance s apart, a rough calculation gives for dR, the breadth of line due to this cause,

$$dR = \frac{Rs}{r}.\frac{\sin^3 a}{\cos a}$$

$$= s \text{ very approximately with the dimensions used.}$$

(*b*) *Finite diameter of source.* If r' is the radius of the source, a similar calculation to the preceding gives

$$dR = \frac{r'R}{r}.\sin^2 a$$

$$= 2r' \text{ with the instrument used.}$$

The above represents the irreducible minimum breadth of the lines, and may be summarized thus:

(*a*) Length of source	... ...	·6 mm.
(*b*) Width of slit	... ...	·25 ,,
(*c*) Diameter of source	... ...	·25 ,,
		1·1 ,,

Errors of adjustment have also to be considered. Of these the two following are to be expected:

1. *Lack of straightness in source and bad centering.* These produce a broadening similar to that considered in the case of the diameter of the source. In this case, however, the effect will be $4r'$ (where r' is now the displacement from the axis due to the above cause), since in the case of the source, half the slit is shielded from any given radioactive atom, by means of the wire supporting it. Hence it is necessary to make the adjustments to about 1/10 mm.

2. *Variation of the current in the solenoid.* Since the range R is inversely proportional to the field H, it varies inversely as the current i.

Therefore
$$\frac{dR}{R} = -\frac{dH}{H} = -\frac{di}{i}.$$

Putting $R = 27$ cms.
$$dR = -270.\frac{di}{i}\text{ mm.}$$

Hence the current must be kept constant to at least 1 in 1000 and better 1 in 10,000.

Apparatus. The apparatus (Fig. 3) consists of an airtight, cylindrical brass box, which can be evacuated by means of a Gaede pump and charcoal tube to a pressure of a thousandth of a millimetre or less. It contains an annular slit, on the axis of which are the radioactive source and a strip of double-sided X-ray film. The source and film are attached to a central rod which can be lifted out of the apparatus. The whole is placed inside a solenoid producing a maximum field of about 500 gauss, for a current of some 5 amperes. The rays come to a focus on the film at a distance of between 24 and 30 cms. from the source, which consisted of the radioactive deposit from radium emanation, on a piece of platinum wire ¼ mm. in diameter. Blocks of lead suitably placed served to screen the film from the action of γ-rays.

In the experiments now to be described attention was confined to the group of lines between $(H\rho) = 1677$ and 1938.

In using the instrument the following adjustments and precautions had to be made.

1. *Axial position of source and film.* The adjustment of the source and film was made by rotating the central rod to which they are attached inside a tube, which fitted it, meanwhile observing with a microscope.

2. *Levelling.* It is necessary to ensure that the magnetic field is accurately parallel to the axis of the apparatus. This was done photographically. When the film does not lie at the focus, it intersects the conical beam of rays in the two branches of a hyperbola.

The distance between these two branches gives the distance of the focus from the film, and two such photographs taken at right angles indicate the required adjustments.

3. *Constancy of the current.* This was accomplished by means of a potentiometer arrangement.

To obtain the group of lines mentioned above a current of 4·6 amps. was required. The exposure given was from one-half to one milligram-hour.

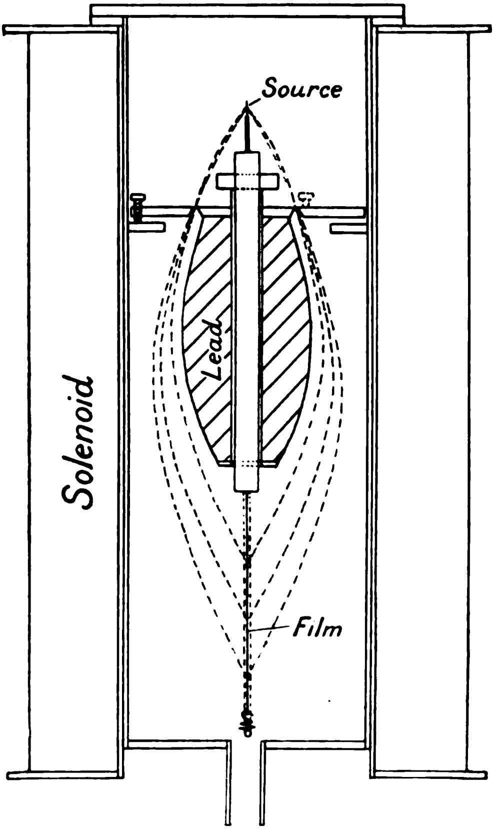


Fig. 3

Results. Examples of the spectra produced by this method are reproduced in Plate III. It will be seen that the spectrum consists of a well-defined line given by the axis of the instrument, all the β-rays crossing the axis within its comparatively narrow limits, beyond which the darkening of the film is very small. The β-ray lines of the ordinary spectra occur as spots on this line.

The dispersion produced is equal to that of a large apparatus of the ordinary type, the distance between the lines $(H\rho) = 1677$ and $(H\rho) = 1938$ being just over four centimetres.

The intensity obtained compares favourably with that of the usual method, it being greater than that of an ordinary instrument of one-quarter its dispersion.

In the spectra produced finally the lines have a thickness very little greater than that to be expected theoretically, namely 1·1 mm. The lines are not sharp, however, but fade off on each side. The main disadvantage attached to this lack of sharpness is the difficulty found in seeing faint lines, which do not stand out from the background. It is because of this that the method has disadvantages for obtaining ordinary β-ray spectra. A second obstacle is that a spot is by no means as easy to see as a line, and is liable to be confused with a blemish on the film.

This want of definition in the lines is compensated to some extent, as far as measurement of $(H\rho)$ is concerned, by the large dispersion obtained. It is possible to estimate this quantity to about one in eight hundred, in the case of the strong lines relatively to one another.

To obtain this accuracy it is necessary to apply a correction to the simple formula (which assumes that $(H\rho)$ is proportional to R) due to a variation of the angle of projection with range. For particles of range R and $R + h$, this consists of the factor

$$\frac{\cos\alpha_1}{\cos\alpha_2} = \sqrt{\frac{1 + \dfrac{\pi^2 r^2}{(R+h)^2 \sin^2\{x'/(R+h)\}}}{1 + \dfrac{\pi^2 r^2}{R^2 . \sin^2(\pi x'/R)}}}.$$

Using the line $(H\rho) = 1677$ as a reference the following numbers were obtained from three films:

Film	$(H\rho)$ observed	Actual value (Ellis and Skinner)
1	1775·5	1774
2	1775·5	1774
3	1775·5	1774
1	1830 (very faint)	1834
2	1834 „	1834
3	1836 „	1834
1	1853 „	1850
2	1857 „	1850
3	(invisible)	—
1	1936	1938
2	1935	1938
3	1936	1938

To obtain an accuracy of one in a thousand it is necessary to measure the lines to a $\frac{1}{4}$ mm. This should be possible even with the absence of a sharp edge to measure to. In the above figures there does not appear to be any systematic error.

Variation in the diameter of the wire supporting the source was tried and the result of finer wire is shown in the photograph (*b*). The whole spectrum is reduced in width and the lines are only slightly narrower.

The remaining photograph illustrates the use of a source on platinum foil, 5 mm. square, placed parallel to the film. The foil, being equivalent to a series of line sources lying in one plane, produces a cross-shaped trace, with a maximum intensity along the edges, as would be expected theoretically. In this form of trace the centre spot seems better defined, and is the form best adapted to measurement.

The Continuous Background. It is of importance, as a means of distinguishing between the two rival theories of the β-ray emission, to decide whether the general darkening of the plate in ordinary β-ray spectra arises from an emission from the radioactive atoms themselves, of electrons with a continuous range of velocity, or whether it is due to a spurious effect of secondary and scattered radiation.

In this apparatus secondary effects can be disentangled from the primary. Hence this form of apparatus may be of use in investigating the continuous background. This result arises from the fact that the secondary rays produce a uniform darkening of the film, while the actual spectrum is confined to the narrow line mentioned before, and applies to both the secondary β-rays and the γ-rays.

It is obvious that the direct action of the γ-rays will produce a uniform darkening of the film.

The primary β-rays disturbed at some point of their trajectory, the secondary β-rays arising from the action of the primary beam and γ-rays on the walls, etc., of the apparatus, will be the only source of secondary radiation to be considered in addition. The latter will be in random directions and will produce a uniform effect, while the former may be considered as follows.

The motion of the particles resolved in a plane at right angles to the magnetic field, is circular. If through causes which we need not specify, the value of $(H\rho)$ is altered, the radius of this circle is also altered, and in general the particle will not recross the axis. If $(H\rho)$ is not altered the particle will still form part of the β-ray line. The only points of the trajectory at which the radius can be altered and the particle still recross the axis, are the beginning and the end. Hence the only sources of secondary rays which could produce the continuous background are the radioactive source and the film.

The secondary rays arising from the film may be eliminated by using the fact that the particles arising from one side of the film will strike it a second time on the other, after having described a complete horizontal circle. Hence if the slit be suitably shielded so that the film is illuminated on one side only by the primary rays, the secondary rays knocked out of the film will illuminate the other. Hence if the background were due to these rays only, it ought to be eliminated by removing the appropriate side of the film.

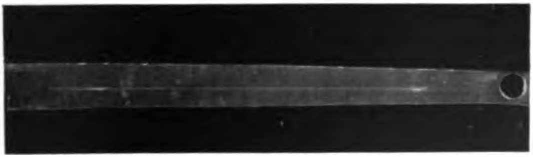
Photographs taken in this manner showed this to be a very minor cause in the production of the background. Hence the background must originate in the source,—i.e. either in the radioactive atoms themselves, or in the material supporting them.

Experiments were tried to see if the intensity of the background varied with the material of the support. Deposits were obtained on platinum wire, very thin platinum leaf, aluminium wire, aluminium foil (which was thinner than the wire) and on thin glass film. In the last case the deposit was very weak and the result uncertain. Aluminium leaf and collodion film were tried without success. In all these cases where a photograph was obtained, the background remained plainly visible. Direct comparison of the intensity in the various cases is difficult on account of the small and variable deposits obtained on the different materials.

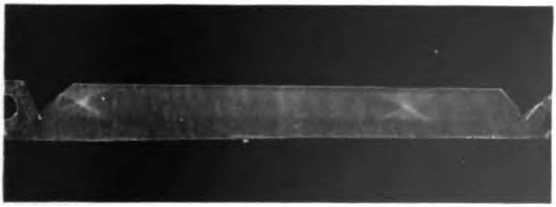
In the case of the aluminium wire, where a good photograph was obtained, it seemed at first that a possible source of the background might be found in the particles which were fired right through the aluminium support. This effect would not be expected with elements of high atomic weight, owing to too great an absorption, nor with very thin supports owing to too little alteration in ($H\rho$). A calculation of the intensity to be expected in such a case was made, using the curves in *The Structure of the Atom* by Andrade, and it was found that the intensity to be expected was about 1/200th of the intensity of the line from which it arose, and fell off very rapidly from this, as a maximum, as the distance from the line increased. Such a falling off is noticeable in the film, but does not appear to be at all rapid enough; also the background seems too intense.

On the whole the evidence seems to point to the continuous emission arising from the radioactive atoms themselves.

(*a*) Spectrum with source on 1/4 mm. Pt. wire.



(*b*) Spectrum using 1/10 mm. wire.



(*c*) Spectrum from a source on a piece of Pt. foil 5 mm. × 5 mm.

The two main lines showing are $(H\rho) = 1677$ and $[H\rho] = 1938$.

SUMMARY.

Experiments with a helical method of focussing β-rays are described. Although the method does not appear to be so generally useful for obtaining ordinary β-ray spectra, it possesses certain advantages, of which may be mentioned:

(1) Large dispersion;
(2) Greater intensity;
(3) Good screening.

The method produces some evidence on the question of the continuous background, and experiments with it in this connection are described.

The above experiments were carried out under the direction of Professor Sir Ernest Rutherford, whom I have to thank for much useful advice. Thanks are also due to Dr C. D. Ellis and to Dr Kapitza for a large number of suggestions.

Some Methods of Estimating the Successive Ionisation Potentials of any Element. By Mr D. R. Hartree, St John's College.

[*Received* 13 *August* 1924.]

§ 1. *Introduction.*

For the calculation of the properties of matter at very high temperatures—of the order of the temperature at the centre of a star—it is necessary to know the successive ionisation potentials of the atoms concerned; the object of this paper is to show how some of these ionisation potentials can be obtained probably fairly accurately, and some others roughly, and to give numerical values for oxygen, iron and silver as typical elements*.

The methods of estimation are based on the relations between corresponding terms of the optical spectra† of different atoms ionised to such an extent that they possess the same electronic structure. If such relations can be found, and if it is assumed that the electron structures of two atoms with the same number of core electrons are similar, then the terms of the optical spectrum of a highly ionised atom can be estimated from the terms of the spectra of less highly ionised atoms with the same number of core electrons. Subject to possible restrictions as to the existence of some terms, the greatest of the calculated terms will correspond to the ionisation potential of the ionised atom in question, but in view of the possible restrictions it is most satisfactory to assume the quantum numbers of the normal orbit of the series electron and calculate the value of the corresponding term from the appropriate relation.

The estimation of the ionisation potential of an ionised atom may therefore be divided into four steps:

1. An assumption as to the electronic structure of the core.
2. An assumption as to the quantum numbers of the normal orbit of the series electron.
3. The determination of the form of the relation between values of the term corresponding to this orbit, for different atoms with the core structure assumed in (1).
4. Determination of the numerical constants in this relation.

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* The work described in this paper was carried out at the request of Mr R. H. Fowler, in connection with his work on the properties of gases at high temperatures.

† The spectrum given out by the most loosely bound electron (series electron) of an atom, ionised or not, will be called the optical spectrum of this atom.

Terms of different spectra which correspond to orbits of the same quantum numbers n_k will be called corresponding terms; and for shortness we will say the "n_k term" instead of "the term corresponding to the n_k orbit."

The two assumptions (1) and (2) are not really independent, as the series electron of an ionised atom becomes part of the core for the capture of the next electron. They will be discussed in § 2. Relations between values of corresponding terms of spectra of different atoms with the same core structure will be treated in § 3, and the numerical results given in § 4.

The rough quantitative theory of spectra of atoms with more than one electron which is at present available is based on the assumption that the field of force of the atom is a central field; it is capable of giving the form of the relation required for some types of core structure, and suggests a rough form for some others, but it is not adequate to give numerical values of the constants, which have either to be determined from known spectra or guessed.

We will write N for the atomic number of an element and C for the core charge, which is greater by 1 than the degree of ionisation; and the spectrum of a given element in a given state of ionisation will be named by writing the number C in Roman numerals after the symbol of this element.

§ 2. *Core Structure and Normal Orbit of the Series Electron.*

There is little experimental data on which to base the assumptions about the normal orbits of successive electrons captured by an atom, but it will be assumed that for different elements the quantum numbers of the normal orbit of each of the first 18 electrons captured by a bare nucleus are independent of the atomic number.

It is certainly probable that the normal orbits of the first four electrons have quantum numbers 1_1, 1_1, 2_1 and 2_1 in this order for all the elements, and that the 10th goes into a 2_2 orbit to complete a neon-like structure, but it is by no means so certain whether the 5th to the 9th electrons fill up the unoccupied 2_1 and 2_2 orbits in the same order for all atoms; similarly it is probable that the 11th and 12th electrons always go into 3_1 orbits and the 18th into a 3_2 orbit to complete an argon-like structure, but it is doubtful if the 13th to 17th electrons captured fill up the unoccupied 3_1 and 3_2 orbits in the same order.

Thus the assumption of a uniform order of filling up for the first 18 electrons is least likely to lead to serious error in the case of the 1st, 2nd, 3rd, 4th, 10th, 11th, 12th and 18th electrons.

It will be assumed that except for K and Ca the normal orbit of the 19th electron is a 3_3 orbit; this is probably the case, and it is probable also that for elements of atomic number 29 (Cu) and over, the 28th electron goes into a 3_3 orbit to complete a structure like the core of the neutral Cu atom, and that the 29th is the first electron whose normal orbit is a 4_1 orbit. The 36th electron probably goes into a 4_2 orbit and forms a krypton-like structure, and the

37th into a 4_3 orbit. For elements of atomic number 47 (Ag) and over, the 46th electron probably goes into a 4_3 orbit and completes a structure like the core of the neutral Ag atom, and the 47th is the first electron which in its normal state is in a 5_1 orbit. This is as far as we need to go in connection with the elements for which numerical values are at present required.

§ 3. *Relations between Values of Corresponding Terms.*

The normal orbits of the series electron of different atoms can be grouped into three main types, for which the relations between values of corresponding terms have to be considered separately. These types comprise:

(1) Circular orbits which lie well within the boundary of the core (e.g. the 2_2 orbit of neutral oxygen).

(2) Non-circular orbits of which the outer part lies in a field which is not much different from the field of a point charge C at the nucleus (e.g. the 3_1 orbit of neutral sodium).

(3) Non-circular orbits over the whole of which the field is always considerably greater than that of a point charge C at the nucleus (e.g. the 3_2 orbit of neutral aluminium).

Circular Orbits. Consider first a k-quantum circular orbit lying well within the core.

The writer has shown elsewhere* that if Z is the effective nuclear charge at the radius of such an orbit, defined as the charge which, placed at the nucleus, would give the same *field* at this radius, and $Z - s$ is the charge which would give the same *potential* there, then the term corresponding to the orbit is given by

$$\nu/R = (Z^2 - 2sZ)/k^2 = [(N - \sigma)^2 - s^2]/k^2, \quad \ldots\ldots(3\cdot1)$$

where $$\sigma = N - Z + s.$$

But it may be expected that both $N - Z$ and s at the radius of a given circular orbit will be approximately the same for all atoms of the same electronic structure, so for such atoms we can put σ and s constant in (3·1). Further, s is often fairly small compared to $N - \sigma$, so we have

$$\nu/R = (N - \sigma)^2/k^2, \qquad \ldots\ldots(3\cdot2)$$

where σ is roughly constant.

The convenience of this formula compared to (3·1) is that it contains only one undetermined constant instead of two, which is important when, as often happens, there is only one atom of given type for which the value of the term in question is known.

It is probable that both $N - Z$ and s tend asymptotically to become constant as the atomic number increases, and s also becomes smaller compared to $N - \sigma$, so if σ is *defined* by (3·2) it also tends asymptotically to become constant.

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* D. R. Hartree, *Proc. Camb. Phil. Soc.*, vol. 21, p. 630 (1923).

Non-circular Orbits. For non-circular orbits, at any rate those which are in a nearly Coulomb field at the external apse, it is most convenient to make use of the quantum defect. If ν is the wave number of the term in the optical spectrum of an atom with core charge C, and R is the Rydberg constant, the quantum defect q is defined by the equation

$$\nu/R = C^2/(n - q)^2. \qquad \text{......(3·3)}$$

Consider first atoms which in the normal state have one series electron only.

It can be shown theoretically that q can be written approximately as the sum of contributions from the groups of core orbits with different principal quantum number $\mathbf{n}$, thus*

$$q = \Sigma_{\mathbf{n}} c_{\mathbf{n}}/(N - \gamma_{\mathbf{n}}), \qquad \text{......(3·4)}$$

where for corresponding orbits of different atoms (not necessarily of the same electron structure) each $c_{\mathbf{n}}$ and $\gamma_{\mathbf{n}}$ depends on the structure of the group of core orbits with principal quantum number $\mathbf{n}$, but not on its absolute dimensions or on the presence or absence of other groups of orbits.

When one of the $c_{\mathbf{n}}$'s is several times larger than the rest, a simple formula

$$q = c/(N - \gamma) \qquad \text{......(3·5)}$$

may be nearly as accurate as the more complicated formula (3·4), and is more convenient to use.

The quantities $c_{\mathbf{n}}$ and $\gamma_{\mathbf{n}}$ or c and γ have to be determined by comparison with observed spectra.

When q is small compared to n, (3·3) can be written

$$\nu/R = (C + Cq/n)^2/n^2, \qquad \text{......(3·6)}$$

and if C is large so that the difference between C and $N - \gamma_{\mathbf{n}}$ can be neglected compared to C we have

$$Cq/n = (\Sigma_{\mathbf{n}} c_{\mathbf{n}})/n,$$

from (3·4), so for C large (3·3) can be written approximately in the form (3·2)

$$\nu/R = (N - \sigma)^2/n^2$$

where

$$\sigma = N - C - (\Sigma_{\mathbf{n}} c_{\mathbf{n}})/n, \qquad \text{......(3·7)}$$

and is independent of N, for atoms of a given electronic structure.

In applying formula (3·2) to orbits of this type, σ must be calculated from (3·7) and not by making (3·2) fit spectra of neutral or slightly ionised atoms.

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* This formula is based on some work by the writer which has not yet been published.

Strictly speaking, (3·4) is only an asymptotic relation for large C, but in the cases where it can be tested the error seems to be much smaller than the second order terms in q in (3·6).

At present there is no clear theoretical indication of the best method to use for atoms which in the normal state have two series electrons in equivalent non-circular orbits, but the values of the singlet 3_1 terms of magnesium-like atoms fit a formula of the type (3·7) fairly closely, and it seems reasonable to expect similar terms of other elements to do the same.

Non-circular orbits lying entirely within the core cannot be treated adequately in the present state of the theory, and except for aluminium-like atoms there are very few experimental data either.

For aluminium-like atoms there are enough experimental data to test whether the 3_2 terms do fit a relation of the type (3·6) approximately, and it is found that they do.

The values of ν/R for the 3_2 terms of aluminium-like atoms suggest another kind of relation for orbits of the series electron which lie entirely within the core.

Let us define σ by

$$\nu/R = (N - \sigma)^2/n^2,$$

and write σ_0 for the limiting value of σ (for atoms with a given core structure) as $N \to \infty$. Then σ_0 is related to the constants c_n of (3·4) by (3·7), viz.

$$\sigma_0 = N - C - (\Sigma_n c_n)/n,$$

so σ_0 can be determined if the c_n's are known. By definition of σ_0, $\sigma - \sigma_0$ must tend asymptotically to zero with increasing C, and for the 3_2 orbit of aluminium-like atoms for which spectra have been observed, it is found that

$$\sigma - \sigma_0 = 5{\cdot}4/(C + 3),$$

very closely.

It will be assumed (for want of anything better) that $\sigma - \sigma_0$ is given by this same formula for any non-circular orbit of the series electron which lies entirely within the core.

Thus the quantum numbers of the term corresponding to the normal orbit of the series electron being assumed, there are four methods which may be used to estimate the value of this term.

I. For circular orbits, assume

$$\nu/R = (N - \sigma)^2/n^2; \qquad \ldots\ldots(3{\cdot}2)$$

determine σ from corresponding terms of known spectra of atoms of the same core structure.

II. For non-circular orbits, assume

$$q \sim \Sigma_n c_n/(N - \gamma_n), \qquad \ldots\ldots(3{\cdot}4)$$

or if it seems justifiable, $q \sim c/(N - \gamma)$. ......(3·5)

Determine the quantities c_n and γ_n (or c and γ) from known spectra, calculate q and hence $\nu/R = C^2/(n-q)^2$ for spectrum required.

This method will be used when sufficient experimental data are available for the determination of constants.

III. For non-circular orbits, assume (3·4) or (3·5) as before, estimate the quantities c_n or c from known spectra, but calculate

$$\sigma = N - C - (\Sigma_n c_n)/n, \text{ or } \sigma = N - C - c/n, \quad \ldots(3{\cdot}7)$$

and calculate the term value from

$$\nu/R = (N-\sigma)^2/n^2.$$

This method will be used when the experimental data are enough to furnish rough values of c_n or c, but not enough to furnish accurate values of the quantities c_n and γ_n or c and γ.

IV. For non-circular orbits which lie entirely within the core, assume

$$\nu/R = [N - \sigma_0 - 5{\cdot}4/(C+3)]^2/n^2;$$

determine σ_0 from known spectra.

This method will be used when the data are not sufficient even for a rough determination of c or the c_n's. It can also be used for aluminium-like atoms, as it is from these that the correcting term $5{\cdot}4/(C+3)$ was determined.

The application of these methods is best shown by considering examples.

Example of Method I.

5th electron captured: normal orbit assumed, 2_2.

The known values of ν/R for the 2_2 term of the spectrum of the 5th electron captured are 0·61 approximately for B I and 1·791 for C II; assuming $\nu/R = (N-\sigma)^2/n^2$, the values of σ obtained are 3·35 and 3·32. The usual tendency is for σ to decrease slightly with increasing N, and the value $\sigma = 3{\cdot}3$ is adopted for later elements.

Example of Method II.

11th electron captured: normal orbit assumed, 3_1.

From the values of the terms of the spectra Na I to S VI we find

$$q \sim 6{\cdot}3/(N - 6{\cdot}15)$$

approximately. There is enough experimental data to give a fairly reliable determination of the constants in this formula, so it is worth using Method II rather than the less accurate Method III.

An alternative, and probably still more accurate method is to assume the complete formula (3·4) for q. Since the core consists of electrons in orbits of principal quantum numbers 1 and 2 only, the formula is

$$q \sim c_1/(N - \gamma_1) + c_2/(N - \gamma_2).$$

The values of c_1 and γ_1 can be determined from the spectra of lithium-like atoms, and then the values of c_2 and γ_2 from those of sodium-like atoms. The formula for q is

$$q \sim 0{\cdot}8/(N - 1{\cdot}0) + 5{\cdot}6/(N - 6{\cdot}4).$$

Example of Method III.

4th electron captured: normal orbit assumed, 2_1.

Rough values of the 2_1 term of the spectrum of two beryllium-like atoms can be obtained. These are 0·71 for Be I and 3·3 for C III. The values of $1/q$ are 1·24 and 2·86 respectively, difference $1{\cdot}62 = 2 \times 0{\cdot}81$, and from the way in which $1/q$ varies with N for magnesium-like atoms, it seems probable that the difference between successive values of beryllium-like atoms is going to become constant at about 0·76 for large N, i.e. that $c = 1{\cdot}3$ in (3·5); so from (3·7), σ is given approximately by

$$\sigma = N - C - c/n = 3 - 0{\cdot}65 = 2{\cdot}35.$$

The experimental data are not accurate or complete enough for anything but a rough estimate of c to be made.

§ 4. *Numerical results.*

In Table I are given:

(1) The assumed normal orbit of the series electron in the order of capture.

(2) The method used for the estimation of each ionisation potential.

(3) The values of constants in the formula used, or in some cases the whole formula.

(4) The experimental data used in the determination of these constants. The information in the last column is given in the following form: spectrum, value of ν/R for term of quantum numbers given in second column, (reference).

Thus the normal orbit of the 4th electron captured is assumed to be a 2_1 orbit and in column 5 the entry is Be I, 0·71 (3); C III, 3·3 (4), meaning that the values of ν/R for the 2_1 term of Be I and C III are taken as 0·71 and 3·3 respectively, and the authorities for these values are given in references (3) and (4) respectively.

The terms whose values are given in this column do not necessarily all correspond to normal orbits, e.g. for Ca II the 3_3 term is given, since for higher elements the 3_3 orbit is assumed to be a normal orbit of the series electron.

Table I.

Number of electron in order of capture	Assumed n_k of normal orbit	Method used	Values of σ or q	Data used in determining σ or q
1	1_1	—	$q=0$	—
2	1_1	I	$\sigma=0{\cdot}65$	He I, 1·82 (1)
3	2_1	II	$q=0{\cdot}81/(N-1{\cdot}05)$	Li I to C IV (2)
4	2_1	III	$c=1{\cdot}3,\ \sigma=2{\cdot}35$	Be I, 0·71 (3); C III, 3·3 (4)
5	2_2	I	$\sigma=3{\cdot}3$	B I, 0·61 (3); C II, 1·791 (5)
6	2_2?	I	$\sigma=4{\cdot}25$	O III, 3·5 (4)
7	?	I	$\sigma=4{\cdot}95$	O II, 2·3 (4)
8	2_2	I	$\sigma=6{\cdot}0$	O I, 1·00 (6)
9	?	—	—	—
10	2_2	I	$\sigma=7{\cdot}5$	Ne I, 1·59 (7)
11	3_1	II	$q\sim\begin{cases}0{\cdot}8\,(N-1)\\+5{\cdot}6\,(N-6{\cdot}4)\end{cases}$	Na I to S VI (2)
12	3_1	III	$c=6{\cdot}0,\ \sigma=9{\cdot}0$	Mg I, 0·5620 (8); Al II, 1·3404 (9) Si III, 2·3371 (10); P IV, 3·559 (3)
13	3_2	IV	$\sigma_0=9{\cdot}65$	Al I, 0·4389 (8); Si II, 1·1986 (10) P III, 2·205 (3); S IV, 3·465 (3)
14	3_2?	IV	$\sigma_0=10{\cdot}5$	S III, 2·3 (4)
15	?	IV	$\sigma_0=11{\cdot}6$	S II, 1·2 (4)
16	3_2	IV	$\sigma_0=12{\cdot}05$	S I, 0·761 (6)
17	?	—	—	—
18	3_2?	IV	$\sigma_0=13{\cdot}5$	A I, 1·133 (7)
19	3_3	I	$\sigma=16$	Ca II, 0·747 (8, 12)
20–25	—	—	—	—
26	3_3	I	$\sigma=23{\cdot}4$	Fe I, 0·60 (11, 12)
27–28	—	—	—	—
29	4_1	II	$q\sim 26/(N-17)$	Cu I, 0·5678 (8); Zn II, 1·2309 (13) Rb IX, Sr X (12)
30	4_1	—	—	Zn I, 0·6904 (8)
31	4_2	IV	$\sigma_0=27{\cdot}0$	Ga I, 0·440 (8)
32–36	—	—	—	—
37	4_3	IV	$\sigma_0=33{\cdot}3$	Sr II, 0·678 (8)

(1) T. Lyman, *Nature*, vol. 110, p. 278 (1922).

(2) D. R. Hartree, *Proc. Camb. Phil. Soc.*, vol. 22, p. 409 (1924).

(3) These values are quoted from rough provisional analyses of the respective spectra by the writer, which are unpublished.

(4) Deduced from stellar absorption spectra, see R. H. Fowler and E. A. Milne, *Monthly Notices, R.A.S.*, vol. 84, p. 499 (1924). See also C. H. Payne, Harvard Coll. Obs., *Circular* 256 (1924).

(5) A. Fowler, *Proc. Roy. Soc.*, vol. 105, p. 299 (1924).

(6) J. J. Hopfield, *Phys. Rev.*, vol. 22, p. 523 (1923).

(7) G. Hertz, *Proc. Kon. Akad. Amst.*, vol. 25, p. 442 (1923).

(8) A. Fowler, *Report on Series in Line Spectra.*

(9) F. Paschen, *Ann. der Phys.*, vol. 71, p. 537 (1923).

(10) A. Fowler, Bakerian Lecture, 1924.

(11) H. Giesler and W. Grotrian, *Zeit. für Phys.*, vol. 25, p. 165 (1924).

(12) See text of this paper.

(13) Deduced from the lines given in Paschen-Götze, *Seriengesetze der Linienspectren*, assuming $\nu=27700$ for the 4_4 term, which is not likely to be far different from the corresponding term of He II.

There are three points which need further explanation than is given in the notes to the table.

The first is that the value of σ for the 3_3 orbit of the 19th electron captured is taken as 16, while for the corresponding term of Ca II it is 17·4. This is done because the 3_3 orbit of Ca^+ lies only a little way within the boundary of the core, and for later elements it is almost certain to lie further in, and therefore in a region where σ is smaller than it is at the radius of the 3_3 orbit of Ca^+.

Another point is that a value of σ for the normal orbit of the 26th electron is deduced from the ionisation potential of iron, though it is very uncertain whether the 26th electron captured always goes into a 3_3 orbit; but whatever the normal orbit of the 26th electron, this process will probably give a rough idea of the ionisation potential required. The value of σ is not at all sensitive to the value assumed for the ionisation potential of iron.

The derivation of the formula for the 4_1 term of the spectrum of the 29th electron captured is the other point which calls for explanation.

In the formula $q = \Sigma_{\mathbf{n}} c_{\mathbf{n}}/(N - \gamma_{\mathbf{n}})$ each term $c_{\mathbf{n}}/(N - \gamma_{\mathbf{n}})$ represents the contribution to q from the core orbits of principal quantum number $\mathbf{n}$, as already mentioned.

From the spectra of lithium-like and sodium-like atoms the values of c_1, γ_1 and c_2, γ_2 in the terms for contributions from complete *K* and *L* groups respectively can be found; then from the spectra K I and Ca II the contribution to q from the incomplete *M* group of these atoms can be found. By making an appropriate allowance for the difference in dimensions, approximate values for the contribution to q for corresponding orbits of Rb I and Sr II from the similar incomplete *N* group can be estimated, and by subtraction from the observed values of the q, the combined contribution to q from the *K*, *L* and (now complete) *M* groups can be estimated. But these are approximately the values of q for the corresponding orbits of Rb IX and Sr X respectively.

By such a calculation we can estimate values of q for the 5_1 terms of Rb IX and Sr X, and then by estimating the difference of q between the 4_1 and 5_1 terms from corresponding differences for Ca I and Zn II, we finally obtain the following values of q for the 4_1 terms, $q = 1·275$ for Rb IX and $q = 1·230$ for Sr X. These may be subject to a systematic and to a random error. From them and the values for Ca I and Zn II the formula given in the table has been deduced.

Approximate values of the successive ionisation potentials of oxygen, iron, and silver, calculated by the methods specified in Table I where possible, are given in Tables II, III and IV; they are given in the form of values of ν/R (i.e. ratios to the ionisation potential of that of atomic hydrogen) and in volts. Values which are simply guessed (there being no better method of estimating them) are given in volts only. The relativity correction has been neglected, since for levels which are important in astrophysical applications it is less than 1 % of the whole, which is less than the possible errors of estimation.

Table II. *Successive Ionisation Potentials, Oxygen* ($N = 8$).

Spectrum	Ionisation potential		Accuracy
	ν/R	volts	
O VIII	64	865	A
O VII	54	730	B
O VI	10·2	140	A
O V	7·8	105	C
O IV	5·5	75	C
O III	3·5	47	obs
O II	2·3	31	obs
O I	1·0	13½	obs

Table III. *Successive Ionisation Potentials, Iron* ($N = 26$).

Spectrum	Ionisation potential		Accuracy	Spectrum	Ionisation potential		Accuracy
	ν/R	volts			ν/R	volts	
Fe XXVI	676	9150	A	Fe XV	32·2	435	A
Fe XXV	640	8650	B	Fe XIV	29	390	B
Fe XXIV	149	2010	A	Fe XIII	26	350	C
Fe XXIII	139	1880	B	Fe XII	22	300	C
Fe XXII	128	1730	C	Fe XI	21	280	C
Fe XXI	118	1590	C	Fe X	—	250	C
Fe XX	110	1490	C	Fe IX	16	220	C
Fe XIX	100	1350	C	Fe VIII	11	150	C
Fe XVIII	—	1250	C	Fe VII	—	average 80	D
				to			
Fe XVII	85	1150	C	Fe II	—		
Fe XVI	35·7	480	A	Fe I	0·60	8·15	obs

To each value a letter is allotted, giving an estimate of the maximum possible error, according to the following scale:

A Error probably less than 3 %,
B „ „ 10 %,
C „ „ 30 %,
D Error may quite possibly be greater than 30 %.

It is interesting to note the great difference between the ionisation energy of an electron when it is the series electron of an atom and the X-ray excitation of the same electron in a neutral atom. For example in the case of silver the L_{I} X-ray term (2_1 orbit) is $\nu/R = 280$, the M_{I} (3_1 orbit) is $\nu/R = 53$ and the N_{I} (4_1 orbit) is $\nu/R = 7\frac{1}{2}$; while the ionisation energy for the first L electron captured (the 3rd) is given by $\nu/R = 516$, the first M electron (the 11th) by $\nu/R = 170$, and the first N electron (the 29th) by $\nu/R = 36$, which are greater than the corresponding X-ray terms by factors of nearly 2, over 3, and nearly 5 respectively.

Table IV. *Successive Ionisation Potentials, Silver* ($N = 47$).

Spectrum	Ionisation potential		Accuracy	Spectrum	Ionisation potential		Accur
	ν/R	volts			ν/R	volts	
Ag XLVII	2210	30000	A	Ag XXXI	—	1700	C
Ag XLVI	2150	29000	B	Ag XXX	120	1600	C
Ag XLV	516	6980	A	Ag XXIX	105	1400	C
Ag XLIV	500	6770	A	Ag XXVIII	—	average 1200	D
				to			
Ag XLIII	480	6500	B	Ag XXIII	—		
Ag XLII	460	6200	C	Ag XXII	62	850	D
Ag XLI	440	6000	C	Ag XXI, XX	—	800	D
Ag XL	420	5700	C	Ag XIX	36	500	B
Ag XXXIX	—	5500	C	Ag XVIII	—	450	C
Ag XXXVIII	390	5300	C	Ag XVII	24	350	D
Ag XXXVII	170	2300	A	Ag XVI	—	average 300	D
				to			
Ag XXXVI	160	2160	A	Ag XII	—		
Ag XXXV	154	2100	B	Ag XI	11	150	C
Ag XXIV	150	2000	C	Ag X	—	average 80	D
				to			
Ag XXXIII	140	1900	C	Ag II	—		
Ag XXXII	135	1800	C	Ag I	0·56	7½	obs

§ 5. *Summary.*

On certain assumptions as to the quantum numbers of the normal orbits of the successive electrons captured by an atom, the successive ionisation potentials can be estimated by making use of the relations between terms of the same quantum number for different atoms of the same electronic structure. Formulae for several of the successive ionisation potentials are given, the form of the formulae being suggested by theoretical reasoning and numerical values of the constants determined from known spectra: the successive ionisation potentials of O, Fe and Ag, calculated from these formulae, are tabulated. It is pointed out how far different these ionisation potentials are from the X-ray excitation potential of the same orbit of a neutral atom.

An Optical Interference Method of Measuring Young's Modulus for Rods. By G. F. C. SEARLE, Sc.D., F.R.S., University Lecturer in Experimental Physics.

[*Read* 19 May 1924.]

§ 1. *Introduction.* The general arrangement of the apparatus is shown in Fig. 1. A vertical steel rod *AA* of circular cross-section is fixed to a heavy cast-iron base *Z* shown diagrammatically in the figure. The rod *AA* is bent by a load applied to the end of the horizontal bar *BC* carried by the upper end of *AA*. The point of application of the load is defined by a transverse groove cut in the bar. The rod *AA* passes through the two metal plates *DD*, *FF*, which are soldered to the rod, the faces of these plates being horizontal.

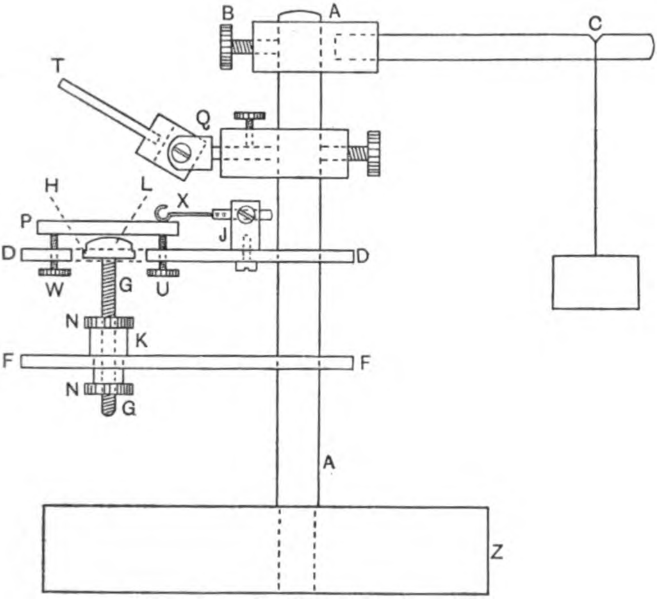

Fig. 1

The upper plate *DD* carries a plane glass plate *P*, of fair quality, which is held against the screws *U*, *V*, *W* (Figs. 1 and 2) by the wire springs *X*, *Y* (only *X* is shown in Fig. 1). These springs are soldered into short brass cylinders held by set screws in the fittings *J*, *J*. The spring *X* presses *P* against the screw *U*; the spring *Y* presses on *P* between the screws *V*, *W*. Thus the screw *U* turns the plate about an axis through the tips of *V* and *W*, while the screw *W* turns it about an axis through the tips of *U*, *V*.

The lower plate *FF* carries a pillar *K* through which passes the screwed rod *GG*. This rod is locked in position by the nuts *N*, *N*, which have knurled edges. The rod carries a small metal table *H*. A plano convex lens *L*, with convex surface upwards, is fixed by black sealing-wax in a recess turned in *H*; the radius of the convex surface may be about 50 cm. An opening in the plate *DD* allows the lens *L* to project above the upper surface of the plate. The optical interference is brought about by the opposed surfaces of *P* and *L*.

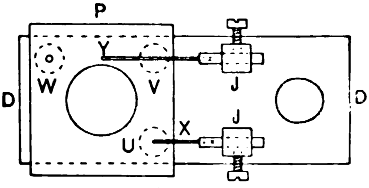

Fig. 2

The plates *DD*, *FF* are about ·25 cm. thick in order to secure adequate stiffness; they are about 4 cm. wide. It is important to pay attention to the manner in which they are fixed to the rod. Care must be taken that the solder "runs" properly on both sides of each joint. It is obvious that, if the soldering be not symmetrical, the "effective" distance between the plates will differ from the distance between their mean planes, with a consequent error in the value obtained for Young's modulus. In the apparatus used in § 5, the distance between the plates was about 2·7 cm.

Light from a sodium flame is reflected on to the lens *L* by a plane glass reflecting plate *T* supported by the vertical rod *AA* and suitably adjusted. A converging lens may be used to concentrate the light upon *T*. When the lens *L*, the plate *P* and the reflector *T* are properly adjusted, Newton's rings will be seen on looking vertically down upon the lens. The rings are most conveniently observed by means of a microscope held, with its axis vertical, in a suitable holder clamped to a rod springing from the base *Z* (not shown in Fig. 1). The microscope should be of low power suitable for viewing objects at a distance of 5 to 10 cm. from the object glass. When the load applied to the bar *BC* is increased, the ring system will contract and rings will disappear one by one at the centre of the pattern. When the load is diminished, the ring system will expand and new rings will start out from the centre.

If the fitting holding the reflector *T* be provided with a hinge *Q* (Fig. 1), the line joining the flame to *T* may be either at right angles to *BC* or parallel to *BC*. The latter arrangement is convenient when the width of the bench is limited.

It is necessary that the load should be increased or diminished in a gradual manner. If the load be increased by placing a mass in a pan, it is impossible to avoid setting up vibrations of the system which cause such rapid motions of the Newton's rings that it is impossible to count how many rings have disappeared. The requisite gradual variation of the load suspended from *BC* from zero to some determinate value, or the reverse process, is easily accomplished by the device shown in Fig. 3. A light frame is formed by rivetting and soldering the ends of a bent stout wire *W* into a cross-piece *L*. Into a block attached to *L* is soldered the lower end of a helical spring such that a load of a kilogramme extends it about 2 cm. The spring is stiff enough to remain upright when the frame is upright, although the upper end of the spring is free. A loop *R* is formed at the upper end of the spring and the bar *BC* (Fig. 1), seen in section in Fig. 3, passes through *R* with plenty of clearance. The frame is suspended by its apex *T* from a hook *H* which can be raised or lowered gradually by means of a screw arrangement. A load, conveniently formed of slotted weights resting on a suitable hanger, is suspended from a hook carried by the cross-bar *L*. When the hook *H* is sufficiently lowered, the whole load of weight, frame and spring is borne by the bar *BC*. When *H* is sufficiently raised, the loop *R* will rise clear of the bar *BC*, which will then be relieved of all load.

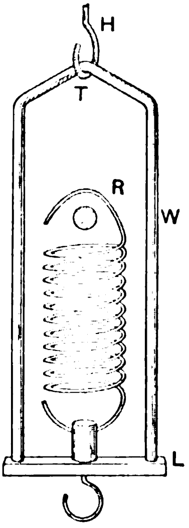

Fig. 3

For convenience, the screw arrangement is supported by a cross-bar carried by two vertical rods springing from the casting forming the base of the instrument. The apparatus is then self-contained.

The effects of the vibrations of the room upon the rings are much diminished when the instrument stands on a stone slab resting on a thick pad of felt.

§ 2. *Optical adjustments.* It is desirable that the lens *L* (Fig. 1) should be so mounted on the table *H* that the centre of curvature of its upper surface lies on the axis of the rod *GG*. This adjustment may be tested by holding the rod in a self-centering chuck in a lathe and observing the image of a flame formed by reflexion at the convex surface. If the image does not remain stationary as the lathe mandrel revolves, the centre of curvature lies off the axis of *GG*. Moderate accuracy in this adjustment suffices.

In assembling the apparatus, the lock nuts *N*, *N* (Fig. 1) are turned so that the lens is sufficiently low to allow the glass plate *P* to be put into place without damage to either lens or plate. The lens is then cautiously raised by the nuts *N*, *N* until the gap between the lens and the plate is a few tenths of a millimetre. The gap is easily observed by aid of a magnifying glass if it be illuminated by a roughly horizontal beam of light. The plate *P* is then adjusted by the screws *U* and *W* so that the plate appears parallel to the tangent plane of the lens at its highest point. If this preliminary setting be carefully made, the rings, if not actually in view, may be brought into view by slight adjustments of the screws *U* and *W*. Once the rings have been seen, it is easy to bring their centre to any desired point by the screws *U* and *W*.

The screws *U*, *V*, *W* may now be slightly readjusted so as to reduce the gap to the order of a few hundredths of a millimetre, care being taken to keep the centre of the rings approximately on the axis of the rod *GG*. The rings can be seen with a wider gap, but the difference of one-tenth per cent. in the wave lengths of the two sodium lines will cause the ring system to be inconspicuous when the gap has the widths ·29, ·58, ·88,... millimetres. The ring system is most vigorous when the gap is very small. If there be not actual contact between lens and plate, a very slight pressure of the finger upon the bar *BC* will cause the rings to contract in a very lively manner. If this slight pressure fail to affect the rings, the observer knows that the lens is *touching* the plate and that the screws *U*, *V*, *W* must be so turned as to slightly increase the gap.

The distance of the centre of the ring system from the nearest point of the rod is measured by a divided scale. The distance *a* cm. from the centre of the rings to the axis of the rod is found by adding to the distance just measured the radius of the rod.

§ 3. *Experimental details.* When the optical adjustments have been made, the observations are taken. A load is hung from the frame, and the hook *H* (Fig. 3) is lowered so far that the whole weight of load and frame is supported by the bar *BC*. If, now, *H* be gradually raised by its screw, rings will begin to move outwards from the centre as soon as the hook *H* begins to take any part of the load. Each new ring which is formed is noted, and the counting is continued until further raising of *H* ceases to cause the formation of fresh rings. It will then be found that the loop *R* (Fig. 3) has been lifted clear of the bar *BC*. The process can, of course, be reversed by imposing the load by lowering *H*, but the formation of a new ring is somewhat easier to observe than the disappearance of an old ring. With a little practice "fractions" of rings can be estimated. Thus, if initially there be a dark spot at the centre and finally there be a light spot at the centre, the number of rings is an integer plus one half.

If a series of observations with different loads be made, it will be found that the number of rings which appear or disappear is proportional to the load. In each case, the "load" includes the device of Fig. 3.

§ 4. *Theory of the Method.* A load of M grammes weight suspended from BC at C (Fig. 4) is equivalent to a load of Mg dynes acting along the axis of the vertical rod AA together with a couple of Mgc dyne cm., where c cm. is the distance of C, the point of support, from the axis of AA. The couple bends the rod into an arc of large radius ρ cm., where

$$EI/\rho = Mgc. \qquad \text{......(1)}$$

Here I is the "moment of inertia" of the cross-section of the rod about an axis through the centre of gravity of the section perpendicular to the plane of bending, and thus $I = \frac{1}{4}\pi r^4$, where r cm. is the radius of the rod. Hence

$$\frac{1}{\rho} = \frac{4Mgc}{\pi E r^4}. \qquad \text{......(2)}$$

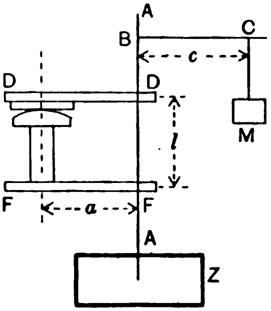

Fig. 4

If the distance between the points in which the mean planes of the plates DD, FF cut the axis of the rod be l cm., the angle between the plates when the rod is bent will be l/ρ radians. If the distance of the central spot of the Newton's ring system from the axis of AA be a cm., the increase of distance between lens and plate due to the bending will be al/ρ cm. or

$$\frac{4Mgacl}{\pi E r^4}\text{ cm.}$$

The load Mg causes the rod to shorten by $Mg/(E\pi r^2)$ cm. per cm. and thus reduces the distance between lens and plate by

$$\frac{Mgl}{\pi E r^2}\text{ cm.}$$

Hence, if x cm. be the resultant increase of distance between lens and plate due to the load Mg applied at C,

$$x = \frac{Mgl}{\pi E r^4}(4ac - r^2). \qquad \text{......(3)}$$

If n new rings be formed when the load Mg is removed, we have $x = \frac{1}{2}n\lambda$. Hence

$$E = \frac{2Mgl\,(4ac - r^2)}{n\pi\lambda r^4}. \qquad \ldots\ldots(4)$$

For sodium light we may take $\lambda = 5{\cdot}893 \times 10^{-5}$ cm.

§ 5. *Practical example.*

The bar under test was of mild steel. The values of the quantities in (4), § 4, were as follows:

$M = 1065$ grm., $g = 981$ cm. sec.$^{-2}$, $l = 2{\cdot}655$ cm.,
$a = 5{\cdot}088$ cm., $c = 12{\cdot}81$ cm., $r = {\cdot}5575$ cm.

Thus $4ac - r^2 = 260{\cdot}7 - {\cdot}3 = 260{\cdot}4$ cm.^{2}

Four counts by Dr Searle and Mr C. G. Tilley agreed in giving $n = 37$. Sodium light, with $\lambda = 5{\cdot}893 \times 10^{-5}$ cm., was used. Hence, by (4), § 4,

$$E = \frac{2Mgl\,(4ac - r^2)}{n\pi\lambda r^4} = \frac{2 \times 1065 \times 981 \times 2{\cdot}655 \times 260{\cdot}4}{37 \times \pi \times 5{\cdot}893 \times 10^{-5} \times {\cdot}5575^4}$$

$$= 2{\cdot}18 \times 10^{12} \text{ dyne cm.}^{-2}.$$

When a bar of the same material and of the same radius was placed on two knife edges, at a distance $2d = 150{\cdot}2$ cm. apart, and was loaded at the centre, the deflexion (h) per kilogramme (M) of load was ${\cdot}442$ cm. Hence

$$E = \frac{Mgd^3}{6h \times \frac{1}{4}\pi r^4} = 2{\cdot}06 \times 10^{12} \text{ dyne cm.}^{-2}.$$

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CONTENTS.

PAGE

Complexes of conics and the Weddle surface. By Mr C. G. F. James, Trinity College 201

Trajectories of small horizontal velocity in a resisting medium. By Mr J. E. Littlewood, Trinity College, Cayley Lecturer . . 217

Two notes on the Riemann zeta-function. By Mr J. E. Littlewood, Trinity College, Cayley Lecturer 234

On the applicability and deformation of surfaces. By B. M. Sen, King's College. (Communicated by Professor H. F. Baker) 243

The quadratic form for radial acceleration, in the theory of relativity. By Mr R. Hargreaves, St John's College 248

The statistical theory of dissociation and ionization by collision, with applications to the capture and loss of electrons by α-particles. By Mr R. H. Fowler, Trinity College 253

Integrals of systems of ordinary differential equations. By Dr T. M. Cherry, Trinity College 273

A system of linear equations with an infinity of unknowns. By E. C. Titchmarsh. (Communicated by Mr G. H. Hardy, Trinity College) 282

On Poincaré's theorem of "the non-existence of uniform integrals of dynamical equations." By Dr T. M. Cherry, Trinity College . 287

On the zeros of the Riemann zeta-function. By Mr J. E. Littlewood, Trinity College, Cayley Lecturer 295

On loci which have two systems of generating spaces. By Mr H. W. Richmond and F. Bath, King's College 319

On integrals developable about a singular point of a Hamiltonian system of differential equations. By Dr T. M. Cherry, Trinity College . 325

On a case of steady flow of a gas in two dimensions. By Mr S. Lees, St John's College. (With One Text-figure) 350

The thallous thallic halides. By Mr A. J. Berry, Downing College . 363

The high energy groups in the magnetic spectrum of the radium C β-rays. By Dr C. D. Ellis, Trinity College. (With One Text-figure) . 369

The relative absorbing powers of the L-levels for radiation of varying wave-length. By H. W. B. Skinner, Trinity College. (With Two Text-figures) 379

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On the measurement of the critical potentials of gases. By Mr E. G. Dymond, St John's College. (With Two Text-figures) 405

The spectra of some lithium-like and sodium-like atoms. By Mr D. R. Hartree, St John's College 409

Improvements in the apparatus for measuring the velocity of very rapid chemical reactions. By Dr H. Hartridge, King's College, and Mr F. J. W. Roughton, Trinity College, (With Three Text-figures) 426

Note on the Doppler principle and Bohr's frequency condition. By P. A. M. Dirac, St John's College. (Communicated by Mr R. H. Fowler) 432

Some experiments with an α-particle counter. By Mr E. V. Appleton, K. G. Emeléus, B.A., and M. A. F. Barnett. (With Eleven Text-figures) 434

A helical method of focussing β-rays. By R. A. R. Tricker, B.A., Trinity College. (Communicated by Professor Sir E. Rutherford.) (With Three Text-figures.) (With Plate III) 454

Some methods of estimating the successive ionisation potentials of any element. By Mr D. R. Hartree, St John's College 464

An optical interference method of measuring Young's modulus for rods. By Dr G. F. C. Searle, Peterhouse. (With Four Text-figures) . 475

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VOL. XXII

PART IV

(WITH PLATES IV AND V)

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On the Vector Algebra of Eight Associated Points of Three Quadric Surfaces.

By Mr H. W. TURNBULL, Trinity College.

[*Received* 11 July 1924. *Read* 14 July 1924.]

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if 1, 2, 3, 4 denote four vectors (1234) is their outer product. Let $(ijkl)$ denote any such outer product.

The following condition is necessary if eight points are associated:

$$\begin{vmatrix} (1256)(3476), & (1258)(3478) \\ (1276)(3456), & (1278)(3458) \end{vmatrix} = 0 \qquad \ldots\ldots(1),$$

the left-hand member of which may be denoted shortly by X or

$$(12\,.\,34\,.\,5678)^2 \qquad \ldots\ldots(2).$$

For if the seven points 1, 2, ... 7 are taken as fixed in general position, this condition implies that the eighth point lies on a quadric surface, since 8 enters twice into each term.

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On the Vector Algebra of Eight Associated Points of Three Quadric Surfaces. By Mr H. W. TURNBULL, Trinity College.

[*Received* 11 July 1924. *Read* 14 July 1924.]

§ 1. Let the eight points in space common to three arbitrary quadric surfaces be called eight *associated* points. These points possess the property that any one of them is uniquely determined if the other seven are given. It is interesting to give the analytical expressions for eight associated points in terms of symbols entirely free from coordinate systems. Thus we take the first eight digits 1, 2, ... 8 to denote the points; a group of two digits to denote the line joining these two; a group of three to denote a plane, and so on. Also a conjunction of two groups denotes the point, line or plane common to the two groups, thus

$$(34, 678)$$

is the point common to the line 34 and the plane 678.

The group of four points (1234) is more specifically defined as the volume of the tetrahedron whose vertices are points 1, 2, 3, 4. Or, algebraically, if four suffixes 1, 2, 3, 4 denote four coordinates of the point i, the symbol (1234) is equivalent to the determinant

$$\begin{vmatrix} 1_1 & 2_1 & 3_1 & 4_1 \\ 1_2 & 2_2 & 3_2 & 4_2 \\ 1_3 & 2_3 & 3_3 & 4_3 \\ 1_4 & 2_4 & 3_4 & 4_4 \end{vmatrix}$$

so that $(1234) = (2314) = \ldots = -(2134) = \ldots$, etc. Or, once again, if 1, 2, 3, 4 denote four vectors (1234) is their outer product. Let $(ijkl)$ denote any such outer product.

The following condition is necessary if eight points are associated:

$$\begin{vmatrix} (1256)(3476), & (1258)(3478) \\ (1276)(3456), & (1278)(3458) \end{vmatrix} = 0 \qquad \ldots\ldots(1),$$

the left-hand member of which may be denoted shortly by X or

$$(12 \,.\, 34 \,.\, 5678)^2 \qquad \ldots\ldots(2).$$

For if the seven points 1, 2, ... 7 are taken as fixed in general position, this condition implies that the eighth point lies on a quadric surface, since 8 enters twice into each term.

Now since $(ijkl)$ is a determinant it vanishes identically if for l_a we write

$$i_a, j_a, k_a \text{ or } \lambda i_a + \mu j_a + \nu k_a \quad (a = 1, 2, 3, 4),$$

a type of substitution which is clear enough even if we drop the suffix a.

Hence by substituting, for 8 in X, any of the following

$$1, 2, 3, 4, 5, 6, 7, \quad \lambda 1 + \mu 2, \quad \lambda 3 + \mu 4,$$

we find that X vanishes. Also X does not vanish identically, as is seen by substituting $\lambda 7 + \mu 3$ for 8. So the quadric surface contains the seven given points and also the lines 12 and 34, which gives the theorem:—

The vanishing of the determinant X implies that the point 8 *lies on the quadric through the other seven points and having* 12 *and* 34 *as generators.*

Since 1, 2, 3, 4 are arbitrary points these generators do not necessarily intersect: they are therefore generators of one regulus on the quadric.

§ 2. Next if the points are associated, all 8! determinants obtained from X by permuting the digits 1, 2, ... 8 in all possible ways must vanish. These 8! conditions are sufficient, but naturally extremely redundant, for geometrically only three such conditions are needed, representing three linearly independent quadrics which define the eight points. The interesting algebraic question arises how to reduce all these 8! determinants to a minimum system. It appears at once that the number of algebraic irreducibles is greater than three; that is to say, if $X_1, X_2, \ldots X_n$ denote n different forms among these 8! in terms of which each other X can be expressed as

$$\sum_{a=1}^{n} \lambda_a X_a,$$

where λ is a numerical multiplier, then n must exceed three.

§ 3. Before considering this further, it is well to show how the determinant (1) may be established geometrically. This is done by use of the elementary identity between outer products

$$(1234)(5\alpha) = (1235)(4\alpha) + (1254)(3\alpha) + (1534)(2\alpha) + (5234)(1\alpha) \qquad \ldots\ldots(3),$$

where α denotes any arbitrary plane.

In fact Hesse* gave a theorem, for eight associated points, that the line (125, 734) intersects the join of (56, 348) and (67, 812).

This may be denoted very conveniently by symbols; for if a denotes a point and β a plane, the point belongs to the plane if

$$(a\beta) = 0.$$

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* Cf. Baker, *Principles of Geometry* (Cambridge, 1923), vol. 3, p. 155.

Also the line through two points a, b cuts the line common to two planes γ, δ if the expression

$$(ab\gamma\delta) \equiv (a\gamma)(b\delta) - (b\gamma)(a\delta)$$

vanishes. Further, if $e + f = a$ then

$$(ab\gamma\delta) = (eb\gamma\delta) + (fb\gamma\delta).$$

Applying these results to Hesse's theorem, the necessary and sufficient condition for the required intersection is

$$\left.\begin{array}{l} (ab\,.\,125\,.\,734) = 0 \\ \text{where} \\ a \equiv (3485)\,6 - (3486)\,5 \text{ and } b \equiv (8126)\,7 - (8127)\,6 \end{array}\right\} \ldots\ldots(4).$$

Substituting these values for a and b and dropping terms containing a vanishing outer product $(iijk)$, we have

$$\begin{aligned}(3485)(6125)(8127)(6734) + \{(3485)(6734) - (3486)(5734)\} \\ \times \{(8126)(7125) - (8127)(6125)\} = 0 \quad \ldots\ldots(5).\end{aligned}$$

$$\left.\begin{array}{ll} \text{But} & (3485)(6734) - (3486)(5734) = (3465)(8734) \\ \text{and} & (8126)(7125) - (8127)(6125) = (7126)(8125) \end{array}\right\} \ldots(6)$$

by use of identities like (3). This reduces (5) to

$$X \equiv (1256)(3467)(1278)(3485) - (1258)(3487)(1276)(3465) = 0$$

which establishes condition (1).

Conversely this proves Hesse's theorem if condition (1) is made the starting-point.

§ 4. The function X may next be considered for its algebraic properties. It is obviously symmetrical in 1, 2 also in 3, 4; and it changes sign if 1, 2 are interchanged with 3, 4 as is directly seen from its definition (1). Further, it changes sign if any two of the remaining symbols 5, 6, 7, 8 are interchanged, and therefore is an alternating function of 5, 6, 7, 8, namely,

$$X = (12, 34, 5678)^2 = (12, 34, 6758)^2 = -(12, 34, 6578)^2 = \text{etc.} \quad \ldots\ldots(7).$$

This last requires proof. In fact if [5678] denote X we have at once

$$[5678] = [7856] = [6587] = [8765] \quad \ldots\ldots(8).$$

But, by the fundamental identity like (6),

$$\begin{aligned}&(1256)(3467)(1278)(3485) \\ &= (1257)(3467)(1268)(3485) + (1276)(3467)(1258)(3485).\end{aligned}$$

Similarly

$$\begin{aligned}&(1258)(3487)(1276)(3465) \\ &= (1258)(3467)(1276)(3485) + (1258)(3457)(1276)(3468).\end{aligned}$$

Hence, by subtraction,

$$\begin{aligned} X &= -(1257)(3476)(1268)(3485) + (1258)(3486)(1267)(3475) \\ &= -(12, 34, 5768)^2 \qquad \text{......(9).} \end{aligned}$$

Hence 6, 7 in X may be interchanged determinantally. But by (1) so also may 6, 8 be interchanged. We infer after including result (8) that all 5, 6, 7, 8 may be so interchanged, which is the required result.

§ 5. It is now convenient to write $\{12, 34\}$ for X so that

$$\{12, 34\} = \{21, 34\} = \{12, 43\} = \{21, 43\} = -\{34, 12\} = \text{etc.} \qquad \text{......(10),}$$

and in general to write $\{ij, kl\}$ for this type of function, where for definiteness the unexpressed symbols $i'j'k'l'$ are placed in the natural ascending order of digits so that, for example,

$$\{62, 83\} \text{ means } (62 \,.\, 83 \,.\, 1457)^2.$$

This reduces the original 8! forms to $210 \left(= \frac{1}{2} \cdot \frac{8 \,.\, 7}{1 \,.\, 2} \cdot \frac{6 \,.\, 5}{1 \,.\, 2}\right)$ forms one of which is given by (10). Among these the following identity holds:

$$(ij \,.\, lm \,.\, kpqr)^2 + (ik \,.\, lm \,.\, jpqr)^2 + (kj \,.\, lm \,.\, ipqr)^2 = 0 \quad \text{......(11).}$$

For by a fundamental identity

$$\begin{aligned} &(1256)(3467)(1278)(3485) \\ &= (1256)(3467)[(5278)(3481) + (1578)(3482) + (1258)(3487)]. \end{aligned}$$

Interchanging 6, 8 throughout and subtracting, we have

$$X = -(25 \,.\, 34 \,.\, 1678)^2 - (15 \,.\, 34 \,.\, 2678)^2$$

which proves (11). Hence we may select any two of the eight symbols, 7, 8 say, and by repeated use of (10) and (11) *express all* 210 *forms* X *in terms of twenty forms*:

(i) *five of type* $\{i6, 78\}$; $i = 1, 2, 3, 4, 5$;

(ii) *fifteen of type* $\{j7, k8\}$, *the values of* j, k *being, for example,* $j = 1, 2, 3, 4, 5$ *and* k *any value such that* $k > j$. This last inequality is simply a convenient notation, for there are 5! alternative sets of fifteen of type (ii) answering to each order in which the five values of j are taken.

The first (i) of these results is at once apparent from (11) by putting $k = 6$, $l = 7$, $m = 8$. The second (ii) follows by first noticing that if type (i) is given and one $(j7, k8)$ of (ii) is given, then $(k7; j8)$ is deducible by using (11).

§ 6. No other linear relation

$$\Sigma\lambda X = 0$$

between these twenty X's appears to exist, if the coefficients λ are numerical or independent of the eight given points. We may in fact prove the first set of five forms linearly independent. For if a relation

$$\sum_{i=1}^{5} \lambda_i \{i6, 78\} = 0 \qquad \ldots\ldots(12)$$

exists, where the coefficients λ_i are numerical, the same relation must hold if 1 and 2 are interchanged. Adding the results and using (9) we deduce

$$(\lambda_1 + \lambda_2)\,[\{16, 78\} + \{26, 78\}] = 0.$$

Here the second factor does not vanish identically, as is seen by writing $\mu 1 + \nu 6$ for 3, so $\lambda_1 = -\lambda_2$. This gives by symmetry

$$\lambda_1 = -\lambda_2 = \lambda_3 = -\lambda_4 = \lambda_5.$$

Adding the relation similar to (12) with 1 and 6 interchanged, and using (11) we deduce

$$\text{six} \times (16\,.\,78\,.\,2345)^2 = 0$$

which is not identically true. Hence the five forms of type (i) are linearly independent.

Such considerations would probably justify the retention of all twenty forms as algebraically irreducible. Without however embarking on this, the further question may be answered: *how are these twenty forms compatible with the geometrical fact that only three are linearly independent?*

The answer is simply this, that further linear relations

$$\Sigma L\,\{ij, kl\} = 0 \qquad \ldots\ldots(13)$$

exist between three or more of the twenty forms, where the coefficients L are invariants of the given eight points. Any of the X's may be so given linearly in terms of the three

$$(12, 34, 5678)^2, \quad (13, 24, 5678)^2, \quad (14, 23, 5678)^2,$$

but the simplest such relations are those which connect three quadrics with a common generator 78, say. One of these relations is

$$a\,(12\,.\,78\,.\,3456)^2 + b\,(13\,.\,78\,.\,4256)^2 + c\,(14\,.\,78\,.\,2356)^2 = 0 \qquad \ldots\ldots(14),$$

where

$$\frac{a}{(1534)\,(2578)} = \frac{b}{(1542)\,(3578)} = \frac{c}{(1523)\,(4578)} = \frac{a+b+c}{(2345)\,(1578)} \qquad \ldots\ldots(15).$$

For if 1, 2, 3, 4, 5, 7, 8 are arbitrary fixed points and if 6 denotes a variable point (x), then the three quadric surfaces

$$(12\,.\,78\,.\,345x)^2 = 0,$$
$$(13\,.\,78\,.\,425x)^2 = 0,$$
$$(14\,.\,78\,.\,235x)^2 = 0,$$

have a common generator 78 and are therefore linearly connected. This requires a relation such as (14), where the coefficients a, b, c are independent of 6. Since this relation will hold for all values of 6, substitute $\mu 1 + \nu 2$ for 6 in (14). The first term then vanishes, and after dividing by (1423) (7812) the ratio $b : c$ emerges as in (15). Similarly for $a : b : c$.

Hence equations (14) and (15) are true for all values of 6 when the other seven points are chosen arbitrarily. They therefore form a syzygy connecting the three given forms of type X.

Other such relations can be constructed but no others quite so simple have been discovered.

§ 7. This method could be used to discuss the allied problem of the configuration of eight common points of two arbitrary bicircular quartics in a plane. This would be effected by considering as before 1, 2, ... 8 to be eight points and x to denote a variable point, the whole to lie in a plane with the original coordinates interpreted as circular coordinates and one of the original quadric equations interpreted as the identical relation between four circular coordinates.

§ 8. *Historical note.* The corresponding expression

$$I = (135)\,(612)\,(234)\,(456) - (246)\,(123)\,(345)\,(561)$$

in ternary forms, which vanishes if six coplanar points lie on a conic, was first given by Reiss* who also gave the similar expression for the condition that ten points should lie on a plane cubic. This condition $I = 0$ implies Pascal's Theorem and also Desargues' Theorem for two coplanar triangles in perspective†.

The analogous problem, expressing by a vanishing invariant J the condition that ten points should lie on a quadric surface, has not yet been solved. According to Hunyady‡ a prize was offered in Belgium so far back as 1825 for a solution of the problem. Among other expressions Hunyady gives the determinant Δ or

$$\left|\begin{array}{cccc} (1237)\,(4567), & (1347)\,(2567), & (1457)\,(2367), & (1257)\,(3467) \\ (\;\text{,,}\;8)\,(\;\text{,,}\;8), & (\;\text{,,}\;8)\,(\;\text{,,}\;8), & (\;\text{,,}\;8)\,(\;\text{,,}\;8), & (\;\text{,,}\;8)\,(\;\text{,,}\;8) \\ (\;\text{,,}\;9)\,(\;\text{,,}\;9), & (\;\text{,,}\;9)\,(\;\text{,,}\;9), & (\;\text{,,}\;9)\,(\;\text{,,}\;9), & (\;\text{,,}\;9)\,(\;\text{,,}\;9) \\ (\;\text{,,}\;0)\,(\;\text{,,}\;0), & (\;\text{,,}\;0)\,(\;\text{,,}\;0), & (\;\text{,,}\;0)\,(\;\text{,,}\;0), & (\;\text{,,}\;0)\,(\;\text{,,}\;0) \end{array}\right|$$

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* *Math. Annalen*, 1870, vol. 2, p. 397.

† Cf. Study, *Vectorenrechnung* (Braunschweig, 1923), vol. 1, p. 68.

‡ *Crelle*, 1880, vol. 89, p. 47.

which vanishes if the ten points 0, 1, ... 9 lie on a quadric. This does not vanish identically and therefore contains J as a factor. It is not J however (which like the functions X and I above is of degree two in each different point involved), for the points 1, 2, 3, 4, 5, 6 here occur to degree four. Hence

$$\Delta = JK,$$

where K is a factor of degree two in 1, 2, 3, 4, 5, 6. Even though K may be evaluated as

$$(1423)(1456)(2356) - (1345)(1326)(4526),$$

yet J has not been found as a rational integral function of $(ijkl)$.

It will be noticed that the second minors of Δ are the functions X above. This is not however how I was first led to consider these functions. They occurred as in § 3 for the expression of Hesse's Theorem. It was not until after discovering the factor K that I found it had been already established by Hunyady.

Why Wireless Electric Rays can bend round the Earth (Abstract*).
By Sir Joseph Larmor, F.R.S., Lucasian Professor.

[*Received* and *read* 27 October 1924.]

Mystery has remained attached to the transmission of free electric rays a long way round the protuberant curvature of the earth, which has recently developed into the greatest sudden practical evolution in signalling since the telephone. The difficulty was already emphasised by the late Lord Rayleigh, as soon as the first signs of transmission across the Atlantic had been detected by the Marconi operators. The effect has been sometimes supposed to be accounted for by a hypothesis that the rays are turned downward by an upper conducting layer in the atmosphere. But conduction, as usually understood, involves dissipation, and thus loss of energy of the rays by absorption: so that a train of radiation travelling along a layer sufficiently conducting to bend the rays could not go far. In fact, by a well-known dynamical principle, if the absorption is small of the first order, the resulting decrease of velocity of the train is small of the second order and so of no account for bending rays in a varying field.

There seems however to be a sufficient and rather striking cause available for transmission of long waves horizontally round the earth at great altitudes, though the rays travelling at lower heights would be gradually quenched. For the synchronous oscillations of free ions of small mass go on undisturbed by collisions in very high regions of the atmosphere, and have time owing to the long period of the exciting waves to get up high speed and so become important under the influence of the field; much as the electrons in a triode valve have time during the electric oscillation affecting them, or rather in this case during their passage, to get deflected on to the grid. In an extremely tenuous atmosphere this free ionic oscillation would be almost the sole conceivable deflecting agency. On the other hand, the absorption that would accompany ionic oscillations in ordinary conduction arises from the shortness of the free path, which leads to dissipation, through collisions with the surrounding molecules, of part of the extra energy acquired in the accelerated motion of the ions and thus abstracted from the electric field. When, as above, the period and amplitude of the oscillatory motions of the ions are much shorter than the time and length of their free path, so that the great majority of the oscillations take place without interference from collisions, this loss of energy will be very much reduced. A rough illustration of this principle is doubtless afforded, at the other extreme, by the remarkable penetration of the very short X-rays into light metals. It is not, of course, that conductance tends to become perfect, for then no waves could travel: the influence must be turned on to the dielectric modulus of the medium.

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* Printed in full in *Philosophical Magazine*, Dec. 1924.

This cause must operate also for long waves travelling in the extreme limits of the atmosphere where the free path is great. The density aloft may be extremely minute: the long swing of the ions can bend the waves. The region in which the *aurora borealis* is displayed extends up to a height of 50 miles or more, and there the free atomic path would be about 2 centimetres. For radiation of wave-length of a kilometre, with oscillating field of amplitude one volt per kilometre and so transmitting $\frac{4}{3}\,10^{-3}$ watts per square km., the amplitude of the oscillations of free electrons would be about 5 centimetres and for heavier ions it would be smaller inversely as their mass: while the time of oscillation for this length of wave is only about a twentieth of the time of the free path at this altitude, for a molecule of the air, or a quarter of the free time for a hydrogen ion. For an electron moving with the speed of thermal equilibrium this time of oscillation would come to ten times the time of the free path at this height; this would be entirely excessive and the electron would be an agent of strong dissipation, were it not a reminder that the electron passes through the atom each time with but slight derangement of its motion.

The long free path would in itself tend to augment the degree of ionisation. There must, of course, be as many positive ions present as negative: but though both influence the velocity of propagation of the waves in the same direction, the effect of the massive ones is negligible. It can readily be shown that the result is to reduce the effective dielectric modulus of the medium, whose value is close to unity, by $\frac{N_0 e^2 \lambda^2}{\pi m}$, where N_0 is the numerical density of the ions of charge e and small inertia m. The amplitude of free oscillation of each ion would be $\frac{e\lambda^2}{4\pi^2 mc^2} F_0$ in a field of amplitude F_0. The velocity of propagation v is increased relatively by half the first of these amounts, which is for the same stratum proportional to the square of the wave-length λ. The curvature of the horizontal rays in any stratum is equal to $d \log v/dh$, the rate of relative increase of the velocity upwards, which is here $\frac{e^2\lambda^2}{2\pi m} \frac{dN_0}{dh}$. The stratum of transmission to great distances is the one for which this is exactly the same as the curvature of the earth, namely $\frac{1}{2}\pi \,.\, 10^{-9}$. For electrons this gives dN_0/dh equal to 10^{-5}, for hydrogen ions 1700 times this; thus if this adaptation of curvature applies to a sheaf of rays extending over the height of as much as a kilometre,

N_0 need not mount to more than one electron per cubic cm. for this wave-length of one kilometre, whereas the number of the atoms would still be as much as 10^{14} per cubic cm. at the height of 50 miles, which gives a sufficient free path for material ions. For hydrogen ions N_0 would have to be increased to $2 . 10^3$ per cubic cm., while the amplitude of oscillation would be reduced 1700 times, so that the effective stratum need not be so high: if free electrons also are present in numbers, in thermal equilibrium, they would introduce dissipation of the rays passing at levels considerably higher than the former rough illustrative estimate of 50 miles—were it not that their motion, more potent on account of their smaller mass, is so slightly deranged by passing through an atom as to compensate for the much greater number of such encounters.

These considerations remain valid in a general way when the strata of equal ionisation are not exactly horizontal. If any cause, such as the influence of sunlight on the ionisation, alters the height of this effective stratum too suddenly, the rays will be bent away upward or downward, and scattered, at the place of dislocation, and may not be able to get adjusted into the new stratum of complete transmission. If, however, the stratum is thick the dislocation will be incomplete, extending over only part of it. The stratum may be expected to be of varying height, some function of the local solar time: but it is too high for any merely meteorological derangement. All rays emitted not very nearly horizontally are lost: a highly conducting local earth assists the antennas here. We can think of the constituent beam of rays connecting the transmitter with a receiver anywhere along the path: it travels most of the way, without loss except by spreading sideways, in this effective stratum, in which all such beams unite to form a nearly horizontal band of rays the rough analogue of an optical caustic, in the almost vacuous region above. Each receiver collects from an area around it of the order of the square of the wave-length: it thus appears from numerical estimate that the amount of energy available need not be at all inadequate to account for the now familiar features of free electric transmission round the earth even to the antipodes and beyond.

Note on the Chemical Constant of Chlorine. By R. R. S. Cox, B.A., Christ's College. (Communicated by Mr R. H. Fowler.)

[*Read* 24 November 1924.]

Since the publication of a previous paper on chemical constants*, some further experimental data upon the dissociation of chlorine by Wohl† have appeared, from which a new and rather more satisfactory value of the chemical constant of diatomic chlorine can be calculated. Wohl concludes that Q_0, the heat of dissociation at absolute zero, is $-57{,}000$ calories. This is in good agreement with Henglein's value $-54{,}000$‡; but differs from that of Trautz and Stäckel§, namely $-71{,}000$, which is the value adopted in the previous paper. Q_0 is necessarily an adjustable constant, and since Wohl's value gives results which are a good deal more concordant with each other and with the theory, we now take $Q_0 = -57{,}000$. Wohl also uses $h\nu_0/k = 902$ instead of 1093, but as the difference made by this change is very small compared with other disagreements, we retain the value 1093. The following table gives the new values of Γ (Cl_2), calculated in the same way as before. The initials at the heads of the columns refer to the results of Henglein, Trautz and Wohl respectively.

T abs.	log K_p			Γ (Cl_2)		
	H.	Tr.	W.	H.	Tr.	W.
987	-6·37	—	—	-0·51	—	—
1050	-5·47	—	—	-0·63	—	—
1127	-4·375	—	—	-0·87	—	—
1473	—	-3·88	-2·72	—	1·33	0·16
1513	—	-3·38	—	—	1·05	—
1553	—	-3·05	-2·27	—	0·95	0·16
1773	—	—	-1·23	—	—	0·17
1973	—	—	-0·50	—	—	0·19

The theoretical value of Γ (Cl_2) as near as can be estimated is 0·9. Henglein's results are obtained at very low pressures (about 10^{-5} atm.) and their discordance with the others is probably connected with this fact. Trautz and Stäckel's experiments were carried out at approximately atmospheric pressure, and the figures deduced from their measurements are nearest to the theoretical values.

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* Cox, *Proc. Camb. Phil. Soc.*, 1923, vol. 21, p. 541.

† K. Wohl, *Zeitschr. für Elektrochem.*, 1924, vol. 30, p. 36.

‡ *Zeitschr. für Anorg. Chem.*, 1922, vol. 123, p. 137.

§ *Ibid.* 1922, vol. 122, p. 112.

Wohl's results, however, obtained by an explosion method with hydrogen, give very good constancy for Γ over a range of 500°. It is difficult to make a judgment as to the most probable value of Γ, and it is evident that more accurate experimental results are needed, for no alteration of Q_0 or $h\nu_0/k$ will remove the discordance. We ignore Henglein's results, and take the mean of Trautz's and Wohl's, and so obtain a provisional value

$$\Gamma\,(\mathrm{Cl}_2) = 0{\cdot}64.$$

The table of chemical constants as calculated now reads:

	$\Gamma\,(X_2)$ (Dissociation)	$\Gamma\,(X_2)$ (Vapour pressure)	$\Gamma\,(X_2)$ (Theoretical)
Iodine	1·84	3·49	2·42
Bromine	1·01	1·84	[1·8]
Chlorine	0·64	0·42	[0·9]
Nitrogen	—	-0·15	-0·16

The general agreement, though still unsatisfactory, appears considerably improved by the change in the value for chlorine.

Note added. Since the above was written, papers by Eucken and others have been published* in which chemical constants are calculated. The results differ from ours; but the authors find similar unsystematic discrepancies between theoretical and experimental values, so that the general conclusion is the same.

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* *Zeits. für Physik*, 1924, vol. 29, pp. 1, 36, and 71.

Dissociative Equilibrium in an External Field of Force. By Mr E. A. Milne, Trinity College.

[*Received* and *read* 8 December 1924.]

1. *The equilibrium of a column of dissociating gas subject to gravity.* Consider a vertical column of gas composed of atoms A of mass M. If the temperature is constant and equal to T, the pressure P at height x is given in terms of the pressure P_0 at the level $x = 0$ by the formula

$$P = P_0 e^{-Mgx/kT}, \qquad \text{......(1)}$$

where k is Boltzmann's constant and g is the acceleration due to gravity. Suppose now that the atom A is capable of dissociating into two atoms B and C, of masses m and m', according to the chemical equation

$$A \rightleftarrows B + C.$$

Let us assume that the gas is initially in the undissociated state and is then allowed to begin dissociating. The pressure being different at every different height, the tendency to dissociate will be different at every different height; the lower the pressure, the more the dissociation. But if m and m' are unequal, separation by diffusion will immediately begin. By Dalton's law, a mixture of independent gases of atomic masses m and m' will arrange itself in equilibrium according to the formulae

$$p = p_0 e^{-mgx/kT}, \quad p' = p_0' e^{-m'gx/kT}, \qquad \text{......(2)}$$

p and p' being the partial pressures. The lighter atoms tend to predominate at the higher levels, the heavier ones at the lower levels. Now a state of dissociative equilibrium is altered by an excess of either of the products of dissociation, and accordingly diffusion may be expected to influence the distribution of the degree of dissociation throughout the column considered. The question arises, what is the final state?

If atoms of the species A, B, C are in dissociative equilibrium at temperature T, whether or no they are present in chemically equivalent amounts, the partial pressures P, p, p' are connected by a relation

$$pp'/P = K, \qquad \text{......(3)}$$

where K, the equilibrium constant, is a function of temperature only. Let us now assume tentatively that in our column the atoms of species A, B, C are arranged as if they were independent, and let us examine the extent to which they then satisfy the condition of dissociative equilibrium. The partial pressures are given by (1) and (2), where now P denotes the *partial* pressure of the atoms A. Since

$$M = m + m',$$

it follows that for all values of x

$$pp'/P = p_0p_0'/P_0. \qquad \text{......(4)}$$

It follows that if $$p_0p_0'/P_0 = K, \qquad \text{......(5)}$$

the condition of dissociative equilibrium is satisfied everywhere. Thus if the atoms are in chemical equilibrium at the base of the column (or at any particular height), they will be in chemical equilibrium everywhere if arranged as if they were independent atoms obeying Dalton's law. The state of equilibrium is therefore given by (1) and (2), combined with condition (5).

Dalton's law thus automatically provides for the simultaneous satisfaction of the conditions both of chemical and of dynamical equilibrium. It must be noticed, however, that the ultimate analysis of the gas at any point of the column will not agree with that of the undissociated gas; there will be an excess of the lighter product above and of the heavier one below.

2. *Willard Gibbs' general condition of chemical equilibrium in an external field of force.* The question just discussed is not merely of theoretical interest. It has an important application to the atmospheres of stars, where atoms are dissociated into electrons and positive ions and are subject to the gravitational pull of the star. This particular problem is complicated by the presence of electric forces, to which we shall return in §§ 4–7. It justifies us, however, in considering the matter in more detail.

The result obtained in the previous section is no mere accident depending on the special circumstances of the case. The general problem was solved about fifty years ago, as indeed were almost all problems of pure thermodynamics, by Willard Gibbs in his classic memoir "The Equilibrium of Heterogeneous Substances*."

We may summarise as follows the results established by Gibbs bearing on the problem. Let the energy U of a portion of a homogeneous substance be expressed as a function of the $n + 2$ variables T, v, $m_1, \dots m_n$, where T is the temperature, v the volume, $m_1, \dots m_n$ the masses of the independent constituents $C_1, \dots C_n$ of which the substance is a homogeneous mixture. The differential dU is connected with the differentials dT, dv, $dm_1, \dots dm_n$ by a relation of the form

$$dU = SdT - pdv + \mu_1 dm_1 + \dots + \mu_n dm_n.$$

The μ's are called the partial potentials, or simply the potentials, of the corresponding constituents.

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* *Trans. Connecticut Acad.* 3, 108 (1875). *Sci. Papers*, 1, 144, 171.

I. Let a number of homogeneous substances be in equilibrium in an enclosure. Then the pressure p and temperature T are constant throughout the enclosure, and for each constituent C_r the corresponding potential μ_r has the same value in all homogeneous bodies of which C_r is a constituent.

II. If reversible chemical reactions between any of the constituents can take place, so that $C_1, \dots C_r$ are no longer independent constituents, let any such reaction be represented by a chemical equation of the type

$$\Sigma a_r C_r = 0, \qquad \text{......(6)}$$

where the a_r's are numbers proportional to the masses taking part in the reaction, written with the positive sign for the products and the negative sign for the reactants. Then in equilibrium, in addition to the conditions I, for each chemical equation (6) there is a corresponding relation between the potentials,

$$\Sigma a_r \mu_r = 0. \qquad \text{......(7)}$$

III. Let the substances be in equilibrium under the action of an external field of potential Ω per unit mass. Then the temperature is constant throughout the enclosure, and corresponding to each constituent C_r a relation of the type

$$\mu_r + \Omega = \text{const.}, \qquad \text{......(8)}$$

holds throughout the enclosure. Gibbs further showed that the equation of mechanical equilibrium

$$dp = -\rho d\Omega,$$

where ρ is the density, is included in conditions (8) in any region in which there are no surfaces of discontinuity of phase. For if $\rho_1, \dots \rho_n$ are the partial densities, from the theory of fundamental equations

$$dp = \Sigma \rho_r d\mu_r$$

and hence by (8) $dp = -(\Sigma \rho_r)\, d\Omega = -\rho d\Omega.$

From (7) and (8) Gibbs deduced that if a system containing constituents $C_1, \dots C_n$ satisfies the conditions of equilibrium in an external field of force when the constituents are treated as independent, and if a condition of chemical equilibrium is satisfied at one point, then the same condition is satisfied at all points. For if we denote by the suffix 0 the point where the condition of chemical equilibrium is satisfied, then

$$\Sigma a_r (\mu_r)_0 = 0.$$

But $$\mu_r + \Omega = (\mu_r)_0 + \Omega_0,$$

and hence $$\Sigma a_r \mu_r = (\Omega_0 - \Omega)\, \Sigma a_r.$$

But by the conservation of mass

$$\Sigma a_r = 0.$$

Therefore $$\Sigma a_r \mu_r = 0.$$

It follows that if the constituents are in fact capable of undergoing a reaction denoted by (6), then a state defined by (8) will be an actual state of equilibrium provided (7) is satisfied at one point.

For an ideal gas-mixture, the potential μ_r of a constituent C_r, whose atomic mass is M_r and specific heat at constant pressure C_r^p, is given by

$$\mu_r = C_r^p T (1 - \log T) + (kT/M_r) \log p_r + U_r^0 - TS_r^0, \quad \ldots(9)$$

where U_r^0, S_r^0 are constants and p_r is given by

$$p_r = \frac{m_r/M_r}{\Sigma (m_r/M_r)} p.$$

Insertion of these values in (7) gives immediately the integrated form of the Van't Hoff "reaction isobar" for a gas-reaction,

$$\log K \equiv \Sigma (a_r/M_r) \log p_r = \text{function of } T. \quad \ldots\ldots(10)$$

Insertion of the same values μ_r in (8) gives immediately the relation usually known as "Boltzmann's principle," thus deduced by Gibbs as a theorem in pure thermodynamics without recourse to kinetic theory. For since T is constant, we can incorporate the terms in μ_r depending only on T in the constant on the right-hand side of (8), obtaining

$$(kT/M_r) \log p_r + \Omega = \text{const.},$$

or $$p_r = \text{const.} \times e^{-M_r\Omega/kT}. \quad \ldots\ldots(11)$$

If equations (11) hold throughout the gas-mixture, and (10) holds at one point, then (10) holds at all points.

It should be noted that (8) gives a precise result whether or no the constituents are perfect gases. It can be shown that conditions (8) can be written in the form

$$\int^{p_r'} \left(\frac{dp}{\rho}\right)_r + \Omega = \text{const.}, \quad \ldots\ldots(12)$$

where the integrand refers to the constituent C_r, at temperature T, when isolated in a pure state, and p_r' is the pressure of C_r in a pure state when in equilibrium with the mixture at the point concerned, e.g. across a semi-permeable membrane. (For an ideal gas-mixture, $p_r' = p_r$.)

3. *Extension to electric forces and ionization.* Suppose that the constituents C_r taking part in a reaction of the type (6) have

associated with them electric charges of amount e_r per unit mass, where the e_r's may be positive, negative or zero. The presence of the charges will give rise to an electrostatic field, and there may be in addition an external electrostatic field. Let the total electrostatic potential be V. It can be shown, using Gibbs' methods, that the conditions of equilibrium (8) are now to be replaced by the conditions

$$\mu_r + \Omega + e_r V = \text{const.} \qquad \text{......(13)}$$

Now suppose that condition (7) is satisfied at one point, which we denote by the suffix 0. Then since

$$\mu_r + \Omega + e_r V = (\mu_r)_0 + \Omega_0 + e_r V_0,$$

we have $\quad \Sigma a_r \mu_r = (\Omega_0 - \Omega)\, \Sigma a_r + (V_0 - V)\, \Sigma e_r a_r.$

But $\Sigma a_r = 0$ as before. And since the net charge generated by the reaction must be zero, we have also

$$\Sigma e_r a_r = 0.$$

Hence (7) is satisfied at all points.

In this it has been assumed that the electrostatic forces acting on the material in the neighbourhood of a point P can be calculated from the field at P as given by the total distribution of charge. (The assumption is made in deducing (13).) Thus the polarisation of the material is neglected.

This concludes the main portion of the paper. We proceed to apply the theory to two problems which present themselves in the study of stellar atmospheres. §§ 4–8 are concerned with the equilibrium of an ionized gas in dissociative equilibrium under gravity. § 9 is concerned with a similar equilibrium under an applied electric field. In the first of these the application of the general theory only occurs incidentally, but the opportunity is taken of working out the mathematical details in full.

4. *Application to stellar atmospheres.* The typical reaction is that of the high-temperature dissociation of an atom into a positive ion and an electron, e.g.

$$\text{Ca} \rightleftarrows \text{Ca}^+ + \text{e}^-.$$

In consequence of the preceding result, in calculating the state of equilibrium of a stellar atmosphere we can treat the ions and electrons as independent constituents. We simply have to make sure that the condition of dissociative equilibrium is satisfied at one point (i.e. we must impose a suitable restriction on the initial conditions affecting the solution of the differential equations concerned), and the dissociative equilibrium will look after itself.

For the dissociation of an atom of mass M into a positive ion of mass m and an electron of mass m', of charges $\pm e$, the conditions of equilibrium give

$$P = P_0 e^{-M\Omega/kT}, \qquad \text{......(14)}$$

$$p = p_0 e^{-(m\Omega + eV)/kT}, \qquad \text{......(15)}$$

$$p' = p_0' e^{-(m'\Omega - eV)/kT}, \qquad \text{......(16)}$$

subject to a relation $p_0 p_0'/P_0 = K$.

It has been shown by Pannekoek[*] and Rosseland[†] that in a stellar atmosphere in equilibrium under gravitational and electrostatic forces we have approximately

$$eV = -\tfrac{1}{2}(m - m')\,\Omega, \qquad \text{......(17)}$$

and thus $$p = p' = (pp')^{\frac{1}{2}} = (p_0 p_0')^{\frac{1}{2}}\, e^{-\frac{1}{2}(m+m')\Omega/kT}. \qquad \text{......(18)}$$

This means that owing to the electrostatic forces between ions and electrons, the tendency of the light electrons to diffuse upwards is almost entirely counteracted. The equilibrium is the same as that of uncharged atoms of mass $\frac{1}{2}(m + m')$ under gravity only. Or, alternatively[‡], the equilibrium can be calculated from the equation

$$\frac{dp}{dx} = -\rho\frac{d\Omega}{dx},$$

where now p denotes the combined pressure of the ions and electrons, namely $2nkT$, where $\rho = n(m + m')$. The ratio of the number of ions to the number of neutral atoms at any level is proportional to $e^{\frac{1}{2}(m+m')\Omega/kT}$, and so increases outwards. The fact that ionization increases as the pressure decreases here appears as a consequence of the circumstance that the ions, having an effective weight of only half that of a neutral atom, necessarily predominate at high levels.

There is no doubt that this result is true in all cases that are of practical importance. But Rosseland's proof, which is very elegant, has reference to one particular case only. If n, n' are the numbers of ions and electrons per unit volume, V and Ω must satisfy

$$\nabla^2 V = -4\pi(n - n')\,e, \qquad \text{......(19)}$$

$$\nabla^2 \Omega = +4\pi G(nm + n'm'). \qquad \text{......(20)}$$

Rosseland observed that if a solution of the form $V/\Omega = \text{const.}$ exists, this requires $n/n' = \text{const.}$, and hence, since $p = nkT$, $p' = n'kT$, the exponential factors in (15) and (16) must be identical. Hence the only solution of this type is given by

$$m\Omega + eV = m'\Omega - eV,$$

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[*] *B.A.N.* 1, 110, No. 19 (1922).

[†] *Monthly Notices*, 84, 720 (1924).

[‡] In this form the principle has been used by the author in a paper in *Monthly Notices*, December, 1924, in considering the equilibrium of an ionized atmosphere under radiation pressure.

which is (17). The resultant force in an ion is then equal to the resultant force on an electron. This solution implies a certain definite volume charge, and therefore a certain total charge on the star. For an atmosphere (in which we can neglect curvature and take $\Omega = gx$), it implies a resultant charge σ per unit area given by

$$4\pi\sigma = -\frac{dV}{dx} = \frac{\frac{1}{2}(m - m')\,g}{e}.$$

There is, however, no reason to suppose that the positive charge per unit area of a star has this value. It is a charge sufficient to neutralise half the weight of the positive ions. The escape of electrons from the surface in virtue of their thermal velocities will however be checked long before the charge is so large as this. I have shown* in the *Philosophical Transactions* of this Society that for a giant star the escape of electrons will proceed until the star has a residual positive charge capable of supporting about one-quarter of the weight of hydrogen ions (and therefore a much smaller fraction of the weight of an ion of any other element); for a dwarf star, where gravity is so much more powerful, the fraction will be very much smaller†, e.g. for the sun 1/65.

Again, Pannekoek's proof, though more general than Rosseland's, may perhaps be criticised on the ground of lack of rigour. Pannekoek's proof assumes in effect that the resultant charge E inside a sphere of radius r concentric with the star is independent of r. We shall see that this is very nearly exactly true, save on the extreme outskirts of the atmosphere. But this point requires proof.

It can be shown that the result of Pannekoek and Rosseland, that the electric field is equal to half the weight of a positive ion, is true save at extremely low pressures, whatever the charge on the star. The question outstanding is merely that of the extent to which the effect of the surface charge penetrates inwards. The physical interest of this result—that the settling out of the atoms by diffusion in the order of their atomic weights is prevented (at least in the interior of a star, where the ionization is large and where the charge on an ion is approximately proportional to its mass)—has been fully expounded by Rosseland. But some mathematical interest attaches to a more general proof. Such a proof will now be sketched. It will be sufficient to give it for an atmosphere in which g is constant and curvature is neglected, since the complications only occur at the boundary. For simplicity, we shall consider only an ion which has lost one electron; the alterations for an ion which has lost several are obvious, and the generalisation of the result for this case is as given by Pannekoek and Rosseland.

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* *Camb. Phil. Trans.* 26, 512 (1923).

† If the sun has evolved from a giant star, it may have retained the charge it possessed in the giant stage. In that case the ratio of electrostatic force to gravity will be the same in the dwarf and giant stages.

5. *Atmosphere with zero resultant charge.* It is simplest first to examine this case. The extension to any charge is immediate and is given in § 8.

The tendency of the electrons to predominate at high levels, though slight, will cause a resultant negative charge at high levels, and there will be in consequence, inside the atmosphere, an electric intensity F positive outwards. The equations of equilibrium may be written (x being measured positive outwards),

$$\frac{kT}{p}\frac{dp}{dx} + mg - eF = 0, \qquad \text{......(21)}$$

$$\frac{kT}{p'}\frac{dp'}{dx} + m'g + eF = 0, \qquad \text{......(22)}$$

where by Poisson's equation

$$\frac{dF}{dx} = 4\pi e\frac{p - p'}{kT}. \qquad \text{......(23)}$$

Since there is no net charge, outside the atmosphere F is zero. Hence $F \to 0$ as $x \to +\infty$. Hence at very high levels the equilibrium is given by

$$\frac{kT}{p}\frac{dp}{dx} + mg = 0, \quad \frac{kT}{p'}\frac{dp'}{dx} + m'g = 0,$$

and thus at very high levels there is settling out according to Dalton's law. This will clearly occur when the ions and electrons are too far apart to give rise to appreciable electrostatic forces.

We shall now prove that as $x \to -\infty$, $F \to \frac{1}{2}(m - m')g/e$. We shall also find asymptotic solutions valid near $x = -\infty$, and $x = +\infty$, and from these we shall show that the transition from $F \sim 0$ to $F \sim \frac{1}{2}(m - m')g/e$ occurs between $p = 10^{-34}$ atmos. and $p = 10^{-24}$ atmos.

Write
$$\frac{eF}{kT} = \frac{\frac{1}{2}(m - m')g}{kT} - f, \qquad \text{......(24)}$$

so that
$$\frac{df}{dx} = \frac{4\pi e^2}{(kT)^2}(p' - p), \qquad \text{......(25)}$$

and (21) and (22) become

$$\frac{1}{p}\frac{dp}{dx} + \frac{\frac{1}{2}(m + m')g}{kT} + f = 0, \quad \frac{1}{p'}\frac{dp'}{dx} + \frac{\frac{1}{2}(m + m')g}{kT} - f = 0.$$

......(26), (27)

The latter equations have solutions which may be written

$$p = p_0 \exp\left(-\frac{\frac{1}{2}(m+m')gx}{kT} - \int_{x_1}^{x} f dx\right), \qquad \ldots\ldots(28)$$

$$p' = p_0' \exp\left(-\frac{\frac{1}{2}(m+m')gx}{kT} + \int_{x_1}^{x} f dx\right), \qquad \ldots\ldots(29)$$

where x_1 is arbitrary and p_0 and p_0' depend on the choice of x_1. For brevity write

$$a = \frac{\frac{1}{2}(m+m')g}{kT}, \quad \beta = \frac{4\pi e^2}{k^2T^2}.$$

Substitute for p and p' in (25), differentiate and eliminate $\exp(\int f dx)$. We find

$$\left(\frac{f'' + af'}{f}\right)^2 - f'^2 = 4\beta^2 p_0 p_0' e^{-2ax}. \qquad \ldots\ldots(30)$$

This is the differential equation satisfied by f. As $x \to -\infty$, the right-hand member $\to +\infty$. It follows that if f has a limit as $x \to -\infty$, that limit must be zero. It is easy to see that there are no other possibilities, at least in the physical problem. Hence from (24), as $x \to -\infty$

$$eF \to \tfrac{1}{2}(m - m')g.$$

Again, since $f \to 0$, we must have $f' \to 0$. Hence $p' - p \to 0$. This shows, from (28) and (29), that $\int^{x} f dx$ converges as $x \to -\infty$. We can therefore choose $x_1 = -\infty$, and it then follows that $p_0 = p_0'$. Hence $p' > p$ for all x, and therefore f steadily decreases as x decreases.

6. *Asymptotic solution for* $x \to +\infty$. Here

$$f \to \tfrac{1}{2}(m - m')g/kT = f_\infty, \text{ say,}$$

and hence we must have

$$f'' + af' \to 0.$$

This suggests trying $\qquad f' = \xi e^{-ax}. \qquad \ldots\ldots(31)$

Inserting in (30), the differential equation for ξ is

$$\xi'^2 - f_\infty^2 \xi^2 = 4\beta^2 p_0^2 f_\infty^2,$$

where we have approximated by writing f_∞ for f. The solution of this is

$$\xi = \beta p_0 (\gamma e^{xf_\infty} - \gamma^{-1} e^{-xf_\infty}),$$

γ being an arbitrary constant. The second term is negligible compared with the first. We have then

$$f' = \beta\gamma p_0 e^{-(a - f_\infty)x}. \qquad \ldots\ldots(32)$$

Now $\alpha - f_\infty = m'g/kT$. Integrating, we have finally for an approximation valid near $x = +\infty$

$$f = \frac{\frac{1}{2}(m - m')g}{kT} - \frac{\beta\gamma p_0 kT}{m'g} e^{-m'gx/kT}. \qquad \ldots\ldots(33)$$

The error can be shown to be $O(e^{-2m'gx/kT})$. From this,

$$\int^x f dx = \frac{\frac{1}{2}(m - m')gx}{kT} + \beta\gamma p_0 \left(\frac{kT}{m'g}\right)^2 e^{-m'gx/kT} + \text{const.}$$

We find then

$$p' = p_0 C h(x) e^{-m'gx/kT}, \quad p = p_0 [C h(x)]^{-1} e^{-mgx/kT}, \ldots(34), (35)$$

where
$$h(x) = \exp\left[\beta\gamma p_0 \left(\frac{kT}{m'g}\right)^2 e^{-m'gx/kT}\right],$$

and C is an arbitrary constant. We observe that $h(x) \to 1$ as $x \to \infty$, as it should. Insertion of these values of p, p' and f in (25) then gives

$$C = \gamma. \qquad \ldots\ldots(36)$$

It now appears that electrons and ions will separate out according to Dalton's law when $h(x)$ becomes approximately unity. This will occur when

$$\beta\gamma p_0 \left(\frac{kT}{m'g}\right)^2 e^{-m'gx/kT}$$

is small compared with unity, i.e. on combining (34) and (36) when

$$\beta \left(\frac{kT}{m'g}\right)^2 p' \quad \text{or} \quad \frac{4\pi e^2}{m'^2 g^2} p'$$

is small compared with unity. For electrons on the sun, the value of $4\pi e^2/m'^2 g^2$ is $4{\cdot}85 \times 10^{27}$. Thus the partial pressure of electrons must be at least as low as 10^{-28} dyne cm.$^{-2}$ or 10^{-34} atmos. before ions and electrons separate out according to Dalton's law.

7. *Asymptotic solution for $x \to -\infty$.* Near $x = -\infty$, equation (30) approximates to

$$f'' + \alpha f' = 2\beta p_0 e^{-\alpha x} f.$$

Putting $f = \alpha e^{-\eta}$, where $\eta \to +\infty$ as $x \to -\infty$, this becomes

$$-\eta'' + \eta'^2 - \alpha\eta' = 2\beta p_0 e^{-\alpha x}.$$

The solution of this is

$$\eta' = -(2\beta p_0)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha x} + \tfrac{1}{4}\alpha + O(e^{\frac{1}{2}\alpha x}),$$

whence
$$f = \alpha \exp\left[-\frac{2}{\alpha}(2\beta p_0)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha x} \times D\right], \qquad \ldots\ldots(37)$$

where $D \to 1$ as $x \to -\infty$. The value of D is approximately

$$1 + \frac{\alpha}{2} (2\beta p_0)^{-\frac{1}{2}} e^{\frac{1}{2}\alpha x} (\tfrac{1}{4}\alpha x + \text{const.}).$$

The determination of the value of the constant would involve a great deal of labour, but it is obvious on general grounds that D may be taken as unity when the term in $e^{-\frac{1}{2}\alpha x}$ in (37) is large. Putting $D = 1$, we find approximately

$$\int_{-\infty}^{x} f dx = \alpha (2\beta p_0)^{-\frac{1}{2}} e^{\frac{1}{2}\alpha x} \exp\left[-\frac{2}{\alpha} (2\beta p_0)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha x}\right]. \quad \ldots\ldots(38)$$

It is clear that the approximations

$$p = p' = p_0 e^{-\alpha x} \qquad \ldots\ldots(39)$$

will be valid provided the left-hand side of (38) is small compared with unity. Using (39), the right-hand side of (38) is seen to be equal to

$$\frac{\alpha}{(2\beta p)^{\frac{1}{2}}} \exp\left[-\frac{2}{\alpha} (2\beta p)^{\frac{1}{2}}\right]$$

which reduces to

$$\frac{(m + m') g}{4 (2\pi p)^{\frac{1}{2}} e} \exp\left[-\frac{8e}{(m + m') g} (2\pi p)^{\frac{1}{2}}\right]. \qquad \ldots\ldots(40)$$

The value of $8e (2\pi)^{\frac{1}{2}}/(m + m') g$, for ionized calcium on the sun, is $5{\cdot}4 \times 10^9$. Therefore expression (40) will be small compared with unity provided say $p > 10^{-18}$ dyne cm.$^{-2} = 10^{-24}$ atmos.

It follows that Rosseland's solution is valid in the interior, and that for this purpose "the interior" means the region in which $p > 10^{-24}$ atmos.

8. *Extension to any charge.* It can now be seen that this result is independent of the charge on the star. If the charge on the atmosphere has a surface density σ, then instead of $F \to 0$ as $x \to +\infty$, we have $F \to 4\pi\sigma$, and hence

$$f_\infty = \frac{1}{kT} [\tfrac{1}{2} (m - m') g - 4\pi e\sigma].$$

The asymptotic solution for $n \to +\infty$ is then slightly different. Equation (32) however is true as it stands, and proceeding as before we find

$$p' = p_0 C h_1 (x) e^{-(\alpha - f_\infty) x}, \quad p = p_0 [C h_1 (x)]^{-1} e^{-(\alpha + f_\infty) x},$$

where $$h_1 (x) = \exp\left[\frac{\beta\gamma p_0}{(\alpha - f_\infty)^2} e^{-(\alpha - f_\infty) x}\right]$$

and $C = \gamma$. These approximations are valid when p' is so small that

$$\frac{\beta p'}{(\alpha - f_\infty)^2} \quad \text{or} \quad \frac{4\pi e^2}{(m'g + 4\pi e\sigma)^2} p'$$

is small compared with unity. The order of magnitude of p at which this occurs is unaltered. The investigation for $x \to -\infty$ is however entirely independent of the value of f_∞, and thus Rosseland's solution holds in the "interior" whatever the resultant charge on the star.

It is of interest to observe that in the solution we have given for an uncharged atmosphere, the resultant charge down to any level whatever is negative. In the interior, F has the constant value $\frac{1}{2}(m - m') g/e$, which implies a distribution of positive charge at infinity. Rosseland's exact solution, which takes into account the effect of change of gravitation and implies a resultant positive volume charge, fits on to ours at infinity, but infinity may be taken to be reached at 10^{-24} atmos. The excess negative charge in the upper layers, for a star neutral on the whole, is excessively minute. By Gauss' theorem, the excess $-\sigma$ per unit area is given by

$$-4\pi\sigma = F_\infty - F_{-\infty} = -\tfrac{1}{2}(m - m') g/e, \quad \text{......(41)}$$

so that the excess number of electrons per unit area in the upper layers of the atmosphere is $\sigma/e = (m - m') g/8\pi e^2$, which for ionized calcium on the sun is 0·3. The steady field in the "interior" which just neutralises half the weight of an ion is $6{\cdot}5 \times 10^{-7}$ volt cm.$^{-1}$.

9. *The neutralisation of an external electric field in an ionized gas at rest.* In an ordinary stellar atmosphere, the ionization is so large that if an electric field happens to be applied between any two points it is immediately annihilated by the flow of electrons. To obtain an ion of the magnitude of the effect, and as an example of the general theory, I propose to consider the following problem*.

Let a mass of ionized gas in dissociative equilibrium be confined between two parallel insulating barriers at a given distance apart, and let an electric field be applied across the space between the barriers. It is required to find the residual field in the gas. The problem is one of the familiar space-charge type, complicated by the presence of particles of both signs in dissociative equilibrium. The full solution involves elliptic functions, but we shall find it possible to obtain simple inequalities which provide the information needed.

We ignore the effect of gravity, and we consider electrons and once-ionized atoms of a single kind in the presence of neutral atoms. With the same notation as before we have

$$P = \text{const.} = P_0,$$

$$p = p_0 e^{-eV/kT}, \quad p' = p_0' e^{+eV/kT}. \quad \text{......(42), (43)}$$

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* The problem has analogies with that considered by Sir J. J. Thomson, "Conduction of Electricity through Gases" (1906), 84. In his problem, however, there is a steady current and a steady production of ions due to an outside agency.

Take as origin of x the point midway between the barriers, supposed distant $2a$ apart. Then by symmetry, if $V=0$ at $x=0$,

$$p_0 = p_0',$$

and we have also
$$p_0^2/P_0 = K. \qquad \text{......(44)}$$

Poisson's equation gives

$$\frac{d^2V}{dx^2} = \frac{4\pi e}{kT}(p'-p). \qquad \text{......(45)}$$

Inserting for p and p', multiplying by dV/dx and integrating, we find

$$\frac{1}{2}\left(\frac{dV}{dx}\right)^2 + C = 4\pi(p'+p), \qquad \text{......(46)}$$

where C is a constant.

Suppose now that the external field is of intensity $-F$ ($F>0$). Then there will be an excess of electrons near $x=a$ and an excess of positive ions near $x=-a$, and V will steadily increase from, say, $-V_1$ at $x=-a$ to V_1 at $x=a$. At $x=\pm a$, $dV/dx=-F$, and hence (46) becomes

$$\left(\frac{dV}{dx}\right)^2 = F^2 + 8\pi p_0\left[e^{eV/kT} + e^{-eV/kT} - e^{eV_1/kT} - e^{-eV_1/kT}\right]. \quad (47)$$

Write
$$z = e^{eV/kT}, \quad z_1 = e^{eV_1/kT}.$$

Then (47) becomes

$$\left(\frac{kT}{e}\cdot\frac{1}{z}\frac{dz}{dx}\right)^2 = F^2 - 8\pi p_0(z_1 - z)\left(1-\frac{1}{zz_1}\right). \quad \text{......(48)}$$

Put
$$\Phi(z) = [F^2 - 8\pi p_0(z_1-z)(1-z^{-1}z_1^{-1})]^{\frac{1}{2}}.$$

Integrating (48) we have

$$\frac{2ea}{kT} = \int_{z_1^{-1}}^{z_1}\frac{dz}{z\Phi(z)} = 2\int_1^{z_1}\frac{dz}{z\Phi(z)}. \qquad \text{......(49)}$$

Let n be the number of positive ions per unit area. Then

$$n = \frac{1}{kT}\int_{-a}^{a} p\,dx = \frac{p_0}{e}\int_{z_1^{-1}}^{z_1}\frac{dz}{z^2\Phi(z)}. \qquad \text{......(50)}$$

Dividing the range of integration into the parts z_1^{-1} to 1 and 1 to z_1, it is readily verified that, in virtue of the form of Φ, (50) can be written

$$n = \frac{p_0}{e}\int_1^{z_1}\left(z+\frac{1}{z}\right)\frac{dz}{z\Phi(z)}. \qquad \text{......(51)}$$

Now let ν be the total number of atoms per unit area, N the number of neutral atoms per unit area. Also let ϖ be the pressure the ν atoms would have if undissociated. Then

$$\nu = 2a\varpi/kT, \quad N = 2aP_0/kT,$$

and
$$\nu = N + n.$$

Hence, using (44),

$$\varpi = \frac{p_0^2}{K} + \frac{p_0 kT}{2ae} \int_1^{z_1} \left(z + \frac{1}{z}\right) \frac{dz}{z\Phi(z)}. \qquad \text{......(52)}$$

Equations (49) and (52) contain the solution of the problem. They determine the values of the unknowns p_0 and z_1. The general behaviour of p_0 and z, as functions of F, can be seen from these equations, or, more physically, as follows.

As F increases, the crowding of electrons near $x = a$ and that of ions near $x = -a$ increases. The potential step from $x = -a$ to $x = a$ increases, and so z_1 increases. Also p_0 decreases, and therefore too P_0 decreases. The degree of dissociation at $x = 0$, which depends on $p_0/P_0 = K/p_0$, increases. At $x = -a$, the pressure of neutral atoms is still P_0, but that of positive ions is $p_0 z_1$. The latter increases with F, and so the degree of dissociation at $x = -a$, which depends on $p_0 z_1/P_0$, increases with F. The pressure of positive ions at $x = a$ is p_0/z_1, and hence the degree of dissociation at $x = a$, which depends on $p_0/z_1 P_0 = K/p_0 z_1$, decreases as F increases. The total dissociation is measured by n/N, which is equal to

$$\frac{KkT}{2aep_0} \int_1^{z_1} \left(z + \frac{1}{z}\right) \frac{dz}{z\Phi(z)}.$$

This increases as F increases. Thus the effect of an increase in the external field is to ionize more of the atoms, though near $x = a$, where there is an excess of electrons, the ionization is diminished.

If all the atoms were dissociated, and the positives completely separated from the negatives, the field in the interior would be less than F by an amount

$$4\pi e \frac{2a\varpi}{kT}.$$

This state of affairs will only occur for very large values of F. Now the expression $(z_1 - 1)(1 - z^{-1} z_1^{-1})$ is greatest when $z = 1$, when its value is

$$z_1 + z_1^{-1} - 2.$$

It follows from (48) that

$$\lim_{F \to \infty} \frac{4\pi p_0}{F} (z_1 + z_1^{-1} - 2) = 4\pi e \frac{2a\varpi}{kT}.$$

Thus for F large, the partial pressures at $x = 0$ and $x = a$ are given by

$$p_0 z_1 \sim \text{const.} \times F,$$
$$p_0 \sim \text{const.} \times F e^{-eaF/kT},$$

since, for large F, V_1 may be taken to be aF.

On the other hand, for values of F which are not too large the residual field in the interior will be almost zero; the potential step $2V_1$ is then very small, and z_1 is nearly unity. Our main problem is to obtain an estimate of the magnitude of V_1 when F is not too large. We can obtain a sufficient inequality as follows.

Since $\Phi(z)$ must be real, F^2 must exceed the greatest value of $8\pi p_0 (z_1 - 1)(1 - z^{-1} z_1^{-1})$ for $1 \leqslant z \leqslant z_1$. As we have seen, this occurs for $z = 1$ and we have accordingly

$$\frac{F^2}{8\pi p_0} > (z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}})^2. \qquad \text{......(53)}$$

For fields not too strong, dV/dx will be very small at $z = 1$, $(x = 0)$, and so $\Phi(1)$ must be very small. Thus (51) may be expected to provide a fairly close upper limit for z_1 for weak fields provided a lower limit for p_0 can be found. To do this we must have recourse to (52).

Denote by ζ the ratio

$$\int_1^{z_1} \frac{1}{2}\left(z + \frac{1}{z}\right) \frac{dz}{z\Phi(z)} \Big/ \int_1^{z_1} \frac{dz}{z\Phi(z)}.$$

Then (52) may be written, on using (49),

$$\varpi = p_0^2/K + \zeta p_0. \qquad \text{......(54)}$$

If z differs from unity by a small quantity of the first order, ζ differs from unity by one of the second order. Hence an approximate value of p_0 may be found, when F is small, by putting $\zeta = 1$ in (54). This however leaves us in doubt as to how small F must be in order that the resulting inequality may be valid. For a precise result, put $\zeta = 1 + \delta$, $(\delta > 0)$. For brevity put $K = \varpi/\lambda$. Solving (54) for p_0 we find

$$\lambda p_0/\varpi = -\tfrac{1}{2}(1 + \delta) + \tfrac{1}{2}[(1 + \delta)^2 + 4\lambda]^{\frac{1}{2}}.$$

It is readily verified that for $\lambda > 0$,

$$[(1 + \delta)^2 + 4\lambda]^{\frac{1}{2}} - [1 + 4\lambda]^{\frac{1}{2}} > \delta (1 + 4\lambda)^{-\frac{1}{2}}.$$

Hence $\quad \lambda p_0/\varpi > \tfrac{1}{2}[(1 + 4\lambda)^{\frac{1}{2}} - 1][1 - \delta(1 + 4\lambda)^{-\frac{1}{2}}].$

Now since $1 \leqslant z \leqslant z_1$, we have

$$\tfrac{1}{2}(z + z^{-1}) \leqslant \tfrac{1}{2}(z_1 + z_1^{-1}) = 1 + \tfrac{1}{2}(z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}})^2,$$

and hence, from the definition of ζ,

$$0 < \delta < \tfrac{1}{2}(z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}})^2.$$

Thus

$$\lambda p_0/\varpi > \tfrac{1}{2}\left[(1+4\lambda)^{\frac{1}{2}} - 1\right]\left[1 - \tfrac{1}{2}(1+4\lambda)^{-\frac{1}{2}}(z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}})^2\right]. \quad \ldots(55)$$

Put $$X = z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}}.$$

Combining (53) and (55) we have

$$F^2/4\pi > (\varpi/\lambda)\left[(1+4\lambda)^{\frac{1}{2}} - 1\right]\left[X^2 - \tfrac{1}{2}X^4(1+4\lambda)^{-\frac{1}{2}}\right]. \quad \ldots\ldots(56)$$

Let us write this inequality in the form

$$X^4 - 2uX^2 + vF^2 > 0,$$

where $u > 0$, $v > 0$. We know that $X = 0$ when $F = 0$, and that X is an increasing function of F. It follows that provided

$$F^2 \leqslant u^2/v, \quad \ldots\ldots(57)$$

X^2 must be less than the smaller of the roots of the quadratic

$$X^4 - 2uX^2 + vF^2 = 0,$$

i.e. $$X^2 < u - (u^2 - vF^2)^{\frac{1}{2}}. \quad \ldots\ldots(58)$$

We note that when vF^2/u^2 is small, this gives approximately

$$(z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}})^2 < \frac{F^2}{4\pi\varpi} \cdot \frac{\lambda}{(1+4\lambda)^{\frac{1}{2}} - 1},$$

which is the approximate result that would have been obtained by putting $\zeta = 1$ in (54) and substituting for p_0 in (53).

Inequality (58) gives an upper limit for X, and consequently for V_1, for values of F sufficiently small to satisfy (57). The significance of these inequalities is best grasped by considering a particular case. Let the temperature be such that in the absence of an electric field the gas is half-ionized. Then $K = (\frac{1}{2}\varpi)^2/(\frac{1}{2}\varpi) = \frac{1}{2}\varpi$, and so $\lambda = 2$. We find that (58) and (57) become

$$(z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}})^2 < 3\left[1 - \left(1 - \frac{2}{3} \cdot \frac{F^2}{4\pi\varpi}\right)^{\frac{1}{2}}\right],$$

for all values of F such that

$$\frac{F^2}{4\pi\varpi} \leqslant \frac{3}{2}.$$

Let us now adopt $\varpi = 10^{-3}$ atmos. $= 10^3$ dyne cm.$^{-2}$. Then the greatest value of F for which our analysis supplies an upper limit to z_1 is

$$F = 137 \text{ E.S.U.} = 41{,}100 \text{ volt cm.}^{-1}.$$

For an applied field of this magnitude we have

$$z_1 + z_1^{-1} - 2 < 3,$$

which gives $$1 < z_1 < \tfrac{1}{2}(5 + \sqrt{21}) = 4{\cdot}8.$$

If $T = 6000°$, this implies

$$V_1 < 0{\cdot}27 \times 10^{-2} \text{ E.S.U.} = 0{\cdot}81 \text{ volt}.$$

The result may be stated as follows. Suppose we have a gas which if undissociated would be at a pressure of 10^{-3} atmos. Suppose that the ionization potential is such that at 6000° half the atoms are dissociated into electrons and positive ions. Let an external electric field of intensity 40,000 volt cm.$^{-1}$ be supposed applied, and let the motion of the charged particles under this field be prevented by insulating barriers. Then the maximum potential difference between any two points of the gas will be of the order of 1 volt. The conditions taken are typical of a stellar atmosphere, and the result indicates the extreme conductivity of a stellar atmosphere.

10. *Summary.* (1) The conditions of dissociative equilibrium in an external gravitational field have been given by Willard Gibbs, as follows: If the state of a system is such that it is in (mechanical and thermodynamic) equilibrium when the constituents are taken to be independent, and if a condition of dissociative equilibrium is satisfied at one point, then the same condition is satisfied at all points, and the state is one of equilibrium if the constituents are actually capable of dissociating.

(2) In particular, the constituents of a column of dissociating gas under gravity settle out according to Dalton's law; if the condition of dissociative equilibrium is satisfied at one height it is satisfied at all heights.

(3) The conditions of equilibrium are generalised so as to take account of external electric fields and of the possibility of the products of dissociation being charged. Result (1) is shown to be unaffected.

(4) The theory is applied to the equilibrium of an ionized gas such as a stellar atmosphere under gravity. Whatever the charge on the star, the tendency of the light electrons to diffuse away from the heavy ions is almost entirely prevented by the electrostatic forces between them, and the result is the production of a field in the interior capable of supporting half the weight of the positive ions (Pannekoek, Rosseland). For the purposes of this statement the "interior" may be taken to commence at a pressure of 10^{-24} atmos. It is only above the level corresponding to 10^{-34} atmos. that electrons and ions separate out according to Dalton's law.

(5) The theory is applied to the equilibrium of an ionized gas under an external electric field. An external applied field of the order of 40,000 volt cm.$^{-1}$ cannot give rise to potential differences exceeding about 1 volt under the conditions of a typical stellar atmosphere.

Integrals developable about a singular point of a Hamiltonian system of differential equations. Part II. By Dr T. M. Cherry, Trinity College.

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This paper completes an investigation, of which the first part has already been published*, into the integrals of a Hamiltonian system which are formally developable about a singular point of the system. Let

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k} \quad (k = 1, \ldots n) \qquad \ldots\ldots(1)$$

be a system of differential equations of which the origin is a singular point of the first type†, i.e. a point at which H is developable in a convergent Taylor series, but at which its first derivatives all vanish. We suppose that H does not involve t, and we consider only integrals not involving t. Let the exponents of this singular point be $\pm \lambda_1, \pm \lambda_2, \ldots \pm \lambda_n$. In Part I, I considered the case in which the constants $\lambda_1, \ldots \lambda_n$ are connected by no *relation of commensurability*, i.e. a relation of the form

$$A_1\lambda_1 + \ldots + A_n\lambda_n = 0,$$

where $A_1, \ldots A_n$ are *integers* (positive, negative or zero) not all zero, and showed that the equations (1) possess n, and only n, integrals not involving t which are formally developable as power series in the x_k, y_k. In this paper I consider the case in which $\lambda_1, \ldots \lambda_n$ are connected by one or more relations of commensurability. Suppose that there are p, and only p, such relations linearly independent ($p > 0$): it will be shown that *the equations* (1) *possess* $(n - p)$ *independent integrals not involving* t, *formally developable about the origin and independent of* H.

The method pursued is similar to that of Part I. To establish the existence of the developable integrals we take (§§ 1, 2) the general solution of the equations (1), in which the x_k, y_k are exhibited as power series in the $(2n + 1)$ arguments

$$\alpha_1 e^{\lambda_1 t}, \beta_1 e^{-\lambda_1 t}, \ldots \alpha_n e^{\lambda_n t}, \beta_n e^{-\lambda_n t}, t,$$

say $$x_k = f_k(\alpha_1 e^{\lambda_1 t}, \ldots t), \quad y_k = g_k(\alpha_1 e^{\lambda_1 t}, \ldots t),$$

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* *Proc. Camb. Phil. Soc.* 22, 325 (1924): "On integrals developable about a singular point of a Hamiltonian system of differential equations." This paper will be referred to as Part I.

† So called to distinguish it from a point (singular point of the second type) at which H is not developable.

where the α_k, β_k are arbitrary constants, and suppose these series reversed so as to obtain the $\alpha_k e^{\lambda_k t}$, $\beta_k e^{-\lambda_k t}$ as power series in the x_k, y_k and t:

$$\alpha_k e^{\lambda_k t} = \phi_k (x_1, \ldots t), \quad \beta_k e^{-\lambda_k t} = \psi_k (x_1, \ldots t).$$

It is then shown that t disappears from certain linear combinations of the products $\phi_1\psi_1, \ldots \phi_n\psi_n$, and since $\phi_k\psi_k = \alpha_k\beta_k$, which is an arbitrary constant, these combinations give developable integrals not involving t.

The condition that a function Z of the x_k, y_k should be an integral is

$$(Z, H) = 0, \qquad \ldots\ldots(2)$$

where (Z, H) denotes the Poisson bracket $\sum_{k=1}^{n} \frac{\partial (Z, H)}{\partial (x_k, y_k)}$. The result just enunciated establishes therefore that the equation (2), regarded as a partial differential equation for Z, possesses $(n - p)$ independent formally developable solutions in addition to the trivial solution $Z = H$. In § 3 it is shown how these solutions may be constructed.

§§ 1–3 are concerned only with the case in which $\lambda_1, \ldots \lambda_n$ are all unequal and non-zero, but in § 4 it is shown that the results may be extended to the case in which these conditions are relaxed.

It has not been possible to discuss the convergence of the formal power series which appear in §§ 1, 2, owing to the fact that the exponents $\pm \lambda_1, \ldots \pm \lambda_n$ are essentially not all of the same sign. This being understood, the text has not been cumbered, on every occurrence of such words as "developable," by making explicit reference to the fact that the developments are of only formal validity.

§ 1. *Formal solution of the equations.*

We shall take at present the case in which $\lambda_1, \ldots \lambda_n$ are all non-zero and unequal, so that by a linear contact transformation H may be brought to the form

$$H \equiv \lambda_1 x_1 y_1 + \ldots + \lambda_n x_n y_n + H_3 + H_4 + \ldots,$$

where H_r is a homogeneous polynomial of the rth degree. The equations (1) are therefore

$$\left.\begin{aligned} \frac{dx_k}{dt} &= \lambda_k x_k + \frac{\partial H_3}{\partial y_k} + \frac{\partial H_4}{\partial y_k} + \ldots \\ \frac{dy_k}{dt} &= -\lambda_k y_k - \frac{\partial H_3}{\partial x_k} - \frac{\partial H_4}{\partial x_k} - \ldots \end{aligned}\right\} (k = 1, \ldots n), \quad \ldots(3)$$

and to a first approximation the solution is

$$x_k = \alpha_k e^{\lambda_k t}, \quad y_k = \beta_k e^{-\lambda_k t} \quad (k = 1, \ldots n), \qquad \ldots\ldots(4)$$

where α_k, β_k are arbitrary constants. Now this solution is such that

$$\left.\begin{array}{rl} [\alpha_r, \beta_r] = 1 & (r = 1, \dots n) \\ [\alpha_r, \beta_s] = 0 & (r, s = 1, \dots n; r \neq s) \\ [\alpha_r, \alpha_s] = [\beta_r, \beta_s] = 0 & (r, s = 1, \dots n) \end{array}\right\}, \quad \dots (5)$$

where $[\gamma, \delta]$ denotes the Lagrange bracket-expression $\sum_{r=1}^{n} \frac{\partial (x_r, y_r)}{\partial (\gamma, \delta)}$. We shall show that *the equations* (3) *may be solved by power series in the arguments*

$$\alpha_1 e^{\lambda_1 t}, \beta_1 e^{-\lambda_1 t}, \dots \alpha_n e^{\lambda_n t}, \beta_n e^{-\lambda_n t}, t \quad \dots\dots (6)$$

(the initial terms being given by (4)) *for which the relations* (5) *are satisfied.*

(i) Substitute in (3)

$$\left.\begin{array}{l} x_k = \alpha_k e^{\lambda_k t} + x_k^{(2)} + x_k^{(3)} + \dots \\ y_k = \beta_k e^{-\lambda_k t} + y_k^{(2)} + y_k^{(3)} + \dots \end{array}\right\} (k = 1, \dots n), \quad \dots\dots (7)$$

and equate separately the sets of terms whose orders, as measured by the sum of the superscripts in their factors, are equal. Then it is easily shown that if the $x_k^{(2)}$, $y_k^{(2)}$, ... $x_k^{(m)}$, $y_k^{(m)}$ have been obtained as homogeneous polynomials in

$$\alpha_1 e^{\lambda_1 t}, \beta_1 e^{-\lambda_1 t}, \dots \alpha_n e^{\lambda_n t}, \beta_n e^{-\lambda_n t} \quad \dots\dots (8)$$

of the appropriate degrees, with coefficients which are constants or polynomials in t, the equations for $x_k^{(m+1)}$, $y_k^{(m+1)}$ are of the form

$\frac{dx_k^{(m+1)}}{dt} - \lambda_k x_k^{(m+1)} =$ homogeneous polynomial of degree $(m + 1)$ in the arguments (8),

$\frac{dy_k^{(m+1)}}{dt} + \lambda_k y_k^{(m+1)} =$ homogeneous polynomial of degree $(m + 1)$ in the arguments (8).

These equations have a general solution of the form

$x_k^{(m+1)} =$ homogeneous polynomial of degree $(m + 1)$ in the arguments (8) $+ c_k e^{\lambda_k t}$,

$y_k^{(m+1)} =$ homogeneous polynomial of degree $(m + 1)$ in the arguments (8) $+ d_k e^{-\lambda_k t}$,

where c_k, d_k are arbitrary constants. Since we already have the $2n$ arbitrary constants α_1, β_1, ... α_n, β_n in the solution we are constructing, we may assign c_k, d_k in any manner we please, and we shall do this in the most general manner consistent with maintaining $x_k^{(m+1)}$, $y_k^{(m+1)}$ as homogeneous polynomials in the arguments (8).

The exponents $\pm \lambda_1, \dots \pm \lambda_n$ are connected by various relations of commensurability. Let

$$a_1 \lambda_1 + b_1 (-\lambda_1) + \dots + a_n \lambda_n + b_n (-\lambda_n) = 0 \qquad (9)$$

be any such relation in which $a_1, b_1, \dots a_n, b_n$ are *positive or zero* integers whose sum is m. Then in $c_k e^{\lambda_k t}$ we may have a term

$$p_k^{a_1 b_1 a_2 b_2 \dots} \alpha_k e^{\lambda_k t} (\alpha_1 e^{\lambda_1 t})^{a_1} (\beta_1 e^{-\lambda_1 t})^{b_1} \dots (\alpha_n e^{\lambda_n t})^{a_n} (\beta_n e^{-\lambda_n t})^{b_n}, \quad \dots(10)$$

and in $d_k e^{-\lambda_k t}$ a term

$$q_k^{a_1 b_1 a_2 b_2 \dots} \beta_k e^{-\lambda_k t} (\alpha_1 e^{\lambda_1 t})^{a_1} (\beta_1 e^{-\lambda_1 t})^{b_1} \dots (\alpha_n e^{\lambda_n t})^{a_n} (\beta_n e^{-\lambda_n t})^{b_n}, \quad \dots(11)$$

where the coefficients p, q are arbitrary constants; and there are in $x_k^{(m+1)}$, $y_k^{(m+1)}$ terms of this sort with arbitrary coefficients, corresponding to each relation of the form (9) in which the sum of the a_k, b_k is m. It will be shown that by proper assignment of these arbitrary coefficients we can ensure that the Lagrange brackets have the simple values (5).

(ii) From the form of the series (7) the Lagrange brackets formed from pairs of the constants of integration α_k, β_k may be shown as in Part I, § 3 (iii) to have the forms

$$\left.\begin{aligned} [\alpha_r, \alpha_s] &= \beta_r \beta_s R \\ [\beta_r, \beta_s] &= \alpha_r \alpha_s R' \\ [\alpha_r, \beta_s] &= \beta_r \alpha_s R'' \quad (r \neq s) \\ [\alpha_r, \beta_r] &= R''' \end{aligned}\right\}, \qquad \dots\dots(12)$$

where R, R', R'', R''' are power series in the arguments (6) which are identically independent of t, so that in their typical term

$$A t^m (\alpha_1 e^{\lambda_1 t})^{a_1} (\beta_1 e^{-\lambda_1 t})^{b_1} \dots (\alpha_n e^{\lambda_n t})^{a_n} (\beta_n e^{-\lambda_n t})^{b_n}$$

m must be zero and $a_1, b_1, \dots a_n, b_n$ positive or zero integers satisfying (9). We shall show that each mth degree term in the series on the right of (12) has a coefficient linear in the arbitrary coefficients p, q which occur in the $x_k^{(m+1)}$, $y_k^{(m+1)}$.

The mth degree terms in $[\alpha_r, \beta_s]$, which we shall write $[\alpha_r, \beta_s]_m$, arise from

$$\sum_{k=1}^{n} \left\{ \frac{\partial (x_k^{(1)}, y_k^{(m+1)})}{\partial (\alpha_r, \beta_s)} + \frac{\partial (x_k^{(2)}, y_k^{(m)})}{\partial (\alpha_r, \beta_s)} + \dots + \frac{\partial (x_k^{(m+1)}, y_k^{(1)})}{\partial (\alpha_r, \beta_s)} \right\},$$

which since $x_k^{(1)} = \alpha_k e^{\lambda_k t}$, $y_k^{(1)} = \beta_k e^{-\lambda_k t}$ can be written

$$e^{\lambda_r t} \frac{\partial y_r^{(m+1)}}{\partial \beta_s} + e^{-\lambda_s t} \frac{\partial x_s^{(m+1)}}{\partial \alpha_r} + \sum_{k=1}^{n} \left\{ \frac{\partial (x_k^{(2)}, y_k^{(m)})}{\partial (\alpha_r, \beta_s)} + \dots + \frac{\partial (x_k^{(m)}, y_k^{(2)})}{\partial (\alpha_r, \beta_s)} \right\}.$$

Thus corresponding to the typical terms (10), (11) in the $x_k^{(m+1)}$, $y_k^{(m+1)}$ we have in $[\alpha_r, \beta_r]_m$ the terms

$$\{(b_r + 1) q_r^{a_1 b_1 a_2 b_2 \dots} + (a_r + 1) p_r^{a_1 b_1 a_2 b_2 \dots}\} \alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n}, \quad \dots(13)$$

and in $[\alpha_r, \beta_s]_m$, where $r \neq s$, the terms

$$\left\{ b_s q_r^{a_1 b_1 a_2 b_2 \ldots} \frac{\beta_r}{\beta_s} + a_r p_s^{a_1 b_1 a_2 b_2 \ldots} \frac{\alpha_s}{\alpha_r} \right\} \alpha_1^{a_1} \beta_1^{b_1} \ldots \alpha_n^{a_n} \beta_n^{b_n}. \quad \ldots\ldots(14)$$

Now from (12) the typical term in $[\alpha_r, \beta_r]_m$ is $\alpha_1^{a_1}\beta_1^{b_1} \ldots \alpha_n^{a_n}\beta_n^{b_n}$, where $a_1, b_1, \ldots a_n, b_n$ are positive or zero integers whose sum is m: comparing with (13) its coefficient is

$$(b_r + 1)\, q_r^{a_1 b_1 a_2 b_2 \ldots} + (a_r + 1)\, p_r^{a_1 b_1 a_2 b_2 \ldots} + h_r, \quad \ldots\ldots(15)$$

where h_r depends only on the coefficients in the $x_k^{(2)}, y_k^{(2)}, \ldots x_k^{(m)}, y_k^{(m)}$. Similarly from (12) the typical term in $[\alpha_r, \beta_s]_m$, where $r \neq s$, is $\beta_r \alpha_s \,.\, \alpha_1^{a_1}\beta_1^{b_1} \ldots \alpha_n^{a_n}\beta_n^{b_n}$, where $a_1, b_1, \ldots a_n, b_n$ are positive or zero integers whose sum is $(m-2)$; comparing with (14) its coefficient is

$$b_s' q_r^{a_1' b_1' a_2' b_2' \ldots} + a_r'' p_s^{a_1'' b_1'' a_2'' b_2'' \ldots} + h_{rs}, \quad \ldots\ldots(16)$$

where h_{rs} depends only on the coefficients in the $x_k^{(2)}, y_k^{(2)}, \ldots x_k^{(m)}, y_k^{(m)}$ and

$$\left.\begin{array}{lll} a_s' = a_s + 1, & a_k' = a_k & (k \neq s), \\ b_s' = b_s + 1, & b_k' = b_k & (k \neq s), \\ a_r'' = a_r + 1, & a_k'' = a_k & (k \neq r), \\ b_r'' = b_r + 1, & b_k'' = b_k & (k \neq r); \end{array}\right\} \quad \ldots\ldots(17)$$

for from (17) each of the sets of positive or zero integers $a_1', b_1', \ldots a_n', b_n'$ and $a_1'', b_1'', \ldots a_n'', b_n''$ satisfies (9) and has m as its sum, and is therefore such as corresponds to a term with an arbitrary coefficient in the $x_k^{(m+1)}, y_k^{(m+1)}$. Since the a_k, b_k are non-negative integers we see that the coefficients of the p's and q's in (15), (16) are never zero.

(iii) We shall now assign the arbitrary coefficients p, q in the $x_k^{(m+1)}, y_k^{(m+1)}$ so that

$$[\alpha_r, \beta_s]_m = [\alpha_r, \alpha_s]_m = [\beta_r, \beta_s]_m = 0 \quad (r, s = 1, 2, \ldots n;\ m > 0).$$

Since there are more coefficients to annul in these $(2n)^2$ Lagrange brackets than we have arbitraries at our disposal, the possibility of this procedure is not immediately obvious, but depends upon the fact that merely from the vanishing of certain of the Lagrange brackets we can infer the vanishing of others, by the following

Lemma. If $[\alpha_r, \beta_r]_m = 0$, $[\alpha_r, \beta_s]_m = 0$, *then* $[\beta_r, \beta_s]_m = 0$. For from the identity

$$\frac{\partial}{\partial \beta_s}[\alpha_r, \beta_r] + \frac{\partial}{\partial \alpha_r}[\beta_r, \beta_s] + \frac{\partial}{\partial \beta_r}[\beta_s, \alpha_r] = 0$$

we may infer

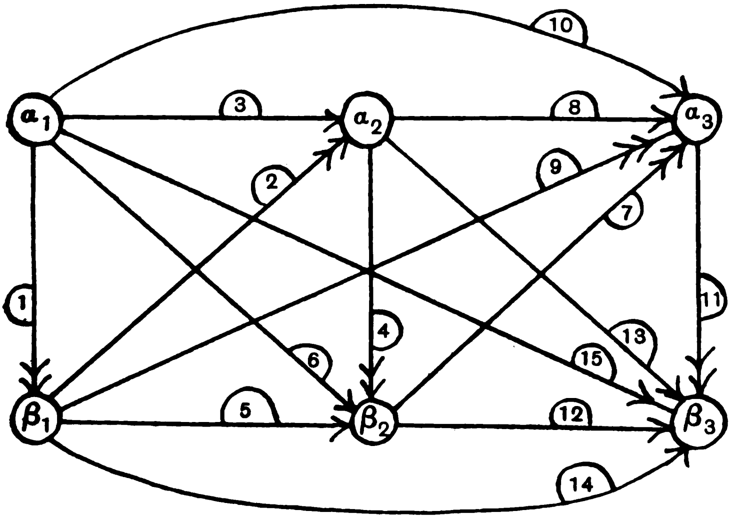
$$\frac{\partial}{\partial \beta_s}[\alpha_r, \beta_r]_m + \frac{\partial}{\partial \alpha_r}[\beta_r, \beta_s]_m + \frac{\partial}{\partial \beta_r}[\beta_s, \alpha_r]_m = 0,$$

and therefore

$$\frac{\partial}{\partial \alpha_r}[\beta_r, \beta_s]_m = 0. \quad \ldots\ldots(18)$$

But from (12), $[\beta_r, \beta_s]_m$ is a polynomial having a_r as a factor, and (18) requires therefore that $[\beta_r, \beta_s]_m = 0$.

In the following diagram the numbered arrows indicate the order in which the mth degree terms in the Lagrange brackets are shown to vanish; a double-headed arrow indicates that to make the corresponding bracket vanish we must suitably assign some arbitrary coefficients in the $x_k^{(m+1)}$, $y_k^{(m+1)}$, while a single-headed arrow indicates that the corresponding bracket vanishes in virtue of those which are already zero. From the Lemma we see that a single-headed arrow may be drawn to complete any right-angled triangle (in which of course the perpendicular sides are respectively vertical and horizontal).



(a) Assign the arbitrary coefficients $q_1^{a_1 b_1 a_2 b_2 \cdots}$ in $y_1^{(m+1)}$ in any manner we please. We may then choose those in $x_1^{(m+1)}$ so that $[\alpha_1, \beta_1]_m = 0$, the relation determining the typical coefficient $p_1^{a_1 b_1 a_2 b_2 \cdots}$ being from (15)

$$(b_1 + 1)\, q_1^{a_1 b_1 a_2 b_2 \cdots} + (a_1 + 1)\, p_1^{a_1 b_1 a_2 b_2 \cdots} + h_1 = 0.$$

(b) Choose the arbitrary coefficients in $y_2^{(m+1)}$ so that

$$[\beta_1, \alpha_2]_m = 0,$$

the relation determining the typical coefficient $q_2^{a_1 b_1' a_2 b_2' \cdots}$ being from (16)

$$b_1' q_2^{a_1 b_1' a_2 b_2' \cdots} + a_2'' p_1^{a_1'' b_1'' a_2'' b_2'' \cdots} + h_{21} = 0.$$

Then, by the Lemma, the mere determination of these two sets of coefficients in this manner ensures that $[\alpha_1, \alpha_2]_m = 0$.

(*c*) Choose the arbitrary coefficients $p_2^{a_1 b_1 a_2 b_2 \dots}$ in $x_2^{(m+1)}$ so that $[\alpha_2, \beta_2]_m = 0$. Then by the Lemma we have $[\beta_1, \beta_2]_m = [\alpha_1, \beta_2]_m = 0$.

(*d*) Choose the arbitrary coefficients in $y_3^{(m+1)}$ so that

$$[\beta_2, \alpha_3]_m = 0,$$

the relation determining $q_3^{a_1' b_1' a_2' b_2' \dots}$ being from (16)

$$b_2' q_3^{a_1' b_1' a_2' b_2' \dots} + a_3'' p_2^{a_1'' b_1'' a_2'' b_2'' \dots} + h_{32} = 0.$$

Then by the Lemma we must have $[\alpha_2, \alpha_3]_m = 0$.

(*e*) From the identity

$$\frac{\partial}{\partial \alpha_2} [\beta_1, \alpha_3]_m + \frac{\partial}{\partial \beta_1} [\alpha_3, \alpha_2]_m + \frac{\partial}{\partial \alpha_3} [\alpha_2, \beta_1]_m = 0 \quad \dots(19)$$

we have, in virtue of (*b*), (*d*) above, $\frac{\partial}{\partial \alpha_2} [\beta_1, \alpha_3]_m = 0$; and similarly $\frac{\partial}{\partial \beta_2} [\beta_1, \alpha_3]_m = 0$. Thus with the arbitrary coefficients in $y_3^{(m+1)}$ as at present determined, $[\beta_1, \alpha_3]_m$ is independent of α_2, β_2. Now these arbitrary coefficients have not all been determined in (*d*), for from (16) any coefficient $q_3^{a_1' b_1' a_2' b_2' \dots}$ for which $a_2' = b_2' = 0$ has a vanishing coefficient* in $[\beta_2, \alpha_3]_m$, and has remained arbitrary in (*d*). Since however $[\beta_1, \alpha_3]_m$ now consists only of terms for which $a_2' = b_2' = 0$, there remains an undetermined coefficient $q_3^{a_1' b_1' a_2' b_2' \dots}$ corresponding to each of these terms, and from the relations

$$b_1' q_3^{a_1' b_1' a_2' b_2' \dots} + a_3'' p_1^{a_1'' b_1'' a_2'' b_2'' \dots} + h_{31} = 0$$

we may so determine these coefficients that $[\beta_1, \alpha_3]_m = 0$. It then follows by the Lemma that $[\alpha_1, \alpha_3]_m = 0$.

(*f*) Choose the arbitrary coefficients in $x_3^{(m+1)}$ so that $[\alpha_3, \beta_3]_m = 0$. Then by the Lemma we have

$$[\alpha_2, \beta_3]_m = [\beta_2, \beta_3]_m = [\alpha_1, \beta_3]_m = [\beta_1, \beta_3]_m = 0.$$

(*g*) Choose the arbitrary coefficients in $y_4^{(m+1)}$ so that

$$[\beta_3, \alpha_4]_m = [\beta_2, \alpha_4]_m = [\beta_1, \alpha_4]_m = 0;$$

the possibility of thus annulling these three Lagrange brackets is shown as in (*e*) by using identities of the form (19).

The process is clearly general, and we have established that with a suitable choice of the arbitrary coefficients in the $x_k^{(m+1)}$, $y_k^{(m+1)}$ we shall have

$$[\alpha_r, \beta_s]_m = [\alpha_r, \alpha_s]_m = [\beta_r, \beta_s]_m = 0 \quad (r, s = 1, 2, \dots n;\ m > 0). \quad \dots(20)$$

(iv) We have already seen that

$$[\alpha_r, \beta_r]_0 = 1 \quad (r = 1, \dots n),$$
$$[\alpha_r, \beta_s]_0 = 0 \quad (r, s = 1, \dots n;\ r \neq s),$$
$$[\alpha_r, \alpha_s]_0 = [\beta_r, \beta_s]_0 = 0 \quad (r, s = 1, \dots n).$$

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* In fact, from (12), no term $\alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n}$ for which $a_2 = b_2 = 0$ is of a type which can occur in $[\beta_2, \alpha_3]$, for this Lagrange bracket has α_2 as a factor.

From the preceding result we can however choose in succession the arbitrary coefficients occurring in the $x_k^{(2)}$, $y_k^{(2)}$; $x_k^{(3)}$, $y_k^{(3)}$; etc., so that the identities (20) are satisfied for $m = 1, 2$, etc. We may thus always suppose that the series (7) are power series in the arguments (6), for which the relations (5) are satisfied.

§ 2. *Existence of formally developable integrals.*

(i) From the form of the linear terms, the Jacobian of (7)

$$\frac{\partial (x_1, y_1, \ldots\ldots\ldots\ldots\ldots\ldots\ x_n, y_n)}{\partial (\alpha_1 e^{\lambda_1 t}, \beta_1 e^{-\lambda_1 t}, \ldots\ \alpha_n e^{\lambda_n t}, \beta_n e^{-\lambda_n t})}$$

has the value 1 when $\alpha_1 = \beta_1 = \ldots = \alpha_n = \beta_n = 0$. Hence the equations may be solved so as to give the $\alpha_k e^{\lambda_k t}$, $\beta_k e^{-\lambda_k t}$ as power series in the x_k, y_k and t, say ϕ_k, ψ_k, so that

$$\left.\begin{aligned} \alpha_k &= e^{-\lambda_k t} \,.\, \phi_k (x_1, y_1, \ldots x_n, y_n, t) \\ \beta_k &= e^{\lambda_k t} \,.\, \psi_k (x_1, y_1, \ldots x_n, y_n, t) \end{aligned}\right\} \quad (k = 1, \ldots n). \qquad \ldots\ldots(21)$$

Writing $\frac{\partial \alpha_k}{\partial t}$, $\frac{\partial \beta_k}{\partial t}$ for the partial derivatives of these series with respect to t, the x_r, y_r remaining constant, it may be shown as in Part I, § 3 (ii) that

$$\left.\begin{aligned} \frac{\partial H}{\partial \alpha_k} + \sum_{r=1}^{n} \left\{\frac{\partial \alpha_r}{\partial t} [\alpha_r, \alpha_k] + \frac{\partial \beta_r}{\partial t} [\beta_r, \alpha_k] \right\} = 0 \\ \frac{\partial H}{\partial \beta_k} + \sum_{r=1}^{n} \left\{\frac{\partial \alpha_r}{\partial t} [\alpha_r, \beta_k] + \frac{\partial \beta_r}{\partial t} [\beta_r, \beta_k] \right\} = 0 \end{aligned}\right\} \ (k = 1, \ldots n),$$

where H is supposed expressed as a function of the $\alpha_k e^{\lambda_k t}$, $\beta_k e^{-\lambda_k t}$, and t by means of (7). In virtue of the identities (5) we therefore have*

$$\frac{\partial \beta_k}{\partial t} = \frac{\partial H}{\partial \alpha_k}, \ \frac{\partial \alpha_k}{\partial t} = -\frac{\partial H}{\partial \beta_k} \quad (k = 1, \ldots n). \qquad \ldots\ldots(22)$$

from which we shall show that certain combinations of the α_k, β_k give power series in the x_k, y_k independent of t.

(ii) Since H is an integral of the equations (1), it must become independent of t when expressed in terms of the α_k, β_k and t by means of (7). Being a power series in the arguments (6), it therefore becomes a series of terms of the form

$$B\alpha_1^{a_1}\beta_1^{b_1} \ldots \alpha_n^{a_n}\beta_n^{b_n}, \qquad \ldots\ldots(23)$$

where B is a constant and the a_k, b_k are positive or zero integers satisfying the relation (9). Suppose that $\lambda_1, \ldots \lambda_n$ are connected by p, and only p, linearly independent relations of commensurability, say

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* It must be remembered that these equations do not give the dependence of α_k, β_k upon t, the equations which do this being $\frac{d\alpha_k}{dt} = \frac{d\beta_k}{dt} = 0$ since the α_k, β_k are constants of integration.

$$A_{k1}\lambda_1 + A_{k2}\lambda_2 + \dots + A_{kn}\lambda_n = 0 \quad (k = 1, \dots p), \quad \dots(24)$$

where $A_{k1}, \dots A_{kn}$ are integers (positive, negative, or zero) not all zero, having no common integral factor. Then the relation (9), written in the form

$$(a_1 - b_1)\lambda_1 + (a_2 - b_2)\lambda_2 + \dots + (a_n - b_n)\lambda_n = 0,$$

must be linearly compounded, with integral multipliers, from (24), so that we can write

$$a_k - b_k = c_1 A_{1k} + c_2 A_{2k} + \dots + c_p A_{pk} \quad (k = 1, \dots n),$$

where $c_1, \dots c_p$ are integers. Using these relations the typical term (23) in H becomes

$$B(\alpha_1{}^{A_{11}}\alpha_2{}^{A_{12}} \dots \alpha_n{}^{A_{1n}})^{c_1} \dots (\alpha_1{}^{A_{p1}}\alpha_2{}^{A_{p2}} \dots \alpha_n{}^{A_{pn}})^{c_p}(\alpha_1\beta_1)^{b_1}(\alpha_2\beta_2)^{b_2} \dots (\alpha_n\beta_n)^{b_n}.$$

Hence writing

$$\left.\begin{aligned} u_k &= \alpha_k\beta_k & (k = 1, \dots n) \\ w_k &= \alpha_1{}^{A_{k1}}\alpha_2{}^{A_{k2}} \dots \alpha_n{}^{A_{kn}} & (k = 1, \dots p) \end{aligned}\right\}, \quad \dots\dots(25)$$

H is a series of positive and negative integral powers of the arguments $u_1, \dots u_n, w_1, \dots w_p$. Expressing the derivatives of H in (22) in terms of derivatives with respect to these arguments we have

$$\left.\begin{aligned} \frac{\partial\beta_k}{\partial t} &= \frac{u_k}{\alpha_k}\cdot\frac{\partial H}{\partial u_k} + A_{1k}\frac{w_1}{\alpha_k}\cdot\frac{\partial H}{\partial w_1} + \dots + A_{pk}\frac{w_p}{\alpha_k}\cdot\frac{\partial H}{\partial w_p} \\ \frac{\partial\alpha_k}{\partial t} &= -\frac{u_k}{\beta_k}\cdot\frac{\partial H}{\partial u_k} \end{aligned}\right\}(k=1,\dots n). \quad (26)$$

(iii) Since the α_k, β_k are constants of integration any function $f(\alpha_k, \beta_k)$, when expressed in terms of the x_k, y_k and t by means of (21), is an integral of the equations (1), and if this integral is to be a power series in the x_k, y_k, independent of t, it must be a power series in the α_k, β_k such that

$$\sum_{k=1}^{n}\left(\frac{\partial f}{\partial\alpha_k}\cdot\frac{\partial\alpha_k}{\partial t} + \frac{\partial f}{\partial\beta_k}\cdot\frac{\partial\beta_k}{\partial t}\right) = 0.$$

We can find $(n - p)$ independent such functions of the α_k, β_k, for we have from (26)

$$\frac{\partial}{\partial t}(\alpha_k\beta_k) = A_{1k}w_1\frac{\partial H}{\partial w_1} + \dots + A_{pk}w_p\frac{\partial H}{\partial w_p},$$

so that $\dfrac{\partial}{\partial t}(\mu_1\alpha_1\beta_1 + \dots + \mu_n\alpha_n\beta_n) = 0$ provided $\mu_1, \dots \mu_n$ are constants so chosen that

$$\mu_1 A_{k1} + \mu_2 A_{k2} + \dots + \mu_n A_{kn} = 0 \quad (k = 1, \dots p); \dots(27)$$

and since the equations (27) are linearly independent, there are $(n-p)$ independent sets of values for the ratios $\mu_1 : \mu_2 : \ldots : \mu_n$. *Thus the equations* (1) *possess* $(n-p)$ *independent developable integrals, say* $Z_1, \ldots Z_{n-p}$, *not involving* t, *whose expressions in terms of the* α_k, β_k *are* $\mu_1^{(r)}\alpha_1\beta_1 + \ldots + \mu_n^{(r)}\alpha_n\beta_n$, *where* $\mu_1^{(r)} : \ldots : \mu_n^{(r)}$, *for* $r = 1, \ldots (n-p)$, *are the independent solutions of* (27). The initial terms in the development of Z_r are evidently

$$\mu_1^{(r)}x_1y_1 + \ldots + \mu_n^{(r)}x_ny_n.$$

It appears to be true that there are in general no developable integrals independent of H, $Z_1, \ldots Z_{n-p}$, but this has not been proved.

(iv) When $p = 0$, H becomes, when expressed in terms of the α_k, β_k and t by means of (7), a function only of the arguments $u_1, \ldots u_n$; the independent solutions of (27) are

$$\left.\begin{array}{ll} \mu_r = 1 & \\ \mu_s = 0 & (s \neq r) \end{array}\right\} \quad (r = 1, \ldots n),$$

and we have n developable integrals starting with the terms $x_1y_1, \ldots x_ny_n$ respectively. This is the result already reached in Part I, § 3, and H is not independent of these n integrals. When $p > 0$ it may happen that H becomes a function only of $u_1, \ldots u_n$, though for a general form for the series $H_3 + H_4 + \ldots$ we must evidently expect H to depend also on $w_1, \ldots w_p$. In the former case we have from (26)

$$\frac{\partial}{\partial t}(\alpha_1\beta_1) = \ldots = \frac{\partial}{\partial t}(\alpha_n\beta_n) = 0,$$

so that there are n developable integrals not involving t, of the same form as when $p = 0$, and H is not independent of these n integrals. In the latter case H must be independent of $Z_1, \ldots Z_{n-p}$, for these integrals are functions only of $u_1, \ldots u_n$; since however from (24) $\lambda_1 : \lambda_2 : \ldots : \lambda_n$ is a set of ratios of the μ's satisfying (27), we may always suppose that one of $Z_1, \ldots Z_{n-p}$ has for its *second* degree terms $\lambda_1x_1y_1 + \ldots + \lambda_nx_ny_n$, which are the same as the *second* degree terms of H.

The existence of developable integrals not involving t is of course a property of the equations (1) in themselves*, and is in no sense dependent upon the special choice we have made for the arbitrary coefficients in the series (7). The form taken by these integrals when expressed in terms of the constants of integration α_k, β_k will however depend upon this choice, and the procedure of § 1 is necessary[*] to ensure that this form shall be the simple one we have found.

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* It arises out of their Hamiltonian form; for equations $\frac{dx_r}{dt} = X_r(x_1, \ldots x_n)$ there is no integral developable about a singular point unless some restriction is placed upon the form of $X_1, \ldots X_n$.

(v) *Examples of the preceding theory.*

Example I. Let $H = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + a x_1 x_2^2$. The equations

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \frac{\partial H}{\partial y_1} = \lambda_1 x_1, & \frac{dy_1}{dt} &= -\frac{\partial H}{\partial x_1} = -\lambda_1 y_1 - a x_2^2, \\ \frac{dx_2}{dt} &= \frac{\partial H}{\partial y_2} = \lambda_2 x_2, & \frac{dy_2}{dt} &= -\frac{\partial H}{\partial x_2} = -\lambda_2 y_2 - 2a x_1 x_2 \end{aligned}\right\}$$

have the general solution

$$\left.\begin{aligned} x_1 &= \alpha_1 e^{\lambda_1 t}, \quad x_2 = \alpha_2 e^{\lambda_2 t}, \\ y_1 &= \beta_1 e^{-\lambda_1 t} - \frac{a}{2\lambda_2 + \lambda_1} \alpha_2^2 e^{2\lambda_2 t}, \\ y_2 &= \beta_2 e^{-\lambda_2 t} - \frac{2a}{2\lambda_2 + \lambda_1} \alpha_1 \alpha_2 e^{(\lambda_1 + \lambda_2)t}, \end{aligned}\right\}$$

provided $2\lambda_2 + \lambda_1 \neq 0$; but if $2\lambda_2 + \lambda_1 = 0$ the general solution is

$$\left.\begin{aligned} x_1 &= \alpha_1 e^{\lambda_1 t}, \quad x_2 = \alpha_2 e^{\lambda_2 t}, \\ y_1 &= \beta_1 e^{-\lambda_1 t} - a\alpha_2^2 e^{2\lambda_2 t} t, \\ y_2 &= \beta_2 e^{-\lambda_2 t} - 2a\alpha_1 \alpha_2 e^{(\lambda_1 + \lambda_2)t} t. \end{aligned}\right\}$$

In both cases the Lagrange brackets have the standard values (5) as required by the theory. Solving for $\alpha_1 e^{\lambda_1 t}$, etc., we find for the products $\alpha_1 \beta_1$, $\alpha_2 \beta_2$ when $2\lambda_2 + \lambda_1 \neq 0$,

$$\alpha_1 \beta_1 = x_1 y_1 + \frac{a}{2\lambda_2 + \lambda_1} x_1 x_2^2, \quad \alpha_2 \beta_2 = x_2 y_2 + \frac{2a}{2\lambda_2 + \lambda_1} x_1 x_2^2,$$

giving two developable integrals independent of t. On the other hand, when $2\lambda_2 + \lambda_1 = 0$, we find

$$\alpha_1 \beta_1 = x_1 y_1 + a t x_1 x_2^2, \quad \alpha_2 \beta_2 = x_2 y_2 + 2a t x_1 x_2^2,$$

so that
$$2\alpha_1 \beta_1 - \alpha_2 \beta_2 = 2x_1 y_1 - x_2 y_2$$

gives an integral independent of t, the second degree terms being the same as those of H; there is no developable integral independent of t beginning with the second degree terms $x_1 y_1$ or $x_2 y_2$.

Example II. Let $H = -2x_1 y_1 + x_2 y_2 + a x_1 y_1^2 + b x_1 x_2^2$, the equations being

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \frac{\partial H}{\partial y_1} = -2x_1 + 2a x_1 y_1, & \frac{dy_1}{dt} &= -\frac{\partial H}{\partial x_1} = 2y_1 - a y_1^2 - 2b x_1 x_2^2, \\ \frac{dx_2}{dt} &= \frac{\partial H}{\partial y_2} = x_2, & \frac{dy_2}{dt} &= -\frac{\partial H}{\partial x_2} = -y_2 - 2b x_1^2 x_2, \end{aligned}\right\}$$

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[*] I.e. necessary when $\lambda_1, \dots \lambda_n$ are connected by relations of commensurability: when there are no such relations it is unnecessary, as shown in Part I, § 3.

which are not soluble in finite terms. The solution constructed as in § 1 is, to the fifth order,

$$\left.\begin{aligned}
x_1 &= \alpha_1 e^{-2t} + a\alpha_1\beta_1 + \tfrac{1}{4}a^2\alpha_1\beta_1{}^2e^{2t} + ab\alpha_1{}^2\alpha_2{}^2e^{-2t}(2t+1)\\
&\qquad + 2a^2b\alpha_1{}^2\beta_1\alpha_2{}^2(1-t),\\
y_1 &= \beta_1e^{2t} - \tfrac{1}{2}a\beta_1{}^2e^{4t} + \tfrac{1}{4}a^2\beta_1{}^3e^{6t} + b\alpha_1\alpha_2{}^2 - \tfrac{1}{8}a^3\beta_1{}^4e^{8t}\\
&\quad - ab\alpha_1\beta_1\alpha_2{}^2e^{2t}(4t+1) + \tfrac{1}{16}a^4\beta_1{}^5e^{10t} + a^2b\alpha_1\beta_1{}^2\alpha_2{}^2e^{4t}(4t-\tfrac{3}{4}),\\
x_2 &= \alpha_2e^t,\\
y_2 &= \beta_2e^{-t} \qquad + b\alpha_1{}^2\alpha_2e^{-3t} - 4ab\alpha_1{}^2\beta_1\alpha_2e^{-t}t - \tfrac{3}{2}a^2b\alpha_1{}^2\beta_1{}^2\alpha_2e^t.
\end{aligned}\right\}$$

where the terms $ab\alpha_1{}^2\alpha_2{}^2e^{-2t}$, $-ab\alpha_1\beta_1\alpha_2{}^2e^{2t}$ in x_1, y_1 respectively are introduced from the complementary function at the fourth approximation in order to make the third order terms in the Lagrange brackets vanish (cp. § 1 (iii)). Solving for α_1e^{-2t}, etc., we find for $\alpha_1\beta_1$, $\alpha_2\beta_2$ the expressions (correct to the sixth degree)

$$\begin{aligned}
\alpha_1\beta_1 &= x_1y_1 - \tfrac{1}{2}ax_1y_1{}^2 - bx_1{}^2x_2{}^2 + abx_1{}^2y_1x_2{}^2(2t+2)\\
&\qquad - a^2bx_1{}^2y_1{}^2x_2{}^2(3t+\tfrac{3}{2}),\\
\alpha_2\beta_2 &= x_2y_2 \qquad\qquad - bx_1{}^2x_2{}^2 + abx_1{}^2y_1x_2{}^2(4t+2) - 6a^2bx_1{}^2y_1{}^2x_2{}^2t.
\end{aligned}$$

By the theory we know that $-2\alpha_1\beta_1 + \alpha_2\beta_2$ must be independent of t, and we find in fact

$$\begin{aligned}
-2\alpha_1\beta_1 + \alpha_2\beta_2 = -2x_1y_1 + x_2y_2 + ax_1y_1{}^2 + bx_1{}^2x_2{}^2\\
- 2abx_1{}^2y_1x_2{}^2 + 3a^2bx_1{}^2y_1{}^2x_2{}^2
\end{aligned}$$

correct to the sixth degree. The series on the right is an integral independent of t, and evidently, from its form, independent of H; the second degree terms are however the same as those of H.

§ 3. *Construction of the formally developable integrals.*

§ 3·1. The condition that Z, a function of the x_k, y_k but not of t, should be an integral of (1) is

$$(Z, H) = 0, \qquad \ldots\ldots(28)$$

where (Z, H) denotes the Poisson bracket

$$\sum_{r=1}^{n}\left(\frac{\partial Z}{\partial x_r}\cdot\frac{\partial H}{\partial y_r} - \frac{\partial Z}{\partial y_r}\cdot\frac{\partial H}{\partial x_r}\right).$$

We shall show how to construct the developable integrals as solutions of (28), regarded as a partial differential equation for Z. For this purpose substitute for H its expansion

$$H = H_2 + H_3 + H_4 + \ldots,$$

and for Z an expansion

$$Z = Z_s + Z_{s+1} + \ldots, \qquad \ldots\ldots(29)$$

where Z_r is a homogeneous polynomial of degree r in the x_k, y_k. Equating separately to zero the terms of each degree we have

$$(Z_s, H_2) = 0, \qquad \ldots\ldots(30)$$

$$(Z_{s+1}, H_2) + (Z_s, H_3) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

from which we are to determine in succession the homogeneous polynomials $Z_s, Z_{s+1}, \ldots$.

Suppose that $Z_s, \ldots Z_{m-1}$ have been determined as polynomials of degrees $s, \ldots (m-1)$ respectively; then the equation for Z_m is of the form

$$\Sigma\lambda_k\left(x_k\frac{\partial Z_m}{\partial x_k} - y_k\frac{\partial Z_m}{\partial y_k}\right) = (Z_m, H_2) = F(x_1, y_1, \ldots x_n, y_n), \quad (31)$$

where F is a known homogeneous polynomial of degree m. Writing

$$F = \Sigma A x_1^{r_1} y_1^{s_1} \ldots x_n^{r_n} y_n^{s_n},$$

the solution is

$$Z_m = \Sigma \frac{A x_1^{r_1} y_1^{s_1} \ldots x_n^{r_n} y_n^{s_n}}{\lambda_1(r_1 - s_1) + \ldots + \lambda_n(r_n - s_n)},$$

provided that for no term does the denominator vanish. Any term $x_1^{r_1} y_1^{s_1} \ldots x_n^{r_n} y_n^{s_n}$ for which

$$\lambda_1(r_1 - s_1) + \ldots + \lambda_n(r_n - s_n) = 0$$

will be called *critical*, and if F contains critical terms logarithms occur in the solution for Z_m. *Thus for* (31) *to be satisfied by a polynomial it is necessary and sufficient that F should contain no critical terms.* If a term ϕ is critical we have $(\phi, H_2) = 0$; hence all the critical terms of degree m in the x_k, y_k may appear in Z_m with arbitrary coefficients, and from (30) the terms of lowest degree in a developable integral must be critical.

When $p = 0$, i.e. when $\lambda_1, \ldots \lambda_n$ are connected by no relations of commensurability, it has been shown in Part I, § 4, that the coefficients of the critical terms in the equations for the terms of successive degrees in an integral vanish identically. When $p > 0$ this is no longer the case, but the critical terms in the equation for Z_m have coefficients which depend upon arbitrary coefficients which occur in the expressions for $Z_{m-1}, Z_{m-2}, \ldots$; and *the disappearance of the critical terms depends upon the suitable assignment of these arbitrary coefficients**. Since however the number of coefficients we require to annul is in general greater than the number of arbitraries at our disposal, the existence proof of §§ 1, 2 is required to establish the possibility of this procedure.

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* Whittaker (*Proc. Roy. Soc. Edin.* 37, 95) uses this principle in a problem very similar to the present one, but does not give a general proof that by its application all the critical terms can be annulled.

§ 3·2. We shall for the sake of definiteness consider fourth order systems in which $\lambda_1 + 2\lambda_2 = 0$. Since $x_1^{r_1} y_1^{s_1} x_2^{r_2} y_2^{s_2}$ is critical if

$$\lambda_1 (r_1 - s_1) + \lambda_2 (r_2 - s_2) = 0,$$

the critical terms of degree m are obtained by finding the common solutions of the equations

$$r_2 - s_2 = 2 (r_1 - s_1),$$
$$r_1 + s_1 + r_2 + s_2 = m$$

in positive or zero integers. We thus find the critical terms:

$$\left.\begin{array}{l}
\text{degree } 2:\ x_1 y_1,\ x_2 y_2;\\
\text{degree } 3:\ x_1 x_2^2,\ y_1 y_2^2;\\
\text{degree } 4:\ x_1^2 y_1^2,\ x_1 y_1 x_2 y_2,\ x_2^2 y_2^2;\\
\text{degree } 5:\ x_1^2 y_1 x_2^2,\ x_1 x_2^3 y_2,\ x_1 y_1^2 y_2^2,\ y_1 x_2 y_2^3;\\
\text{degree } 6:\ x_1^3 y_1^3,\ x_1^2 y_1^2 x_2 y_2,\ x_1 y_1 x_2^2 y_2^2,\ x_2^3 y_2^3,\ x_1^2 x_2^4,\ y_1^2 y_2^4;\\
\text{degree } 7:\ x_1^3 y_1^2 x_2^2,\ x_1^2 y_1 x_2^3 y_2,\ x_1 x_2^4 y_2^2,\ x_1^2 y_1^3 y_2^2,\ x_1 y_1^2 x_2 y_2^3,\\
\hspace{20em} y_1 x_2^2 y_2^4.\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right\} \quad (32)$$

We know by § 2 that there is always a developable integral independent of H, starting with second degree terms, and that in general these second degree terms are the same as those of H. Since in certain cases however these two sets of second degree terms are independent, we assume for the proposed integral a series

$$Z = c_1 x_1 y_1 + c_2 x_2 y_2 + Z_3 + Z_4 + \ldots, \qquad \ldots\ldots(33)$$

where c_1, c_2 are at present arbitrary constants. The equations to determine Z_3, Z_4, ... are

$$(Z_3, H_2) + (Z_2, H_3) = 0, \qquad \ldots\ldots(34)$$
$$(Z_4, H_2) + (Z_3, H_3) + (Z_2, H_4) = 0, \qquad \ldots\ldots(35)$$
$$(Z_5, H_2) + (Z_4, H_3) + (Z_3, H_4) + (Z_2, H_5) = 0, \quad \ldots(36)$$

……

The arbitrary coefficients c_1, c_2 and those which occur in Z_3, Z_4, ... are to be so assigned that the critical terms vanish from these equations in succession; but we proceed on the principle that no arbitrary is to have its value fixed before this purpose makes it necessary. The result of § 2 assures us that we shall be able to construct in this manner an integral independent of H.

We shall illustrate the working of this principle by constructing for the system whose characteristic function is

$$H = -2x_1 y_1 + x_2 y_2 + a x_1 y_1^2 + b x_1^2 x_2^2$$

an integral independent of H, a and b being non-zero constants.

The equation (34) is here

$$(Z_3, H_2) + (c_1 x_1 y_1 + c_2 x_2 y_2,\ a x_1 y_1^2) = 0,$$

i.e. $$(Z_3, H_2) + c_1 a x_1 y_1^2 = 0,$$

giving $$Z_3 = -\tfrac{1}{2} c_1 a x_1 y_1^2 + c_3 x_1 x_2^2 + c_4 y_1 y_2^2,$$

where c_3 and c_4 are arbitrary constants. Equation (35) is then

$$(Z_4, H_2) + (-\tfrac{1}{2} c_1 a x_1 y_1^2 + c_3 x_1 x_2^2 + c_4 y_1 y_2^2,\ a x_1 y_1^2)$$
$$+ (c_1 x_1 y_1 + c_2 x_2 y_2,\ b x_1^2 x_2^2) = 0.$$

i.e. $$(Z_4, H_2) + 2c_3 a x_1 y_1 x_2^2 - c_4 a y_1^2 y_2^2 - 2b (c_1 + c_2) x_1^2 x_2^2 = 0,$$

giving

$$Z_4 = -c_3 a x_1 y_1 x_2^2 + \tfrac{1}{2} c_4 a y_1^2 y_2^2 - b (c_1 + c_2) x_1^2 x_2^2 + c_5 x_1^2 y_1^2$$
$$+ c_6 x_1 y_1 x_2 y_2 + c_7 x_2^2 y_2^2,$$

where c_5, c_6, c_7 are arbitrary constants. The equation (36) is then

$$(Z_5, H_2) - c_3 a^2 x_1 y_1^2 x_2^2 - c_4 a^2 y_1^3 y_2^2 - 4ab (c_1 + c_2) x_1^2 y_1 x_2^2$$
$$+ 2ac_5 x_1^2 y_1^3 + ac_6 x_1 y_1^2 x_2 y_2 + 2abc_1 x_1^2 y_1 x_2^2 - 2bc_4 x_1 x_2^2 y_2^2$$
$$- 4bc_4 x_1^2 y_1 x_2 y_2 = 0,$$

where the critical term $x_1^2 y_1 x_2^2$ has the coefficient $-2ab (c_1 + 2c_2)$. Since a constant factor in Z is immaterial we must therefore put $c_1 = -2$, $c_2 = 1$, so that the second degree terms Z_2 are identical with H_2. We then have

$$Z_5 = \tfrac{1}{4} c_3 a^2 x_1 y_1^2 x_2^2 + \tfrac{1}{4} c_4 a^2 y_1^3 y_2^2 - ac_5 x_1^2 y_1^3 - \tfrac{1}{2} ac_6 x_1 y_1^2 x_2 y_2$$
$$- bc_4 x_1 x_2^2 y_2^2 - 4bc_4 x_1^2 y_1 x_2 y_2 + c_8 x_1^2 y_1 x_2^2 + c_9 x_1 x_2^3 y_2$$
$$+ c_{10} x_1 y_1^2 y_2^2 + c_{11} y_1 x_2 y_2^3,$$

where $c_8, \dots c_{11}$ are arbitrary constants. The equation for Z_6 is

$$(Z_6, H_2) + (Z_5, H_3) + (Z_4, H_4) = 0,$$

i.e.

$$(Z_6, H_2) - \tfrac{3}{4} c_3 a^3 y_1^4 y_2^2 - a^2 c_5 x_1^2 y_1^4 - 2abc_4 x_1 y_1 x_2^2 y_2^2$$
$$- 12abc_4 x_1^2 y_1^2 x_2 y_2 + 3ac_8 x_1^2 y_1^2 x_2^2 + 2ac_9 x_1 y_1 x_2^3 y_2 - ac_{11} y_1^2 x_2 y_2^3$$
$$+ 2c_3 abx_1^2 x_2^4 - 2c_4 abx_1 y_1 x_2^2 y_2^2 - 2c_4 abx_1^2 y_1^2 x_2 y_2 - 4bc_5 x_1^3 y_1 x_2^2$$
$$- 2bc_6 x_1^2 x_2^3 y_2 - 2bc_6 x_1^3 y_1 x_2^2 - 4bc_7 x_1^2 x_2^3 y_2 = 0.$$

Here the critical terms are $x_1 y_1 x_2^2 y_2^2$, $x_1^2 y_1^2 x_2 y_2$; $x_1^2 x_2^4$, whose coefficients vanish only if $c_3 = c_4 = 0$. We then have

$$Z_6 = \tfrac{1}{4} a^2 c_5 x_1^2 y_1^4 - \tfrac{3}{2} ac_8 x_1^2 y_1^2 x_2^2 - ac_9 x_1 y_1 x_2^3 y_2 + \tfrac{1}{2} ac_{11} y_1^2 x_2 y_2^3$$
$$- b (2c_5 + c_6) x_1^3 y_1 x_2^2 - bc_6 x_1^2 x_2^3 y_2 - 2bc_7 x_1^2 x_2^3 y_2 + c_{12} x_1^3 y_1^3$$
$$+ c_{13} x_1^2 y_1^2 x_2 y_2 + c_{14} x_1 y_1 x_2^2 y_2^2 + c_{15} x_2^3 y_2^3 + c_{16} x_1^2 x_2^4 + c_{17} y_1^2 y_2^4.$$

The critical terms in $(Z_6, H_3) + (Z_5, H_4)$ are found to be

$$-4ab (c_5 + c_6) x_1^3 y_1^2 x_2^2 - 2ab (c_6 + 4c_7) x_1^2 y_1 x_2^3 y_2,$$

which require for their vanishing $c_5 = -c_6 = 4c_7$. Hence the arbitrary terms in Z_4 must be a constant multiple of H_2^2, and we lose no generality* by putting $c_5 = c_6 = c_7 = 0$. We then find from

$$(Z_7, H_2) + (Z_6, H_3) + (Z_5, H_4) = 0,$$

$$\begin{aligned} Z_7 = {} & \tfrac{3}{4}a^2 c_8 x_1^2 y_1^3 x_2^2 + \tfrac{1}{4}a^2 c_9 x_1 y_1^2 x_2^3 y_2 + \tfrac{1}{4}a^2 c_{11} y_1^3 x_2 y_2^3 - b(c_8 + c_9) x_1^3 x_2^4 \\ & - b(2c_{10} + 3c_{11}) x_1^2 y_1 x_2^2 y_2^2 - 2bc_{10} x_1^3 y_1^2 x_2 y_2 - bc_{11} x_1 x_2^3 y_2^3 \\ & + \text{terms in } c_{12}, \dots c_{17} + \text{critical terms with arbitrary coefficients.} \end{aligned}$$

The critical terms in the equation for Z_8, which arise from $(Z_7, H_3) + (Z_6, H_4)$, are

$$\begin{aligned} - 4abc_9 x_1^3 y_1 x_2^4 + 2abc_9 x_1^2 x_2^5 y_2 - 6ab(c_{10} + 2c_{11}) x_1^2 y_1^2 x_2^2 y_2^2 \\ - 8abc_{10} x_1^3 y_1^3 x_2 y_2 - 4abc_{11} x_1 y_1 x_2^3 y_2^3, \end{aligned}$$

which vanish only if $c_9 = c_{10} = c_{11} = 0$. Thus c_8 remains arbitrary, and the series for Z as at present determined is

$$\begin{aligned} Z = {} & - 2x_1 y_1 + x_2 y_2 + ax_1 y_1^2 + bx_1^2 x_2^2 \\ & + c_8 (x_1^2 y_1 x_2^2 - \tfrac{3}{2}ax_1^2 y_1^2 x_2^2 + \tfrac{3}{4}a^2 x_1^2 y_1^3 x_2^2 - bx_1^3 x_2^4) \\ & + \text{terms with undetermined coefficients of the 6th and higher orders.} \end{aligned}$$

This is the integral whose existence has already been demonstrated in § 2 (v), Example II, and the result of that section assures us that c_8 will remain arbitrary throughout the succeeding stages of the process.

The integral $Z - H$ has the expansion

$$Z - H = c_8 (x_1^2 y_1 x_2^2 - \tfrac{3}{2}ax_1^2 y_1^2 x_2^2 + \tfrac{3}{4}a^2 x_1^2 y_1^3 x_2^2 - bx_1^3 x_2^4 + \dots),$$

illustrating the principle that the terms of lowest degree in any developable integral must be critical.

§ 3·3. The system which has just been considered is not of general type, because many terms are missing from the series $H_3 + H_4 + \dots$, and as a consequence the arbitrary coefficients introduced at any stage are not determined until the third succeeding stage. We shall now consider a general fourth order system for which $\lambda_1 + 2\lambda_2 = 0$, so that the critical terms of the first few degrees are given by (32).

(i) We shall use the abbreviated notation introduced in Part I, § 4 (ii). A term

$$x_1^{r_1^a} y_1^{s_1^a} x_2^{r_2^a} y_2^{s_2^a}$$

(a being a superscript, not an exponent) will be written ϕ_a, and the corresponding coefficient in H will be called $A_a^{(m)}$, where m

---

* From an integral $Z = -2x_1 y_1 + x_2 y_2 + ax_1 y_1^2 + bx_1^2 x_2^2 + cH_2^2 + Z_5 + Z_6 + \dots$ we may derive by combination with H the integral $Z - cH^2$, which is of the same form as Z with $c = 0$.

denotes the degree of the term. viz. $r_1^\alpha + s_1^\alpha + r_2^\alpha + s_2^\alpha$; the appropriate denominator will be called D_α, so that

$$D_\alpha = \lambda_1 (r_1^\alpha - s_1^\alpha) + \lambda_2 (r_2^\alpha - s_2^\alpha),$$

and we shall write $$D_\alpha + D_\beta = D_{\alpha\beta}.$$

The term ϕ_α is critical if $D_\alpha = 0$, and conversely.

The denominator appropriate to $\phi_\alpha\phi_\beta$ is $D_{\alpha\beta}$, and so for higher products. We have moreover

$$(\phi_\alpha, \phi_\beta) = \phi_\alpha\phi_\beta \sum_{k=1}^{2} \frac{r_k^\alpha s_k^\beta - r_k^\beta s_k^\alpha}{x_k y_k}, \qquad \text{......(37)}$$

and the right-hand side is the sum of terms for each of which the appropriate denominator is $D_{\alpha\beta}$. There is a similar result for Poisson brackets of third and higher orders.

Lemma on critical terms. If ϕ_α and ϕ_β are critical terms, the product $\phi_\alpha\phi_\beta$ is evidently critical; and conversely if $\phi_\alpha\phi_\beta = \phi_\gamma$, where ϕ_α and ϕ_γ are critical, ϕ_β must also be critical.

Moreover if ϕ_α and ϕ_β are critical, the right-hand side of (37) is the sum of terms each of which is critical; and conversely if $(\phi_\alpha, \phi_\beta) = \psi$, where ϕ_α is a critical term and ψ is a series of critical terms, ϕ_β must be critical.

(ii) Writing (33) for the proposed integral, and $H_3 = \Sigma A_\alpha^{(3)}\phi_\alpha$, the equation (34) is

$$(Z_3, H_2) - \Sigma A_\alpha^{(3)} \{c_1 (r_1^\alpha - s_1^\alpha) + c_2 (r_2^\alpha - s_2^\alpha)\} \phi_\alpha = 0. \qquad \text{...(38)}$$

If the critical terms $x_1 x_2^2$, $y_1 y_2^2$ have the coefficients a, b respectively in H_3, their coefficients in (38) are $-a(c_1 + 2c_2)$, $b(c_1 + 2c_2)$ respectively. Since in general a and b are not zero we must have $c_1 + 2c_2 = 0$, so that the second degree terms in Z are to a constant factor (which may of course be suppressed) the same as those in H. We shall confine attention to the general case in which a and b are not both zero; the integral $Z - H$ will then start with terms of higher degree than 2, and this is the integral we shall construct. Since its lowest degree terms must be critical we write (calling now this proposed integral Z)

$$Z = Z_3 + Z_4 + \dots,$$

where $$Z_3 = \alpha x_1 x_2^2 + \beta y_1 y_2^2$$

and α, β are constants. The equations for Z_4, Z_5, ... are then

$$(Z_4, H_2) + (Z_3, H_3) = 0, \qquad \text{......(39)}$$

$$(Z_5, H_2) + (Z_4, H_3) + (Z_3, H_4) = 0. \qquad \text{......(40)}$$

....

(iii) *Determination of* Z_4. If (39) is to be soluble by a polynomial for Z_4 the critical terms must vanish from (Z_3, H_3). The critical terms in H_3 are

$$a x_1 x_2^2 + b y_1 y_2^2, = H_3', \text{ say;}$$

hence by the Lemma in (i), since Z_3 consists entirely of critical terms, the required critical terms are

$$(\alpha x_1 x_2^2 + \beta y_1 y_2^2,\ a x_1 x_2^2 + b y_1 y_2^2) = (\alpha b - \beta a)(x_2^2 y_2^2 + 4 x_1 y_1 x_2 y_2).$$

These vanish only if $\alpha/\beta = a/b$; since a constant factor is immaterial we must accordingly put

$$Z_3 = a x_1 x_2^2 + b y_1 y_2^2 = H_3', \qquad \text{......(41)}$$

since we have supposed that a and b are not both zero. The equation (39) is now

$$(Z_4, H_2) + \Sigma A_\alpha^{(3)} (H_3', \phi_\alpha) = 0,$$

where the denominator appropriate to each term in the expansion of (H_3', ϕ_α) is D_α, since H_3', being critical, has zero for the appropriate denominator; thus

$$Z_4 = \Sigma \frac{A_\alpha^{(3)} (\phi_\alpha, H_3')}{D_\alpha}, \qquad \text{......(42)}$$

where the terms with vanishing denominators are to be replaced by the terms

$$A x_1^2 y_1^2 + B x_1 y_1 x_2 y_2 + C x_2^2 y_2^2, \qquad \text{...... (43)}$$

where A, B, C are arbitrary coefficients.

(iv) *Determination of Z_5.* We consider first the critical terms in

$$(Z_4, H_3) + (Z_3, H_4).$$

Since $Z_3 = H_3'$, the sum of two critical terms, the critical terms in (Z_3, H_4) are, by the Lemma,

$$(H_3',\ p x_1^2 y_1^2 + q x_1 y_1 x_2 y_2 + r x_2^2 y_2^2), \qquad \text{......(44)}$$

where p, q, r are the coefficients of the indicated terms in H_4. The critical terms in (Z_4, H_3) are of two types: we have, firstly, those which arise from the Poisson bracket of a critical term of Z_4 with a critical term of H_3, viz. from (43)

$$(A x_1^2 y_1^2 + B x_1 y_1 x_2 y_2 + C x_2^2 y_2^2,\ H_3'); \qquad \text{......(45)}$$

and, secondly, we have those arising from non-critical terms of Z_4, H_3, viz. from (42)

$$\Sigma \frac{A_\alpha^{(3)} A_\beta^{(3)} ((\phi_\alpha, H_3'), \phi_\beta)}{D_\alpha}, \qquad \text{......(46)}$$

the summation being over permutations of pairs of terms $A_\alpha^{(3)}\phi_\alpha$, $A_\beta^{(3)}\phi_\beta$ such that the expansion of $((\phi_\alpha, H_3'), \phi_\beta)$ consists of critical terms. Now if these terms are critical it is easily seen that the product $\phi_\alpha H_3' \phi_\beta$ must be critical, and hence by the Lemma that $\phi_\alpha \phi_\beta$ must be critical. Thus if ϕ_α is critical, so also in ϕ_β, and moreover if $\phi_\alpha = \phi_\beta$ both are critical; hence since ϕ_β in (46) is not critical we must have $\phi_\alpha \neq \phi_\beta$ and $D_\alpha \neq 0$. The terms in (46) involving the product $A_\alpha^{(3)} A_\beta^{(3)}$ are accordingly

$$A_\alpha^{(3)} A_\beta^{(3)} \left\{ \frac{((\phi_\alpha, H_3'), \phi_\beta)}{D_\alpha} + \frac{((\phi_\beta, H_3'), \phi_\alpha)}{D_\beta} \right\}$$

$$= \frac{A_\alpha^{(3)} A_\beta^{(3)}}{D_\alpha} \{((\phi_\alpha, H_3'), \phi_\beta) - ((\phi_\beta, H_3'), \phi_\alpha)\}^*$$

$$= \frac{A_\alpha^{(3)} A_\beta^{(3)}}{D_\alpha} ((\phi_\alpha, \phi_\beta), H_3'),$$

from the identity

$$((\phi_\alpha, \phi_\beta), H_3') + ((\phi_\beta, H_3'), \phi_\alpha) + ((H_3', \phi_\alpha), \phi_\beta) = 0.$$

But from (37), since $\phi_\alpha \phi_\beta$ is critical and of the sixth degree, every term in $(\phi_\alpha, \phi_\beta)$ is critical and of the fourth degree; hence the terms (46) are of the form

$$(p' x_1^2 y_1^2 + q' x_1 y_1 x_2 y_2 + r' x_2^2 y_2^2, H_3'). \qquad \text{......(47)}$$

From (44), (45), (47) we see that all the critical terms in $(Z_4, H_3) + (Z_3, H_4)$ can be collected into the expression

$$((A - \xi)\, x_1^2 y_1^2 + (B - \eta)\, x_1 y_1 x_2 y_2 + (C - \zeta)\, x_2^2 y_2^2, H_3'),$$

where ξ, η, ζ are determinate constants, and A, B, C are arbitrary constants. Putting then

$$A = \xi, \quad B = \eta, \quad C = \zeta,$$

we make these critical terms vanish; being of the fifth degree they are, from (32), four in number, but their coefficients can be annulled by proper choice of the three arbitraries A, B, C. In point of fact we can leave one of A, B, C arbitrary, for since $(H_2, H_3') = 0$, we have

$$(H_2^2, H_3') = 2H_2 (H_2, H_3') = 0,$$

so that the terms (43) remain indeterminate to an arbitrary constant multiple of H_2^2.

This reasoning breaks down if $H_3' = 0$ and

$$Z_3 = \alpha x_1 x_2^2 + \beta y_1 y_2^2,$$

where α, β remain arbitrary, for then the critical terms involving the arbitraries A, B, C vanish from (Z_4, H_3).

Having shown that the critical terms in (40) vanish we may find for Z_5 a fifth degree polynomial which satisfies the equation, the critical terms, say

$$A x_1^2 y_1 x_2^2 + B x_1 x_2^3 y_2 + C x_1 y_1^2 y_2^2 + D y_1 x_2 y_2^3,$$

having their coefficients A, B, C, D arbitrary.

---

* Since the product $\phi_\alpha \phi_\beta$ is critical we have $D_{\alpha\beta} = 0$, i.e. $D_\alpha + D_\beta = 0$.

(v) *Determination of Z_6.* In the equation for Z_6 the critical terms involving the arbitraries A, B, C, D arise from (Z_5, H_3), and are

$$(Ax_1^2y_1x_2^2 + Bx_1x_2^3y_2 + Cx_1y_1^2y_2^2 + Dy_1x_2y_2^3, H_3').$$

It may be shown without much difficulty that the remaining critical terms can be collected into an expression of the form

$$(\xi x_1^2y_1x_2^2 + \eta x_1x_2^3y_2 + \zeta x_1y_1^2y_2^2 + \omega y_1x_2y_2^3, H_3'),$$

where ξ, η, ζ, ω are determinate constants; the proof involves the transformation of sums of third and fourth order Poisson brackets in a manner somewhat similar to that in Part I, § 4 (iv). Putting then

$$A = \xi, \quad B = \eta, \quad C = \zeta, \quad D = \omega,$$

we annul all the critical terms, and Z_6 may be determined as a sixth degree polynomial. Moreover, since

$$(H_2H_3', H_3') = H_2(H_3', H_3') + H_3'(H_2, H_3') = 0,$$

the constants A, B, C, D are not completely determinate, but Z_5 remains indeterminate to a constant multiple of the fifth degree polynomial H_2H_3'.

The functions Z_7, Z_8, ... are to be determined in a similar manner: but the preceding verification that the critical terms may be made to disappear from the equations for Z_4, Z_5, Z_6 does not appear to be susceptible of generalisation.

(vi) *There is no developable integral independent of Z and H.* We have seen that Z_4 and Z_5 are indeterminate only to polynomials which are functions of H_2 and H_3'; we shall now show that the terms Z_p of the pth degree are indeterminate only to the same extent.

The arbitrary coefficients in Z_p are determined so that the equation

$$(Z_{p+1}, H_2) + (Z_p, H_3) + \dots + (Z_3, H_p) = 0$$

may contain no critical terms, and hence since the critical terms in H_3 are H_3', Z_p is indeterminate to a homogeneous polynomial, Φ say, of critical terms of degree p such that

$$(\Phi, H_3') = 0. \qquad \text{......(48)}$$

Since it consists of critical terms Φ is a function only* of the arguments u_1, u_2, w, where

$$u_1 = x_1y_1, \quad u_2 = x_2y_2, \quad w = x_1x_2^2,$$

and by means of the relations

$$H_2 = \lambda_1u_1 + \lambda_2u_2, \; H_3' = ax_1x_2^2 + by_1y_2^2 = aw + \frac{bu_1u_2^2}{w}$$

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* Compare § 2 (ii) above.

we may express Φ as a function of H_2, H_3', u_2. Hence (48) becomes

$$\frac{\partial \Phi}{\partial H_2}(H_2, H_3') + \frac{\partial \Phi}{\partial H_3'}(H_3', H_3') + \frac{\partial \Phi}{\partial u_2}(u_2, H_3') = 0,$$

i.e.
$$2(by_1y_2^2 - ax_1x_2^2)\frac{\partial \Phi}{\partial u_2} = 0;$$

thus Φ is independent of u_2, and therefore a function only of H_2, H_3'.

Having seen that the only terms which can remain arbitrary in $Z_4, Z_5, \ldots$ are functions of H_2 and H_3', we may show by a method precisely similar to that of Part I, § 5 (i) that the most general solution by power series of the equation $(Z, H) = 0$ is a function of any two solutions of this sort of which the initial terms are H_2, H_3' respectively. Thus there is no developable integral independent of Z and H.

§ 4. *Existence of developable integrals when the exponents are not all non-zero and unequal.*

When two or more of $\lambda_1, \ldots \lambda_n$ are equal, or when one or more of these quantities is zero, it is impossible in general by a linear contact transformation to reduce the second degree terms of H to the form $\lambda_1x_1y_1 + \ldots + \lambda_nx_ny_n$. The conditions will be sufficiently illustrated* by supposing that the system is of the tenth order and that the singular point has four zero exponents, two pairs of equal exponents $\pm\lambda_1, \pm\lambda_1$, and the remaining exponents $\pm\lambda_3$ unequal to $\pm\lambda_1$; then the simplest form to which H_2 is reducible by a linear contact transformation is

$$\begin{aligned} H_2 = \lambda_1(x_1y_1 + x_2y_2) + \mu x_1y_2 + \lambda_3x_3y_3 \\ + \tfrac{1}{2}px_4^2 + x_4(qx_5 + ry_5) + \tfrac{1}{2}sx_5^2, \end{aligned}$$

where μ, p, q, r, s are constants which do not in general vanish. Now the equations

$$\frac{dx_k}{dt} = \frac{\partial H_2}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial H_2}{\partial x_k} \quad (k = 1, \ldots 5)$$

are found to have the general solution

$$\begin{array}{ll} x_1 = \alpha_1 e^{\lambda_1 t}, & y_1 = \beta_1 e^{-\lambda_1 t} - \mu t\beta_2 e^{-\lambda_1 t}, \\ x_2 = \alpha_2 e^{\lambda_1 t} + \mu t\alpha_1 e^{\lambda_1 t}, & y_2 = \beta_2 e^{-\lambda_1 t}, \\ x_3 = \alpha_3 e^{\lambda_3 t}, & y_3 = \beta_3 e^{-\lambda_3 t}, \\ x_4 = \alpha_4, & y_4 = \beta_4 - p\alpha_4 t - q\alpha_5 t - r\beta_5 t + \tfrac{1}{2}rs\alpha_5 t^2 \\ & \qquad + \tfrac{1}{6}r^2s\alpha_4 t^3, \\ x_5 = \alpha_5 + r\alpha_4 t, & y_5 = \beta_5 - q\alpha_4 t - s\alpha_5 t - \tfrac{1}{2}rs\alpha_4 t^2, \end{array}$$

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* The argument which follows is clearly of a general type.

whence we may show

$$\left.\begin{aligned} &[\alpha_r, \beta_r] = 1, && (r = 1, \dots 5), \\ &[\alpha_r, \beta_s] = 0, && (r, s = 1, \dots 5; r \neq s), \\ &[\alpha_r, \alpha_s] = [\beta_r, \beta_s] = 0, && (r, s = 1, \dots 5). \end{aligned}\right\} \quad \dots\dots(49)$$

Moreover, the equations

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k} \quad (k = 1, \dots 5)$$

can be formally solved by power series in the arguments

$$\left.\begin{matrix} \alpha_1 e^{\lambda_1 t}, \ \beta_1 e^{-\lambda_1 t}, \ \alpha_2 e^{\lambda_1 t}, \ \beta_2 e^{-\lambda_1 t}, \ \alpha_3 e^{\lambda_3 t}, \ \beta_3 e^{-\lambda_3 t}, \\ \alpha_4, \beta_4, \alpha_5, \beta_5, t. \end{matrix}\right\} \quad \dots\dots(50)$$

It may be shown by a method precisely similar to that of § 1 that these series may always be supposed such that the relations (49) are satisfied. The demonstration turns upon the fact that only terms of certain types can appear in the series for the Lagrange brackets, and that corresponding to each such term there is a term with an arbitrary coefficient in the series for the x_k, y_k. From considerations of space the details are omitted.

Now suppose that from the general solution we solve for the arguments (50) (omitting t) as power series in the x_k, y_k, t, thus obtaining for the α_k, β_k series of the form (21) with $\lambda_1 = \lambda_2$ and $\lambda_4 = \lambda_5 = 0$. The partial derivatives of these series with respect to t are then, in virtue of (49), given by the equations (22), where H is a power series in the arguments (50) which must be independent of t.

Case I. *λ_1 and λ_2 incommensurable.* The typical term in H,

$$A\,(\alpha_1 e^{\lambda_1 t})^{a_1}\,(\beta_1 e^{-\lambda_1 t})^{b_1} \dots (\beta_3 e^{-\lambda_3 t})^{b_3}\,\alpha_4{}^{a_4}\,\beta_4{}^{b_4}\,\alpha_5{}^{a_5}\,\beta_5{}^{b_5} t^m,$$

where the a_k, b_k are positive or zero integers, is independent of t only if $m = 0$ and

$$(a_1 + a_2 - b_1 - b_2)\,\lambda_1 + (a_3 - b_3)\,\lambda_3 = 0,$$

so that we must have $\qquad a_3 = b_3,$

$$a_1 + a_2 = b_1 + b_2;$$

the typical term therefore becomes

$$A\,(\alpha_1\beta_1)^{b_1}\,(\alpha_2\beta_2)^{a_2}\,(\alpha_1\beta_2)^{b_2 - a_2}\,(\alpha_3\beta_3)^{b_3}\,\alpha_4{}^{a_4}\,\beta_4{}^{b_4}\,\alpha_5{}^{a_5}\,\beta_5{}^{b_5},$$

showing that H is a function only of the eight arguments u_1, u_2, u_3, v, α_4, β_4, α_5, β_5, where

$$u_k = \alpha_k\beta_k \quad (k = 1, 2, 3),$$
$$v = \alpha_1\beta_2.$$

The equations (22) are thus

$$\left.\begin{array}{ll}
\dfrac{\partial\alpha_1}{\partial t} = -\alpha_1\dfrac{\partial H}{\partial u_1}, & \dfrac{\partial\beta_1}{\partial t} = \beta_1\dfrac{\partial H}{\partial u_1} + \beta_2\dfrac{\partial H}{\partial v}, \\
\dfrac{\partial\alpha_2}{\partial t} = -\alpha_2\dfrac{\partial H}{\partial u_2} - \alpha_1\dfrac{\partial H}{\partial v}, & \dfrac{\partial\beta_2}{\partial t} = \beta_2\dfrac{\partial H}{\partial u_2}, \\
\dfrac{\partial\alpha_3}{\partial t} = -\alpha_3\dfrac{\partial H}{\partial u_3}, & \dfrac{\partial\beta_3}{\partial t} = \beta_3\dfrac{\partial H}{\partial u_3}, \\
\dfrac{\partial\alpha_k}{\partial t} = -\dfrac{\partial H}{\partial\beta_k}, & \dfrac{\partial\beta_k}{\partial t} = \dfrac{\partial H}{\partial\alpha_k} \quad (k = 4, 5),
\end{array}\right\} \quad \ldots\ldots(51)$$

whence, eliminating the derivatives of H,

$$\frac{\partial}{\partial t}(\alpha_1\beta_1 + \alpha_2\beta_2) = \frac{\partial}{\partial t}(\alpha_3\beta_3) = 0.$$

Hence there are two developable integrals not involving t, their terms of lowest degree being $x_1y_1 + x_2y_2$, x_3y_3 respectively. These integrals are in general independent of H, for in the contrary case H is in terms of the α_k, β_k a function only of $u_1 + u_2$ and u_3; should this be so however the equations (51) show that

$$\alpha_1\beta_1,\ \alpha_2\beta_2,\ \alpha_3\beta_3,\ \alpha_1\beta_2,\ \alpha_2\beta_1,\ \alpha_4,\ \beta_4,\ \alpha_5,\ \beta_5$$

give power series in the x_k, y_k independent of t, and we have eight* independent developable integrals, seven of which are independent of H.

Case II. *λ_1 and λ_3 commensurable.* If $A_1\lambda_1 + A_3\lambda_3 = 0$, where A_1, A_3 are integers mutually prime, H is in general a function of the argument $\alpha_1^{A_1}\alpha_3^{A_3}$ as well as of u_1, u_2, u_3, v, α_4, β_4, α_5, β_5. The first six of (51) must then be replaced by equations which lead as in § 2 to

$$\frac{\partial}{\partial t}\{A_3(\alpha_1\beta_1 + \alpha_2\beta_2) - A_1\alpha_3\beta_3\} = 0.$$

Hence there is a developable integral independent of t, its terms of lowest degree being $A_3(x_1y_1 + x_2y_2) - A_1x_3y_3$; this integral is in general independent of H.

In Case I the exponents $\pm\lambda_1, \ldots \pm\lambda_5$ are connected by the three independent relations of commensurability

$$\lambda_1 - \lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_5 = 0,$$

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* Since $\alpha_2\beta_1 = \alpha_1\beta_1 . \alpha_2\beta_2/\alpha_1\beta_2$, the fifth series is a function of the first, second and fourth.

and we have at least two developable integrals independent of H. In Case II we have the additional relation

$$A_1\lambda_1 + A_3\lambda_3 = 0,$$

and have found only one such integral. In both cases therefore we verify the general result that when there are p independent relations of commensurability there are, besides H, $(n-p)$ developable integrals not involving t.

The Acceleration of β-particles in Strong Electric Fields such as those of Thunderclouds. By Professor C. T. R. Wilson, Sidney Sussex College.

[*Read* 24 November 1924.]

As is well illustrated by photographs of their tracks, the majority of β-particles of given initial velocity have approximately the same range, measured along the path. When the track shows a visible branch, indicating that an exceptionally close encounter has resulted in the ejection of an electron with kinetic energy amounting to an appreciable fraction of that of the primary particle, the main track is naturally of less than the average range. A diminution of range may also be expected to follow such of the nuclear deflections as result in the emission of X radiation; this decrease in range indeed affords a possible method of measuring the energy of the radiation emitted in individual cases and so testing theories such as that of Kramers* as to the nature and origin of the continuous X-ray spectrum.

In air at atmospheric pressure a typical β-ray track without visible branches or large nuclear deflections has a length of 1 cm. when the initial velocity corresponds to about 20,000 volts†. The average rate of loss of energy by the particle, as deduced from the change of range for a given change of initial energy, is about 10,000 volts per cm. when the kinetic energy of the β-particle corresponds to 20,000 volts. Thus a field of 10,000 volts per cm. applied in the direction of motion will supply to a β-particle, of which the velocity corresponds to 20,000 volts, energy just sufficient to counterbalance the average losses. In a field of 20,000 volts per cm., i.e. twice the critical value (still only two-thirds of the sparking limit), the energy supplied to β-particles of velocity corresponding to 20,000 volts will exceed the average loss by the equivalent of about 10,000 volts per cm., so that the particle will be continually accelerated until some accident occurs, i.e. until there is some exceptionally close encounter with an electron or a nucleus.

The mean rate of loss of energy of a β-particle varies, in accordance with Thomson's theory‡, inversely as the square of the velocity; the rate of loss of energy per cm. by the accelerated particle will thus continually diminish, approaching however a limit of rather less than 1000 volts per cm. as the velocity becomes comparable with that of light. Thus an accelerated particle which has travelled more than a comparatively small number of centimetres under the action of a field of 20,000 volts per cm. will gain in each centimetre energy equivalent to the greater part of the whole 20,000 volts; and even if it arrives in a region where the field is only 2000 volts per cm. it will still continue to gain energy at the rate of about 1000 volts per cm.

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* *Phil. Mag.* 46, p. 836, 1923.

† *Proc. Roy. Soc.* A, 104, p. 192, 1923.

‡ *Phil. Mag.* 23, p. 449, 1912.

If the direction of motion is not along the direction of the electric field, the particle will be accelerated if the component of the electric force resolved along the path exceeds the critical value corresponding to the velocity of the particle. Thus a 20,000 volt particle will be accelerated in a field of 20,000 volts per cm. if its direction of motion lies anywhere within 60° of that of the field. A fast particle (e.g. one which requires only 1000 volts per cm. to accelerate it) will be accelerated in a field of 20,000 volts per cm., even if its divergence from the most favourable direction falls only a few degrees short of 90°; even if this divergence should initially be equal to or exceed 90°, a β-particle of large kinetic energy may after deflection into a forward direction by the field still retain more than the critical energy required to cause acceleration.

Consider now the extent to which close electron encounters, such as result in the formation of branch tracks, may interfere with the continuous acceleration of the β-particle. Direct examination of the tracks of β-particles showed that the number of electrons ejected with velocity corresponding to more than 3300 volts by a β-particle of velocity 9×10^9 cm. per sec. (corresponding to 25,000 volts) amounts to about 0·2 per cm. This is in good agreement with Thomson's theory. According to this theory the number of electrons ejected, with a given velocity, per cm. of the path varies inversely as the square of the velocity of the β-particle. In the field we have been considering, of 20,000 volts per cm., it is thus comparatively rarely that the excess of gain over loss of energy in 1 cm. of the track of the β-particle will fall short of 10,000 volts by more than a few thousand volts. When the energy of the β-particle exceeds 40,000 volts, then even in the extreme case of an encounter in which the original particle gives half its energy to the ejected electron and is diverted through 45°, the energy remaining will exceed the critical value required for acceleration; there will in this case be two β-particles after the encounter, for each of which the component of the field along the path is more than sufficient to make the gain of energy per cm. exceed the average rate of loss.

Experimental data are lacking as to the frequency of ejection of electrons with given energy by β-particles of which the velocity approaches that of light; but let us assume that the Thomson-Bohr theory continues to be applicable. Then the chance of an encounter resulting in a given loss of energy will be about 10 times as small for such a particle as for one of velocity 9×10^9; an encounter in which the loss exceeds 3300 volts will now occur about once in 50 cm. In 50 cm. the excess of energy supplied by the field over the average loss exceeds 50 × 10,000 volts.

We may conclude that the occurrence of exceptional electron encounters has no important effect in preventing the acquisition of large kinetic energy by particles in a strong accelerating field.

Consider next the possible effect of nuclear deflections in preventing the continuous gain of energy by a particle under the action of the field.

Observations on the tracks show that up to velocities approaching 10^{10} cm. per second the chance of a β-particle being deflected through more than a right angle by a single encounter in the course of 1 cm. of its path is in good agreement with Rutherford's theory of scattering. There is a chance of about 1 in 15 that a β-particle moving with a velocity of 9×10^9 cm. per second (corresponding to about 25,000 volts) will suffer a sudden deflection exceeding 90° in 1 cm. of its path in air at normal pressure.

According to Rutherford's theory of scattering* in its simplest form, the deflection of the β-particle by the nucleus is given by

$$\tan\frac{\theta}{2} = \frac{Ze^2}{mu^2p},$$

where m and u are the mass and velocity of the particle, p is the perpendicular distance of the original undeflected path from the nucleus, Z the atomic number and e the electronic charge. Any effect due to change of mass of the electron as it approaches the nucleus is neglected. In order that a nitrogen nucleus should deflect a particle of velocity 9×10^9 cm. per sec. (corresponding to 25,000 volts) through 90° or more, p must not exceed 2×10^{-11} cm. This closeness of approach to a nucleus may be expected to occur in air at atmospheric pressure about once in 15 cm. of path. This, as we have seen, is in agreement with observations on the tracks. Now in a field of 20,000 volts per cm., the accelerated particle is gaining energy at the rate of more than 10,000 volts per cm., and before 15 cm. has been traversed its kinetic energy will exceed 170,000 volts, its velocity 2×10^{10} cm. per sec. and its mass $1{\cdot}33m_0$. Thus mu^2 will have been increased 5 times and the radius of the target which has to be hit to cause deflection through a right angle is in consequence diminished to 4×10^{-12} cm.; the probable length of path before such an encounter exceeds 3 metres. Thus as the β-particle gains energy under the accelerating action of the field its chance of suffering a nuclear deflection through so large an angle that it ceases to gain energy from the field diminishes rapidly. To be deflected through 90° a particle of energy corresponding to a million volts would, if the above expression continued to hold, have to pass within about 7×10^{-13} cm. of the nucleus of a nitrogen atom, a distance comparable with the diameter of an electron. Thus if its energy exceeds the equivalent of a few million volts it is only when a particle actually "hits" a nucleus (i.e. when its line of flight passes within a distance of the nucleus so small that the inverse square law of force ceases to hold) that large deflections can occur. In air at atmospheric pressure an approach within 3×10^{13} cm. of a nucleus of nitrogen or oxygen will occur about once in 600 metres. The path of a β-particle will approach within this distance of one of the electrons of an atom about 7 times as frequently.

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* *Phil. Mag.* 21, p. 669, 1911.

The sudden deflection of a particle may interfere with the accelerating action of the field not only by turning the particle into an unfavourable direction but also by causing loss of energy in radiation. The question of the probability of emission of a quantum of any given frequency has been considered by Kramers (for the case of comparatively small velocities). The chance of a large loss of energy by a nuclear deflection is small, in accordance with the small efficiency of an X-ray tube in producing the radiation of the continuous spectrum. The number of β-particles put out of action by losing so much energy in radiation that their velocity falls below the critical value corresponding to the field is likely to be small. Some X and γ radiation must be produced and the upper limit of frequency will depend on the maximum energy which it is possible for a particle to acquire in the field.

The general effect of an accelerating field is that a β-particle, instead of dying as it were a natural death by gradual loss of energy, is continually acquiring more and more energy and increasing its chance of surviving all accidents other than direct encounters with the nuclei of atoms.

In the strongest parts of the field of a thundercloud the electric force must approach very near to the sparking limit. The whole potential difference may be of the order of 10^9 volts, and a large part of the potential fall will occur in the strong part of the field. A particle starting with suitable velocity and direction in the stronger part of the field may be expected to continue to acquire kinetic energy at the rate of many thousand volts per cm. until put out of action by some exceptional encounter; and the chance of this occurring otherwise than as a result of actual collisions with electrons or nuclei will continually diminish as the energy of the particle increases. It becomes an interesting question whether some of the particles accelerated by the field may gain energy approaching the equivalent of 10^9 volts. The energy acquired by such a particle would be comparable with the energy Mc^2 of the hydrogen nucleus*; its effective mass would thus also be comparable with that of the nucleus. It is not impossible that such a particle may cause nuclear disintegrations like an α-particle, or unite with a nucleus.

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* This is the equivalent of 9×10^8 volts.

The air carried up from the lower atmosphere contains radium emanation and its products. It follows from the results of measurements on the radium content of the lower atmosphere* that the number of β-particles emitted from the radioactive products of the emanation must be of the order of 10 per sec. per cubic metre. These β-particles are emitted with velocities greatly exceeding the minimum required for acceleration in the strongest parts of the field of a thundercloud; they will be accelerated even if they have initially a direction nearly at right angles to the field. In addition to the primary β-particles there must be many secondary β-particles of which some will have velocities and directions such that acceleration occurs. It would be of interest to test by direct experiment whether a thundercloud does emit any measurable amount of extremely penetrating radiation of β- or γ-ray type. There is, as is well known, some evidence of the existence of such penetrating radiation in the atmosphere; possibly some small portion of it may originate in the electric fields of thunderclouds.

It has been assumed throughout that the air is at normal pressure. It is obvious, however, that the results are not essentially different at other pressures; at any pressure a field of two-thirds of the sparking limit will give results identical with those which follow from a field of 20,000 volts per cm. at normal pressure, the linear scale of the whole system of β-rays varying inversely as the pressure. Many of the β-rays which originate in the electric field above a thundercloud may thus have extremely long paths.

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* Satterly, *Phil. Mag.* 16, p. 584, 1908; Eve, *Phil. Mag.* 16, p. 622, 1908.

The Determination of the Frequency of an Alternating Current Supply by the Vibrations of Rods. By Dr G. F. C. SEARLE, Peterhouse.

[*Read* 24 November 1924.]

§ 1. *Introduction.* One method of measuring frequency depends upon resonance. If a body of unit mass, subject to elastic constraints and experiencing a damping resistance proportional to its speed, be acted on by a periodic force $F \sin pt$, the equation of motion may be written

$$\frac{d^2x}{dt^2} + 2s\frac{dx}{dt} + q^2x = F\sin pt.$$

The amplitude of the steady vibration ultimately set up is

$$F\{(q^2 - p^2)^2 + 4s^2p^2\}^{-\frac{1}{2}},$$

the maximum value occurring when $p^2 = q^2 - 2s^2$. If the motion, in the absence of the driving force $F \sin pt$, be only slightly damped, s^2 is very small compared with q^2, and little error will be made if the value of p giving maximum amplitude be taken as equal to q. When s/q is small, the resonance is "sharp," i.e. a small discrepancy between p and q, or, more strictly, between p and $(q^2 - 2s^2)^{\frac{1}{2}}$, causes the amplitude to fall far below its maximum value. Thus, when the damping is small, maximum amplitude, for variation of p, indicates that p is very nearly equal to q.

In the following experiments, resonance is used in measuring the frequency of an alternating current supply. The method of calibrating the vibrating rod by means of tuning forks affords a useful exercise for students desiring to acquire skill in acoustical experiments.

§ 2. *Method.* A thin steel rod AB (Fig. 1), assumed uniform, is held in a firm clamp C, so that the part OA projects. A rod of the "silver" steel, about 0·25 cm. in diameter, used for making small tools, is suitable for the experiment. Near the rod is placed a small electromagnet M, adjustable in position, and the alternating current, whose frequency is to be found, traverses the windings on M. The current from the mains passes through the primary of a transformer; the secondary, giving about 8 volts, is connected to the windings on M. If the core of M be formed of a bundle of iron wires, eddy current heating will be diminished. A non-conducting bobbin should be used for the coil of the electromagnet; if a conducting bobbin be used, powerful currents will be set up in it.

A coil G, shewn in section, is threaded over the rod when necessary and can be joined to a battery, a resistance and a key. The coil does not touch the rod. This coil is used to magnetise the rod when desired.

A self-centering three-jaw lathe chuck forms a convenient clamp and allows the length OA to be easily adjusted. The chuck may be fixed to a massive support by bolts screwing into the holes drilled and tapped in the chuck by the makers.

When the length of OA is suitably adjusted, the rod will be thrown into strong vibration by the action of the electromagnet. The resonance is sharply marked and may be missed unless the length OA be adjusted by small steps. If the rod be just not clamped, so that it can slide through the chuck, it exhibits marked "uneasiness" when OA is approximately of the proper length for resonance. The engines at the power station do not run at an absolutely constant speed, and it may happen that a length which gives strong resonance at one moment fails to do so shortly afterwards. The changes of frequency are, however, generally small.

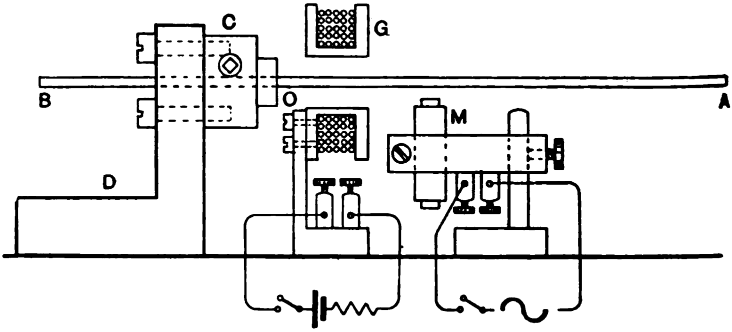

Fig. 1

When the magnetising coil G is not used, the steel rod is magnetised by induction by the electromagnet M and consequently will be strongly attracted to M whenever the strength of M, irrespective of sign, has a maximum value. The action on the rod can thus be represented by a constant attraction on which is superposed an alternating force making twice as many cycles per second as the alternating current. If the current have frequency N, so that it makes N cycles per second, the alternating force will have frequency $2N$.

If a continuous current be sent through G, the rod will be subjected to two magnetising forces, one constant and the other alternating. Hence the mechanical action on the rod will be the resultant of three forces, one constant, one of frequency N and one of frequency $2N$. If the rod be suitably adjusted, it will respond to the force of frequency N, the response increasing with the current in G.

Sometimes the rod is sufficiently magnetised to vibrate with frequency N under the action of the electromagnet, though no current flows through G. This state does not, as a rule, continue long, since the alternating magnetic force shakes out the "permanent" magnetism. A current sent through G makes the response certain.

If n be the frequency of transverse vibrations of a uniform rod of circular section, clamped at one end, as in Fig. 1,

$$n = m^2 \cdot \frac{r}{4\pi l^2} \sqrt{\frac{E}{\rho}}, \qquad \ldots\ldots(1)$$

where r is the radius of the cross-section, l is the length of OA, E is Young's modulus, ρ is the density and m is a numerical constant (see § 8). The number m is a root of the equation

$$\cos m \,.\, \cosh m = -1. \qquad \ldots\ldots(2)$$

The successive values of m, viz. $m_0, m_1, m_2, \ldots$, correspond to no node, one node, two nodes, ... Table I gives $m_0, m_1, \ldots$ to four decimal places. The values of $m_3, m_4, \ldots$ do not, to this accuracy, differ from odd integral multiples of $\frac{1}{2}\pi$.

Table I.

$m_0 = \frac{1}{2}\pi + \cdot 3043 = 1\cdot 1937 \times \pi/2 = 1\cdot 8751$ $\quad m_3 = \frac{7}{2}\pi$ $\quad m_6 = \frac{13}{2}\pi$

$m_1 = \frac{3}{2}\pi - \cdot 0183 = 2\cdot 9884 \times \pi/2 = 4\cdot 6941$ $\quad m_4 = \frac{9}{2}\pi$ $\quad m_7 = \frac{15}{2}\pi$

$m_2 = \frac{5}{2}\pi + \cdot 0008 = 5\cdot 0005 \times \pi/2 = 7\cdot 8548$ $\quad m_5 = \frac{11}{2}\pi$ $\quad m_8 = \frac{17}{2}\pi$

If Young's modulus and the density be known, the frequency of the rod, when vibrating with a given number of nodes, can be calculated by (1) with the aid of Table I. Since, however, the frequency, for a given number of nodes, is inversely proportional to l^2, if the length l_0 for a known frequency n_0 be determined, we can at once find the value of n corresponding to any other observed length l, for $n = n_0 l_0{}^2/l^2$. Thus, if the length be l_0 when the rod is in unison with a tuning fork of frequency n_0, the frequency of the current can be calculated as soon as the length giving resonance has been found.

§ 3. *Calibration by tuning forks.* The electromagnet M and the coil G are removed, and the rod is made to vibrate by plucking, or, better, by a light blow with a "hammer" formed by fixing a piece of thick-walled rubber tube about 2 cm. long to one end of a stout aluminium wire about 10 cm. long. The tuning fork and the rod are struck with the rubber-headed hammer and the pitch of the rod is compared with that of the fork. With a little care, the rod can be tuned to give the octave below the fork and the octave above the fork as well as the fundamental note of the fork. The observations are made with a number of forks. Table II, covering three octaves, will facilitate the calculations.

Table II.

Note	n	$1/\sqrt{n}$	n	$1/\sqrt{n}$	n	$1/\sqrt{n}$
C	128	·08839	256	·06250	512	·04420
D	144	·08333	288	·05892	576	·04167
E	160	·07906	320	·05590	640	·03953
F	170·67	·07655	341·33	·05413	682·67	·03827
G	192	·07217	384	·05103	768	·03608
A	213·33	·06847	426·67	·04841	853·33	·03423
B	240	·06455	480	·04564	960	·03228
C	256	·06250	512	·04420	1024	·03125

According to the simple theory, l should be proportional to $1/\sqrt{n}$, for a given number of nodes. But there is some yielding of the rod in the clamp itself, and thus the effective length is somewhat greater than the distance from the end A of the rod to O, the face of the clamp. When OA, or l, is plotted against $1/\sqrt{n}$, as in Fig. 2, the points will lie on a straight line PR not passing exactly through the origin. The constants a, b in the linear relation

$$l = a + b/\sqrt{n}, \qquad \ldots\ldots(3)$$

between l and $1/\sqrt{n}$, can be obtained from the straight line. The frequency for any given value of l is then given by

$$n = b^2/(l - a)^2. \qquad \ldots\ldots(4)$$

We may, of course, read off $1/\sqrt{n}$ from the graph and thence find n by mathematical tables.

If the rod be struck with the little hammer, it will give out a note due to vibration with one node, in addition to the lower note due to vibration with no node. By adjusting the length of OA, the rod can be tuned so that the higher of the two notes is in unison with the fundamental note of the fork. With care, the rod, vibrating with one node, can be tuned also to the octave above the fork. For forks of higher frequency, a little practice may be needed before the observer can distinguish the note corresponding to one node in the presence of the note corresponding to no node.

If, at the time when the rod is struck, it be lightly touched at a point H, where AH is about $\frac{1}{5}AO$, the amplitude of the vibration with no node will be much diminished. With a little skill, it is possible to leave the rod vibrating with the node clearly visible.

Let the length of OA be now denoted by k. When k is plotted against $1/\sqrt{n}$, as in Fig. 2, the points will lie on a straight line QS not passing exactly through the origin. The constants c, d in the equation

$$k = c + d/\sqrt{n} \qquad \text{......(5)}$$

can be obtained from the line. The frequency, for one node, corresponding to any given value of k is given by

$$n = d^2/(k - c)^2. \qquad \text{......(6)}$$

The value of $1/\sqrt{n}$ may also be read off from the graph.

For a given frequency, the length OA is, by (1), proportional to m. Thus we can determine experimentally the ratio of m_1 to m_0 by finding the lengths sounding a given note, first with one node (k) and then with no node (l). Accuracy will be gained if we find the ratio of k' to l', where k' and l' are the *effective* lengths in the two cases. If k and l be plotted on the same scale against $1/\sqrt{n}$, as in Fig. 2, and the straight lines QS, PR make angles ϕ and θ with the axis of $1/\sqrt{n}$,

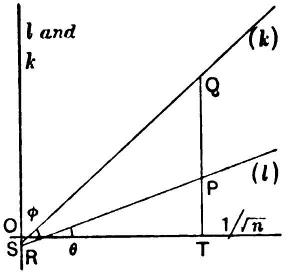

Fig. 2

$$\frac{k'}{l'} = \frac{QT + OS}{PT + OR} = \frac{(QT + OS)/OT}{(PT + OR)/OT} = \frac{\tan\phi}{\tan\theta}. \qquad \text{......(7)}$$

The value of k'/l' so obtained may be compared with the theoretical value of m_1/m_0, viz. 4·6941/1·8751 or 2·503.

§ 4. *Determination of frequency of alternating current.* When the calibration curves for no node and for one node have been drawn, the frequency of the current can be found. The electromagnet M and the coil G are put into place, and a constant current is sent through G. The length of OA is adjusted so that the rod, vibrating with no node, responds, with frequency N, to the magnet, and the length (l) of OA is measured. Then N is calculated by (4), or from the value of $1/\sqrt{N}$, which is read off from the l-line.

While the current flows in G, the length of OA is re-adjusted so that the rod, vibrating with one node, responds to the electromagnet, and the length (k) of OA is measured. Then N is found by (6) or from the k-line.

The current in G is now stopped, or the coil is removed, and the rod, vibrating first with no node and then with one node, is adjusted to respond, with frequency $2N$, to the electromagnet. The length of OA is measured in each case. The value of $2N$ for the cases of no node and of one node are found from (4) and (6) or from the l-line and the k-line respectively.

The distances AO, AH, where H is the node, are measured in each case and the value of h, where $h = AH/AO$, is compared with the theoretical value $h = \cdot 2261$. It is perhaps best to take the *effective* length of AO.

§ 5. *Practical example.* The diameter ($2r$) of the "silver" steel rod was ·2538 cm., its whole length 40·0 cm., and its mass 15·87 grm. Hence density $= \rho = 7 \cdot 8424$ grm. cm.^{-3}.

Results with no node. Four forks C, D, E, F were used; the frequency of C was 256. The rod was tuned to unison with these forks and to the octaves below them.

Fork	n	$1/\sqrt{n}$	l	n	$1/\sqrt{n}$	l
C	128	·08839	12·20 cm.	256	·06250	8·33 cm.
D	144	·08333	11·26	288	·05892	7·85
E	160	·07906	10·61	320	·05590	7·41
F	170·67	·07655	10·36	341·33	·05413	7·15

When l was plotted against $1/\sqrt{n}$, the straight line lying most evenly among the plotted points cut the axis of l at $l = -\cdot 42$ cm., and l was 13·53 cm. when $1/\sqrt{n}$ was 0·1. Thus, for the constants a, b in (3), we have

$$a = -\cdot 42, \quad b = (13 \cdot 53 + \cdot 42)/0 \cdot 1 = 139 \cdot 5.$$

Hence, by (4), $$n = 139 \cdot 5^2/(l + \cdot 42)^2.$$

When the rod, vibrating with no node, responded to the alternating current, l was 14·02 cm., and $l + \cdot 42 = 14 \cdot 44$ cm. Hence

$$N = 139 \cdot 5^2/(14 \cdot 44)^2 = 93 \cdot 33 \text{ cycles per sec.}$$

When the rod, vibrating with no node, responded to the octave above the current, l was 9·88 cm., and $l + \cdot 42 = 10 \cdot 30$ cm. Hence

$$2N = 139 \cdot 5^2/(10 \cdot 30)^2 = 183 \cdot 43, \qquad N = 91 \cdot 71 \text{ cycles per sec.}$$

Results with one node. The rod was tuned to unison with eight forks and also to the octaves above four of the forks.

Fork	n	$1/\sqrt{n}$	k (cm.)	Fork	n	$1/\sqrt{n}$	k (cm.)
C	256	·06250	21·30	B	480	·04564	15·38
D	288	·05892	19·78	C	512	·04420	14·69
E	320	·05590	18·80	D	576	·04167	13·99
F	341·33	·05413	18·21	E	640	·03953	13·62
G	384	·05103	17·49	F	682·67	·03827	13·27
A	426·67	·04841	16·27	G	768	·03608	12·07

When k, the length of OA in this case, was plotted against $1/\sqrt{n}$, the straight line cut the axis of k at $k = -\cdot 40$ cm., and k was 23·80 cm. when $1/\sqrt{n}$ was ·07. For the constants c, d in (5) we have

$$c = -\cdot 40, \quad d = (23 \cdot 80 + \cdot 40)/\cdot 07 = 345 \cdot 7.$$

Hence, by (6), $$n = 345 \cdot 7^2/(k + \cdot 40)^2.$$

When the rod, vibrating with one node, responded to the current, k was 35·31 cm., and $k + \cdot 40 = 35{\cdot}71$ cm. Hence

$$N = 345{\cdot}7^2/(35{\cdot}71)^2 = 93{\cdot}72 \text{ cycles per sec.}$$

When the rod, vibrating with one node, responded to the octave above the current, k was 24·86 cm., and $k + \cdot 40 = 25{\cdot}26$ cm. Hence

$$2N = 345{\cdot}7^2/(25{\cdot}26)^2 = 187{\cdot}30, \qquad N = 93{\cdot}65 \text{ cycles per sec.}$$

Ratio of m_1 *to* m_0. The effective lengths for no node and for one node are $l' = l + \cdot 42$, $k' = k + \cdot 40$. For a given frequency, $n = 139{\cdot}5^2/l'^2 = 345{\cdot}7^2/k'^2$. Thus

$$k'/l' = 345{\cdot}7/139{\cdot}5 = 2{\cdot}478.$$

The theoretical value is 2·503.

Young's modulus. By (1), $E = \rho\,(4\pi n l^2)^2/(m^2 r)^2$. Referring to the results for no node, and using the effective length $l' = l + \cdot 42$, we have $nl'^2 = 139{\cdot}5^2$. Further $m_0 = 1{\cdot}8751$, $\rho = 7{\cdot}8424$. Hence

$$E = 7{\cdot}8424\,(4\pi \times 139{\cdot}5^2)^2/(1{\cdot}8751^2 \times \cdot 1269)^2 = 2{\cdot}36 \times 10^{12} \text{ dyne cm.}^{-2}.$$

Position of node. When the rod vibrated with one node in response to the current, AH was 7·6 cm. and the *effective* length was 35·71 cm. Hence $h = 7{\cdot}6/35{\cdot}71 = \cdot 213$. When the rod responded to the octave above the current, AH was 5·3 cm. and the effective length was 25·26. Hence $h = 5{\cdot}3/25{\cdot}26 = \cdot 210$. The theoretical value is ·2261.

§ 6. *Experiment with several nodes.* A long rod can be adjusted to vibrate with 0, 1, 2, ... nodes under the action of the alternating current, the number of nodes being limited by the finite length of the rod. Equation (1) shews that, when n is given, l/m is constant. When there are many nodes, the rod will be too long to remain approximately horizontal when clamped as in Fig. 1. The clamp is, therefore, placed so that the part OA of the rod is vertically below the clamp. The electromagnet may be placed opposite the middle of the "loop" which is next the clamp. Gravity now comes into play, and, in effect, aids the stiffness of the rod, the increase in effective stiffness increasing with l, and causing l/m, for a given frequency, to increase slightly as l is increased. The effect only becomes appreciable when the tension, T dynes cm.$^{-2}$, is appreciable compared with Young's modulus. In the experiment of § 7, T/E was always less than 5×10^{-7}.

Measurements, similar to those described in § 3, may be made with a short length of rod, using a set of tuning forks, to determine the correction due to imperfect clamping. The quotient $\pi l'/2m$ may then be calculated for the cases of 0, 1, 2, ... nodes. We use $2m/\pi$ in preference to m, because, as Table I shews, $2m/\pi$ may be taken as an integer for m_3, m_4, ...

As an alternative, we may plot the actual length $l = OA$ against $2m/\pi$ for 0, 1, 2, ... nodes. The points will lie on a straight line passing nearly through the origin.

Any given rod is not necessarily uniform along its length, and thus some irregularities may occur in the results.

§ 7. *Practical example.* A steel rod, ·204 cm. in diameter, was used. An allowance of 0·3 cm. was made for imperfect clamping. If l be the observed length OA, and l' be the effective length, $l' = l + 0{\cdot}3$.

Nodes	l'	$2m/\pi$	$\pi l'/2m$	Nodes	l'	$2m/\pi$	$\pi l'/2m$
0	9·05	1·194	7·58	5	82·0	11	7·45
1	22·10	2·988	7·40	6	96·8	13	7·45
2	37·51	5	7·50	7	112·6	15	7·51
3	52·4	7	7·49	8	128·4	17	7·55
4	67·1	9	7·46				

§ 8. *Proof of equation* $\cos m \,.\, \cosh m = -1$. We consider a rod OA (Fig. 3) of circular section of radius r, and of density ρ. At the fixed end O, the axis of x touches the axis of the rod.

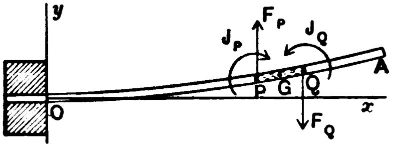

Fig. 3

Let F be the shearing force at G, the centre of gravity of the element PQ, and let J be the bending moment. We count F positive when it acts in the positive direction of y upon the part of the rod which lies on the positive side, with regard to x, of the section through G, and J is positive when, in the case of Fig. 3, it tends to make the part of the rod which lies on the positive side of G rotate in the direction of the hands of a watch. The angle between the axis of the rod at G and the axis of x is always so small that we may replace it by its tangent dy/dx, and may equate the length of PQ to dx, its projection on Ox.

Considering the acceleration of G parallel to Oy, we have

$$\pi r^2 \rho\, dx \frac{d^2y}{dt^2} = F_P - F_Q = \left(F - \frac{1}{2}\frac{dF}{dx}dx\right) - \left(F + \frac{1}{2}\frac{dF}{dx}dx\right),$$

or

$$\pi r^2 \rho \,.\, d^2y/dt^2 = -dF/dx. \qquad \ldots\ldots(8)$$

We next consider the angular acceleration of PQ about an axis through G perpendicular to the plane of bending. Since the moment of inertia of PQ about this axis is {mass of PQ} × $\{\frac{1}{4}r^2 + \frac{1}{12}(dx)^2\}$, we have

$$\pi r^2 \rho\, dx \{\tfrac{1}{4}r^2 + \tfrac{1}{12}(dx)^2\} \frac{d^2}{dt^2}\left(\frac{dy}{dx}\right) = -J_P + J_Q - F_P \,.\, \frac{dx}{2} - F_Q \,.\, \frac{dx}{2}$$

$$= -\left(J - \frac{1}{2}\frac{dJ}{dx}dx\right) + \left(J + \frac{1}{2}\frac{dJ}{dx}dx\right) - \left(F - \frac{1}{2}\frac{dF}{dx}dx\right)\frac{dx}{2} - \left(F + \frac{1}{2}\frac{dF}{dx}dx\right)\frac{dx}{2},$$

or, in the limit,

$$\tfrac{1}{4}\pi r^4 \rho \frac{d^3y}{dt^2dx} = \frac{dJ}{dx} - F. \qquad \ldots\ldots(9)$$

Now $J = E \,.\, \frac{1}{4}\pi r^4 \times (\text{curvature}) = E \,.\, \frac{1}{4}\pi r^4 \,.\, d^2y/dx^2$, and thus, differentiating (9) with respect to x, and using (8), we have

$$\tfrac{1}{4}\pi r^4 \rho \frac{d^4y}{dt^2dx^2} = E \,.\, \tfrac{1}{4}\pi r^4 \frac{d^4y}{dx^4} + \pi r^2 \rho \frac{d^2y}{dt^2}. \qquad \ldots\ldots(10)$$

When the motion of each element is simple harmonic, with periodic time $2\pi/p$, $d^2y/dt^2 = -p^2y$, and thus (10) becomes

$$(E/\rho)\, d^4y/dx^4 + p^2 . d^2y/dx^2 - 4p^2y/r^2 = 0. \qquad \text{......(11)}$$

Now $y = Ae^{qx}$ will be a solution of (11), and $y = Ae^{qx} . \sin pt$ will be a solution of (10), if

$$Eq^4/\rho + p^2q^2 - 4p^2/r^2 = 0,$$

or

$$q^2 = \left\{ -1 \pm \sqrt{1 + \frac{16E}{\rho p^2 r^2}} \right\} \frac{\rho p^2}{2E}. \qquad \text{......(12)}$$

In the experiment, $n < 1000$ and hence $p < 2000\pi$; thus $p^2 < 4 \times 10^7$. For the rod $E = 2{\cdot}36 \times 10^{12}$, $\rho = 7{\cdot}8$, $r = {\cdot}127$. Hence $16E/\rho p^2 r^2 > 7{\cdot}5 \times 10^6$, and $(16E/\rho p^2 r^2)^{\frac{1}{2}} > 2750$. Thus, with considerable accuracy,

$$q^2 = \pm \frac{2p}{r} \sqrt{\frac{\rho}{E}} = \pm f^2. \qquad \text{......(13)}$$

Hence

$$q = \pm f \text{ or } q = \pm f\sqrt{-1} = \pm if. \qquad \text{......(14)}$$

If we neglect, from the beginning, the rotational inertia of the element, we obtain

$$F = dJ/dx \qquad \text{......(15)}$$

in place of (9). Then (15) with (8) gives

$$(E/\rho)\, d^4y/dx^4 - 4p^2y/r^2 = 0, \qquad \text{......(16)}$$

which yields (14) at once.

Thus the solution is $y = \{A'e^{fx} + B'e^{-fx} + C'e^{ifx} + D'e^{-ifx}\} \sin pt$, or, changing the constants,

$$y = \{A \cosh fx + B \sinh fx + C \cos fx + D \sin fx\} \sin pt.$$

When $x = 0$, i.e. at the clamped end of the rod, $y = 0$ and $dy/dx = 0$. Hence $A + C = 0$ and $B + D = 0$. At the free end, where $x = l$, both the bending moment and the shearing force vanish and thus $J = 0$, and, by (15), $dJ/dx = 0$. Hence $d^2y/dx^2 = 0$ and $d^3y/dx^3 = 0$, when $x = l$. If we write $fl = m$, we find, on substituting for C and D,

$$A (\cosh m + \cos m) + B (\sinh m + \sin m) = 0,$$

$$A (\sinh m - \sin m) + B (\cosh m + \cos m) = 0.$$

On eliminating the ratio B/A, we find that m is a root of

$$\cosh m . \cos m = -1,$$

where

$$m^2 = f^2l^2 = (2pl^2/r)\sqrt{\rho/E}.$$

Thus, as in (1),

$$n = p/2\pi = m^2 . \frac{r}{4\pi l^2} \sqrt{\frac{E}{\rho}}.$$

Photographic plates for the detection of mass rays. By Dr F. W. ASTON, Trinity College.

(With Plate IV.)

[*Read* 24 November 1924.]

Introduction.

The difficulties met with in work on positive rays or, to use the more general term, mass rays, fall into two entirely different groups: those connected with their production and those connected with their analysis. It is clear that improvement in the means of detecting and recording the rays will alleviate both groups for, on the one hand, it will enable feebler beams to be employed, and. on the other, will reduce or remove the numerous technical obstacles in the way of very prolonged exposures. I propose to give in this paper some account of my own experiments undertaken to select and improve the photographic plates used. Very little has been published on the subject. so it will be convenient to follow the enquiry from its commencement.

The first photographic records of the parabolas of positive rays were obtained with the apparatus described by Sir J. J. Thomson*. The plates employed were fairly rapid ones in general use for landscape photography. One or two different brands were tried and of these the Sovereign made by the Imperial Plate Co. was chosen as the most convenient. For several years all the positive ray results obtained by Sir J. J. Thomson, including those showing the double parabolas of neon, were recorded on Imperial Sovereign plates. It was early suspected that the swift mass rays of hydrogen had an abnormally strong effect as compared with the heavier ones of other elements and compounds. This was confirmed by electrical measurements of the total charge on the beams of rays and is clearly to be ascribed to the higher penetrating power of the lighter and swifter hydrogen particles. Sir J. J. Thomson realised that the penetrating power of the heavier rays was exceedingly small and that therefore the layer of sensitive silver salt should be as concentrated as possible†. He therefore endeavoured to make use of the old Daguerreotype process, which should give a maximum concentration, but without success.

In 1913 Eagle succeeded in making at the Cavendish Laboratory some Schumann plates by the method described by their inventor‡. These were far and away more sensitive than any plates tried before but were difficult to make and highly inconvenient to use.

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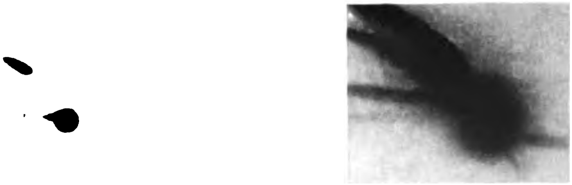
* *Rays of Positive Electricity*, 2nd ed., Fig. 16, p. 32.

† *Loc. cit.* p. 4.

‡ Baly's *Spectroscopy*, p. 359.



A B

Comparative test made in 1919:

A. Imperial Sovereign Plate.
B. Paget Half-Tone Plate.

C D

Comparative test made in 1924 with very short exposure:

C. Half-Tone Plate untreated.
D. Schumannised Plate.

Comparative tests made in 1919.

It is possible to calculate the efficiency of a photographic plate for recording mass rays if we know the energy of the beam and the minimum exposure necessary to produce a visible image. The former can be calculated from the total charge and the potential through which the charged particles have fallen, while the latter could be roughly estimated from general experience. Using figures obtained in this way F. A. Lindemann and I calculated in 1918 that for rays of mass 28—the CO parabola—Sovereign plates were about 2000 times less sensitive than to light. This result was very encouraging in so far as it indicated great possibilities of improvement so that when I returned to Cambridge in 1919 to start the preliminary work which led to the construction of the mass-spectrograph it was clearly time to make a provisional survey of the plates available. By this time Sir J. J. Thomson had abandoned Schumann plates in favour of a process plate called Half-Tone put on the market by the Paget Plate Co. which was very sensitive and reliable*. A simple form of parabola apparatus was set up in which four samples of plate could be exposed in succession under conditions as uniform as possible. It was found that short exposures with a weak discharge could be repeated with a constancy sufficient for grading the plates in order of efficiency. A large selection of fast, slow, lantern and process plates on the market were compared and, as was expected, showed extreme differences of sensitivity to mass rays which had no relation whatever to their sensitivity to light. The results fully justified Sir J. J. Thomson's choice, for the Paget Half-Tone plate was far the most sensitive of all. A comparison between this plate and an Imperial Sovereign is reproduced on Plate IV. These were taken with identical exposures and leave no doubt as to the great advantage in sensitivity to the rays in general possessed by the Half-Tone notwithstanding the fact that it is seven times less sensitive to light.

Experience with Paget Half-Tone plates.

The Half-Tone plates which were adopted as a result of these tests are manufactured especially for use in the production of half-tone blocks for engraving. They appear to possess almost every merit possible for detecting and recording mass rays. Although highly sensitive to the rays they may be manipulated in bright yellow light without fear of fog, which is a great convenience. They have an extremely fine grain and owing to the thinness of their sensitised layer can be fixed, washed and dried much more rapidly than ordinary plates. Their principal drawback, particularly of the quality in which they are now being put on the market, is that as the exposure to the rays is increased the blackness of the developed image rapidly reaches a maximum and then no longer increases. This maximum is by no means high in opacity so that, whatever the exposure, with mass rays other than those of hydrogen, it is impossible to get a really black image at all.

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* *Loc. cit.* p. 5.

All the early work with the mass-spectrograph was done with one particular batch of Half-Tone plates which had been in the Cavendish Laboratory for a considerable time. When these were exhausted, during the investigations on boron and silicon in the summer of 1920 and a new batch procured, I was dismayed to find that the fully exposed mass lines were only about half as black as before, although in their behaviour to light the new plates were absolutely identical with the old ones. Consideration of results already obtained showed that only a very small part of this large change could be put down to ageing so that it was clear that some change had been made in the process of making the emulsion since the original batch of plates were obtained. I communicated with the Paget Plate Co. who informed me that they had actually changed the brand of gelatine during that period. They very kindly offered me all the assistance in their power and prepared a batch of plates especially for my use from a small quantity of the old type of gelatine remaining. These lasted for some years until, owing to unfavourable storage conditions, they deteriorated too much for use.

First experiments with schumannised plates.

The fact that such a minor alteration in manufacture as that described above is capable of producing so large a difference in the efficiency of the plates had a hopeful significance, for it indicated that if a process could be devised for increasing the concentration of the sensitive silver bromide particles on the surface much greater improvements might be expected. I was informed by Professor T. R. Merton in the summer of 1922 that such a process had originated in France by which ordinary plates could be made sensitive to the extreme ultraviolet—in other words, transposed into Schumann plates—by dissolving the gelatine away in dilute acid. I tried this process on half-tone plates and the first plates treated were successful beyond all expectations and led to very important discoveries in the case of the isotopes of tin*. Further efforts were not so successful and it soon became apparent that the previous success was of the nature of a lucky accident. The process resulted in a disheartening series of failures and uniformity seemed an impossible achievement, so that when the method of accelerated anode rays was developed the commercial half-tone plate was returned to for general use†.

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* *Nature*, 1922, vol. 109, p. 813.

† *Phil. Mag.*, vol. 47, p. 392.

Comparative tests made in 1924.

As the search for mass-spectra reached the heavier metallic elements and the rays became more and more difficult to produce and analyse it became evident that progress would soon cease altogether unless more efficient plates could be obtained. I therefore again set up a parabola apparatus in which plates treated in various ways could be compared against the standard untreated ones. The mass-spectrograph from its construction is not suitable for this work. During these experiments a thorough test was given to the Schumann plates now put on the market by Messrs Adam Hilger, samples of which were kindly sent me by Mr Twyman. These plates are much more sensitive than untreated half-tone plates and for the same exposure give lines almost as black as those given by the best schumannised plates, but these lines are not so sharp and, most important of all, I never found it possible to obtain a perfectly clear or uniform background which is so essential for the detection and measurement of faint lines.

In searching for a satisfactory treatment for removing the gelatine to the extent desired to make the plate sufficiently sensitive without rendering it too difficult in manipulation, the time, temperature, nature and strength of the acid bath were all varied. It will not be necessary to detail the results of the numerous experiments performed, for in the end I was driven to the conclusion that success is dependent on causes for the most part unknown and out of control and so remains in the region of pure chance. On successive days two halves of the same plate subjected to what appeared to be absolutely identical series of operations gave results to all outward appearance the same, yet when tested the first was found to be insensitive both to light and mass rays, partially fogged and perfectly worthless, while the other was one of the best plates I have succeeded in making. It follows that every single plate has to be tested by actual exposure to the rays, a laborious operation, before it can be used. The procedure which I have now adopted as most likely to yield good results is as follows:

Schumannising procedure.

The Half-Tone plate is slid into a bath of dilute sulphuric acid, I use ordinary pure battery acid of density 1·225 diluted with an equal volume of water. 10 × 8 plates can be handled but I prefer them cut to 5 × 8 as being less wasteful. The plate is left lying at the bottom of the dish absolutely quiescent overnight at a temperature which should not be below 16° C. It is then extracted, drained for a few seconds to remove the bulk of the acid which drips off as a milk fully charged with bromide grains, and slid into a large dish of cold tap water in a far corner of which a tap is kept dribbling in order to wash the plate as gently as possible. After washing for an hour or two it is again drained and dried leaning against the wall of the dark room.

The operations of removal of the plate from the acid and later from the water are very delicate. It is slowly raised from the bottom by means of a spatula until it can be held by the tips of the fingers. These must be scrupulously clean and already wet with the liquid in which the plate lies. A touch with a greasy finger or one wet with a different fluid removes the film as if by magic, for the latter is simply a mud of bromide particles adhering mostly by surface tension. Once dry, the plates though more sensitive to abrasion than the untreated ones are more robust than Schumann plates. They can be laid face down on velvet and cut with a diamond without any pressure pattern showing on development. Viewed with transmitted light they show wide variations in the thickness of the film, but since the mass rays only affect the surface this does not matter. The more transparent they are the more insensitive they will be to ordinary light, a definite advantage except as regards photographing the fiducial spot.

For development of these treated plates the standard hydroquinone developer recommended for Half-Tone plates is too strong. I use an ordinary metol hydroquinone developer diluted to half strength and with sufficient bromide added to bring the time of development to about a minute.

Photographs C and D, Plate IV, show comparative results obtained with exceedingly short exposures in the parabola apparatus on untreated and treated plates. Such tests are usually made with a high magnetic field since the sensitivity to rays of heavy particles is the important point. This comparison was made with the magnet off, leaving a small residual field. It is reproduced here because it includes the parabola of the hydrogen atom—faint upper streak—and shows that this is hardly altered at all by the treatment whereas that of the heavier particles—parabolas all bunched together in lower streak—is enormously blacker.

Characteristics of schumannised plates.

As already emphasised success cannot be assured, but when it is attained the resulting plates are wonderfully effective. They can be used in bright yellow light with even greater safety than the untreated ones and they fix, wash and dry almost instantaneously. The developed image is however sensitive to abrasion so that it is advisable to protect it by varnishing as in the case of Schumann plates. Unfortunately their keeping qualities seem very doubtful. Packed film to film, the method recommended for Schumann plates, they were found completely fogged after a few weeks. Kept loose in a box, film upwards, they last very much better, but deterioration is generally noticeable a few days after preparation. Although the sensitivity seems to increase rather than decrease the developed image becomes more granular and a coarse grained and uneven fogging begins to appear which is fatal for their use in the mass-spectrograph. If used when fresh, however, they give extremely good results.

One of their great merits is the absence of halation and fog in the close neighbourhood of a very strong and over-exposed line. This they probably owe to their comparative insensitivity to secondary radiation of the type of light or X-rays. Both Half-Tone and Hilger schumann plates are bad in this respect. The latter are indeed specially designed to be sensitive to this kind of radiation.

The outstanding property which makes them so superior to the untreated Half-Tone plate for use in mass-spectrum work is the enormously increased range of density in the developed image. For the detection of a single isolated line in a mass-spectrum the faintest of images is sufficient if the background is clear or uniform so that the important factor in such a case is the threshold value of the plate which does not appear to be improved much by schumannising, indeed sensitivity in this respect may be actually impaired by the necessity of a weaker and more restrained developer. But in general, so far from being isolated, the lines of the isotopes of the heavier elements now being studied are barely resolved from their neighbours. Furthermore, since each line is the image formed by projection of a beam of rays passing through two parallel slits and restricted only by their walls, its middle part will be slightly more exposed than its outer parts. In this way on a plate of high range of tone a visual resolution actually higher than theoretical may be obtained. Thus a group such as is given by tin or cadmium though it appears as an almost uniform blurr on a Half-Tone plate on a good schumannised plate shows an amazing gradation and detail which enables isotopes to be detected with certainty where otherwise they would be unsuspected or ascribed to halation effects of neighbouring strong lines.

Conclusion.

From these statements it will be seen that though some progress has been made the state of affairs is by no means satisfactory and further research is needed. It is obvious that we cannot expect much from the manufacturers. The plates required are those of maximum sensitivity to mass rays but relatively unaffected by other types of radiation. The demand for such plates could hardly be more than infinitesimal and it is also highly probable that superlative surface sensitivity is essentially incompatible with good keeping qualities. The matter is of importance however as further improvement in the plates will not only facilitate the analysis of the remaining elements but may be an essential factor in the application of mass ray analysis to the solution of other fundamental problems.

Summary.

The sensitivity of photographic plates to mass rays bears practically no relation whatever to their sensitivity to light. A short account is given of tests on the relative merits of different brands of plates since the first experiments on positive ray analysis. The process of improving the qualities of plates by schumannising is discussed and details are given of the process found likely to give the best results. Success cannot be guaranteed even with the most careful manipulation, but if achieved yields plates of outstanding value for use in such methods of analysis as the mass-spectrograph.

The kinetic theory of metallic conduction. By E. L. DAVIES and Mr G. H. LIVENS.

[*Read* 24 November 1924.]

1. The kinetic theory of metallic conduction is usually supposed to lead to formulae for the conductivity coefficients which in some respects are very much at variance with the actual phenomena. In spite of the important successes achieved by the theory, which in themselves are almost sufficient evidence that the fundamental basis is correct, these discrepancies are regarded in some quarters as so serious as to warrant the complete rejection of the theory as being inadequate to account properly for the facts.

The object of the present note is to show that such sweeping modifications are not in reality so necessary. It is in fact possible to frame the kinetic theory in a manner which makes it almost perfectly consistent—so far as that is possible with a theory of this type—with the experimental data. The analytical method adopted does not differ appreciably from that expounded in detail by one of the present authors on a previous occasion, so we shall be content with simply quoting results. References will not be given to the authors and works quoted because these are now mostly too well known to need reference and in any case the results are excellently summarised in the article "Elektronentheorie der Metalle" by R. Seeliger in the *Encyklopädie der Mathematischen Wissenschaften*, Bd. v. 2.

In all forms of this theory the carriers of electricity and energy are assumed to be the free electrons. The atoms, comparatively heavy nuclei of positive electricity surrounded by electrons, are regarded as occupying positions in the body which to a first approximation are fixed. The electrons normally associated with the atoms are divided into two classes, the bound electrons describing closed orbits about the central nucleus and the free electrons wandering about in the spaces between. To these latter alone are due the conduction phenomena proper; the former will in all probability have their orbits modified or orientated on the application of an electric or magnetic field; but they will not play an important part in the transportation of electricity or energy, except in modifying the field about the atom and thereby altering the driving force on the electrons.

2. Assuming then that the underlying idea is that of the diffusion of a number of electrons through a space lattice of active force centres, whether these be atomic nuclei or not does not matter, we can simply apply in its most general form the original statistical

analysis of Lorentz. This leads in the usual way to the following formulae for the electrical (κ) and thermal (γ) conductivities, viz.

$$\kappa = \frac{8Ne^2q^{\frac{5}{2}}}{3m\sqrt{\pi}}\int_0^\infty \tau_m u^4 e^{-qu^2}\,du,$$

and
$$\gamma = \frac{2Nmq^{\frac{5}{2}}}{3\sqrt{\pi}}\cdot\frac{dq}{d\vartheta}\int_0^\infty \tau_m u^6 e^{-qu^2}\,du.$$

Here N is the total number of electrons per unit volume at any point in the metal and q is connected with their average (mean square) velocity u_0 by the formula

$$q = \frac{3}{2u_0^2}.$$

The τ_m occurring in these formulae is the fundamental function of the theory. It is introduced in the first place so that its reciprocal is a proportional factor measuring the disorganising effect of the dispersing centres in the metal on the motions of the electrons diffusing through it. It is naturally a function of the velocity components of the electrons themselves and perhaps also, but to a much less extent, of the physical conditions of the metal. In the general aeolotropic medium it would necessarily vary with direction in the metal, that is it would involve the different components of the velocity in different ways, but in isotropic media it would in general be a function of the resultant velocity alone.

Unfortunately there appears to be very little direct evidence to help one in a suitable choice of function. The absorption coefficients for the penetration of β-rays through thin sheets of metal suggest a relation somewhat of the form

$$\tau_m = \mu u^p,$$

where p is certainly > 5; but it appears that this relation is out of the question in the present connexion. Of course it has only been established for very large velocities ($> 10^9\ c/s$) so that it is perhaps not surprising to find it inapplicable to the comparatively slow thermal motions.

In the theory of Lorentz where the atoms are the dispersive centres and repel the electrons as though both were hard elastic spheres it is found that

$$\tau_m = \mu u^{-1},$$

whilst in the modifications of this theory by Bohr and Richardson where the more general case of an inverse nth power repulsion from the atomic centres is used it is found that

$$\tau_m = \mu u^{\frac{5-n}{n-1}}.$$

Lorentz's value being the special case of this when $n = \infty$. There is a certain amount of evidence that the inverse third power law is somewhat near the facts and in this case

$$\tau_m = \mu u,$$

but most of the results deduced on this basis still involve important discrepancies.

It cannot yet be said that we know much of the mechanics of the collision of an electron with an atom beyond what the present type of theory tells us. We must therefore proceed along more tentative lines by assuming a formula for τ_m and attempting to fit the results in with those determined experimentally. Acting on the suggestions which have so far been put forward we may start by assuming that

$$\tau_m = \mu u^p,$$

where μ is a constant characteristic of the conducting substance but not necessarily varying much with its physical condition.

We then get

$$\kappa = \frac{8}{3\sqrt{\pi}} \frac{Ne^2\mu}{mq^{\frac{p}{2}}} \Gamma\left(\frac{p+5}{2}\right),$$

$$\gamma = \frac{2}{3\sqrt{\pi}} \frac{Nm\mu}{q^{\frac{p+2}{2}}} \frac{d}{d\vartheta}\left(\frac{1}{q}\right) \Gamma\left(\frac{p+7}{2}\right).$$

3. If we now make the usual assumption that the mean energy of the electrons has its ordinary thermal value or that

$$\tfrac{1}{2}mu_0^2 = \frac{3m}{4q} = \tfrac{3}{2}R\vartheta,$$

wherein ϑ is the absolute temperature and R the usual gas constant, then the electrical conductivity will vary as $\vartheta^{\frac{p}{2}}$ and the thermal conductivity as $\vartheta^{\frac{p+2}{2}}$. But the main part of the temperature variation of the electrical conductivity at ordinary temperatures appears to arise from a proportional factor ϑ^{-1} whilst the thermal conductivity is practically constant. We are therefore induced to try the relation

$$p + 2 = 0,$$

or

$$\tau_m = \mu u^{-2},$$

wherein μ is a constant practically independent of the physical conditions of the metal.

With this value for p we have

$$\kappa = \frac{2Ne^2\mu q}{3m}, \qquad \gamma = \frac{N\mu m}{4}\frac{d}{d\vartheta}\left(\frac{1}{q}\right),$$

so that
$$\frac{\gamma}{\kappa} = \frac{3}{8}\frac{m^2}{e^2}\cdot\frac{1}{q}\frac{d}{d\vartheta}\left(\frac{1}{q}\right).$$

At ordinary temperatures therefore when

$$\frac{m}{q} = 2R\vartheta,$$

we have
$$\frac{\gamma}{\kappa} = \frac{3}{2}\frac{R^2}{e^2}\vartheta.$$

But we know that
$$\left(\frac{R}{e}\right)^2 = 8{\cdot}26\,.\,10^{-14}$$

so that for an absolute temperature of 300°

$$\frac{\gamma}{\kappa} = 37{\cdot}2\,.\,10^{-12}.$$

That this ratio, although of the correct order of magnitude, is smaller than any value obtained experimentally may be due to either or both of two obvious deficiencies in the above argument.

The law of dependence of τ_m on u, viz.

$$\tau_m = \mu u^{-2},$$

may be only partially correct; in other words the μ may still be a function of the velocity u. If, for example, we take instead

$$\tau_m = \mu u^{-2} f(qu^2),$$

where $f(qu^2)$ is a dimensionless function of its argument, the only change in the above formulae would be a modification in the numerical value of the coefficients in both cases. It may therefore be that such a law is nearer the truth than the simpler one first assumed.

Again, we have neglected entirely the effect of any local field on the complete electromotive force in the metal. If such fields existed then their average effect would be strictly proportional to the intensity of the applied field, in conformity with our general experience with dielectric media, so that the effective field would be linearly related to the applied field, being a mere proportionality in the case of the isotropic media now under review. The electrical conductivity would be altered in a similar ratio and the ratio γ/κ would be decreased in this ratio. The fact that the theoretical value of this latter ratio is less than the experimental one indicates that the necessary induced fields are in fact in the opposite direction to the applied field, so that the effective field is always smaller than the actual inducing field and the actual value of γ/κ is increased from the above theoretical value.

4. The behaviour of the conductivities at low temperatures is generally supposed to be inconsistent with a kinetic theory. Apart however from the question of the proper value of the thermal energy, this cannot be said to be the case with the present form of the theory. Let us interpret our results in another manner; the total kinetic energy of the electrons per unit volume in the metal is

$$T = \tfrac{1}{2} N m u_m^2$$
$$= \frac{3Nm}{4q},$$

so that in terms of this energy we have

$$q = \frac{3Nm}{4T}.$$

In the special case when $p = -2$ we have therefore

$$\kappa = \frac{N^2 e^2 \mu}{4T},$$

whilst

$$\gamma = \frac{\mu}{3} \frac{dT}{d\vartheta}$$

with a possible reservation regarding the numerical coefficients. Thus now

$$\frac{\gamma}{\kappa} = \frac{3e^2}{4N^2} T \frac{dT}{d\vartheta}.$$

Now it appears from experiment that the temperature variations of γ at low temperatures are parallel with those of the specific heat of the metal as a whole; but since T is the total heat content of the electrons, $\frac{dT}{d\vartheta}$ is the specific heat of the electrons in the metal and this may reasonably be expected to vary in a manner similar to that of the metal as a whole, so that in this form our results are not inconsistent with experience.

As we have no direct knowledge of the value of T we cannot obtain direct comparison with the theoretical value of κ, but Herzfeld concludes from a detailed discussion of the results for temperatures down to 12° abs. that the value of the ratio

$$\frac{\gamma}{\kappa} = \frac{3e^2}{N^2} T \frac{dT}{d\vartheta}$$

agrees well with experimental results if T has the value

$$T = \frac{N\epsilon}{e^{\frac{\epsilon}{k\vartheta}} - 1}.$$

suggested by Planck's thermal radiation theories. Such a value would lead to a formula for the electrical conductivity

$$\kappa = \frac{Ne^2}{4\epsilon}\left(e^{\frac{\epsilon}{R\vartheta}} - 1\right)$$

which agrees with the suggestion of Kammerlingh-Onnes that κ actually increases exponentially at very low temperatures, where the so-called super-conductivity exists.

In a discussion along somewhat novel lines Wien obtains a formula for the electrical conductivity which is practically equivalent to our own if we take the value for T suggested by Debye's specific heat theory, viz.

$$T = C\int_0^{x_0} \frac{x^3 dx}{e^{\frac{x}{R\vartheta}} - 1},$$

in which x_0 is a characteristic elastic constant of the metal. His formula he shows to be in very close agreement with experimental data, both as regards its temperature variation and its absolute value as a function of x_0, whose value can be obtained from specific heat or optical data. The theory however appears to involve somewhat artificial hypotheses and does not admit of any explanation of thermal conduction phenomena. The present suggested form of theory has therefore the advantage over his in this respect and it will give the correct variation of γ, the thermal conductivity, at low temperatures, just as far as Debye's theory gives proper account of the variations of the specific heat.

At extremely low temperatures (4° abs. or thereabouts, where super-conductivity occurs) all theories seem to fail in explaining the variations. The results obtained by Kammerlingh-Onnes however suggest that if we take Wien's value of T, or any other value varying similarly, decreased by a small constant independent of the temperature, viz.

$$T = C\int_0^{x_0} \frac{x^3 dx}{e^{\frac{x}{R\vartheta}} - 1} - T_0,$$

then much closer agreement could be obtained. This alteration would not affect the thermal conductivity whilst the electrical conductivity would approach exponentially to an infinite value at a certain small temperature. Physically this points to the fact that the average kinetic energy of the electron falls short of its full thermal equivalent by a small amount representing a real deficiency or energy of some other type. Below the small critical temperature the electrons would all be practically at rest just as they are at this temperature and the super-conductivity phenomena would persist.

5. The present form of the purely kinetic theory thus leads to results which in practically all respects are perfectly consistent with the experimental data in the simple cases where regularity persists, whilst at the same time it points quite clearly the directions in which we must look for the explanation of the irregularities.

The only problem remaining unsolved is the determination of a mechanical or physical explanation of the law for the dispersion of the electrons by the nuclear centres of force, which is required to secure consistency. It was at first hoped to avoid this problem by retaining the older view of dispersing centres with a given law of force and assuming that the intensity of force varies in a definite manner with the temperature. The inverse cube law favoured by Richardson was found to be even better than the above theory if the intensity of force varies as $\vartheta^{-\frac{3}{2}}$; but then we were faced with the difficulty of explaining (i) the occurrence of this law when the inverse square law is the only one consistent with direct experiments on the scattering of α- and β-particles, and (ii) the extraordinary law of variation of its intensity with the temperature. The inverse square law, for which there is a good deal of direct evidence, is unfortunately incompatible with the usual statistical argument, the integral coefficients all becoming infinite. The alternative method, adopted above, of neglecting for the time being the actual law of force and assuming a suitable law of scattering, seemed therefore to hold the only hope of success. When it is possible to formulate a kinetic theory suitable for the inverse square law, it may appear that our suggested law of scattering is the one appropriate to it, but at present this is nothing more than a pious hope.

The decay constant of radium emanation. By L. BASTINGS. (Communicated by Prof. Sir E. RUTHERFORD.)

[*Read* 24 November 1924.]

Introduction.

For many years, the value of the decay constant of radium emanation generally accepted has been that obtained by Rutherford* in 1911, viz 0·1802 day^{-1}, corresponding to a half-period of 3·846 days; a value in close agreement with that obtained by Mme Curie† in 1910, viz 0·1801 day^{-1}.

Rutherford's method consisted in comparing the activity of emanation in a sealed tube, in equilibrium with active deposit, with that due to a standard radium salt, using a γ-ray electroscope. Then after about 100 days, the emanation is pumped out of the tube into an emanation electroscope, and the ionisation compared with that produced in the same electroscope by the emanation pumped from a radium solution containing a known fraction of the original standard.

Mme Curie adopted a direct method, and measured the ionisation current due to the emanation, or its active deposit in equilibrium, over a period of from 20 to 40 days.

Recently these values have been called in question by two series of investigations. Bothe and Leechner‡ adduce evidence to show that the constant is as high as 0·1819; though in a later paper. Bothe§ applies corrections which reduce this to 0·1812. Also Mlles I. Curie and C. Chamié || have arrived at a value 0·1813.

Bothe employed a method similar to Rutherford's, but introduced a number of precautions, particularly in the handling of the solutions, dictated by the accumulated experience of recent years.

Curie and Chamié have evolved an entirely new method of attack, enabling an accurate estimate of the constant to be obtained over the time of a single half-period. The general principle involved is similar to that adopted in the present investigation, but the method of measurement demanded a balance being maintained between two ionisation currents, an electrometer and a piezo-electrique being used to measure the deviation from balance.

The divergence between these different values is sufficient justification for a further measurement, and a method has been developed in which the apparatus and technique employed have been much simplified.

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* *Wien. Ber.*, 1911, vol. 120, p. 303.

† *Le Radium*, 1910, vol. 7, p. 33.

‡ *Zeit. fur Phys.*, 1921, vol. 5, p. 338.

§ *Zeit. fur Phys.*, 1923, vol. 16, p. 266.

|| *Comptes Rendus*, 1924, vol. 78, p. 1809; *Journal de Phys.*, 1924, vol. 5, p. 238.

Principle of Method.

A comparison between two radioactive sources of approximately equal activity can be made with a very high degree of accuracy by means of an ordinary electroscope of simple type. As has already been shown*, in the case of the β-ray electroscope, consistency to within about 1 in 500 is possible under the best conditions; while here, with a γ-ray electroscope, agreement of the same order has been obtained between averages of sets of comparisons.

An accurate comparison is made between an emanation tube in equilibrium with Ra B and C, and a standard radium tube, also in equilibrium, each being placed in turn in such a position with regard to the electroscope as to produce initially about the same ionisation in the instrument. If a series of comparisons between the two be made over a period of about two days, in the vicinity of their equality, the time of their exact equality can be deduced, correct to within a very few minutes. The same process is meanwhile repeated during the same period for another nearly equal emanation tube. The two tubes are then placed together, and after about three days, when the activity of each has fallen to nearly $\frac{1}{2}$, another similar series of comparisons is made between the two together and the radium standard. The interval between the time when they are together equal to the standard, and the time when they average separately the same value, will be the required half-period for the emanation.

If four tubes are employed, first they are compared separately with the standard; then in pairs, after one half-period; and finally, at the end of two such periods, all four are measured together. Thus a number of independent values of the constant can be obtained in quite a simple manner.

Experimental Details.

The electroscopes employed consisted of the usual γ-ray type, a cubical box being used in the two earlier sets of measurements, and a cylindrical one in the later set. (It was of interest to find out if the shape of the electroscope appreciably affected the measurements.) Both were covered with 3 mm. of lead to exclude β-rays.

The emanation tubes were made as nearly of equal activity as possible by constructing a series of connected bulbs of approximately equal volume in 4 mm. glass tubing, and filling the whole with the required amount of emanation. After sealing off the end of the tube, it was left to cool for some time, and then each bulb was quickly sealed off and detached with a small flame, the rest of the tube being allowed to cool in a horizontal position before the next sealing. In this way it was generally found possible to obtain tubes differing in activity by not more than 5 per cent. The tubes were mounted on brass blocks fitting into a vertical fixed slot placed some 40 cm. from the side of the electroscope. Thus they could, after removal, be replaced with considerable precision. When measurements were made with several tubes, they lay close together in a vertical plane; and when single tubes were used, the brass blocks removed above or below were replaced by exactly similar blocks bearing blank tubes made from the same glass. So all measurements were carried out under conditions as nearly identical as possible.

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* *Phil. Mag.*, 1924, vol. 48, p. 1075.

Each comparison consisted of at least 20 timings of the ionisation due to the source, with a like number for that due to the standard, the latter being distributed before and after the former. The rate of the leaf was always in the neighbourhood of 25 divisions per minute, and leakage over 20 divisions was allowed. The natural leak of the electroscope was about 1 division per minute, and remained very constant throughout.

The results for each comparison were plotted on a large scale as log of relative activity against time, and the most probable value of the activity of each source at any required moment deduced. Values were read off for each single source at a specified time: and for each pair of sources 3·82 and 3·85 days later. The exact time when the two sources together were equal to the mean of their original separate values was deduced by linear interpolation. The error introduced by using linear instead of logarithmic interpolation was of the order of 1 in 10,000. The same process was repeated between pairs and quartets, and between singlets and quartets.

A Typical Experiment.

The following is a typical example of the process:

Observations (reduced).

Source	Time in days	$R = \frac{\text{Source}}{\text{Standard}}$	$\log R + 1$
(1)	0·640	1·0749	1·0314
	1·411	0·9343	0·9705
	1·625	0·8991	0·9538
	2·419	0·7780	0·8910
(2)	0·668	1·1728	1·0692
	1·438	1·0137	1·0059
	1·653	0·9759	0·9894
	2·446	0·8440	0·9263
(1 and 2)	4·600	1·0988	1·0409
	5·406	0·9490	0·9773
	5·628	0·9135	0·9607
	6·452	0·7855	0·8951

Deductions from large-scale graph.

Source	Time in days	log $R+1$	R
(1)	1·600	0·9557	0·9031
	1·630	0·9532	0·8979
(2)	1·600	0·9938	0·9858
	1·630	0·9915	0·9806
(1 and 2)	5·450	0·9741	0·9421

Conclusions.

Average of (1) and (2) at 1·600 days = ·9444

,, ,, 1·630 ,, = ·9392

Difference for ·030 day = ·0052

Difference between average of (1) and (2) at 1·600 days and (1 and 2) at 5·450 days = ·0023

$\therefore \Delta t$ (linear interpolation) = $\cdot 013_3$

∴ time at which average of (1) and (2) is ·9421

= 1·600 + 0·013 = 1·613

∴ half-period deduced = 5·450 − 1·613 = 3·837 days.

Results.

Three sets of measurements were carried out, in each case with five sources, each set extending over 10 days, i.e. covering two half-periods. The results (half-period in days) are summarised thus:

	1st set	2nd set	3rd set
Singlets to doublets	3·835	3·833	3·831
	3·837	3·830	3·830
Singlets to quartets	3·840	3·828	3·838
	3·839	3·830	3·830
Doublets to quartets	3·838	3·832	3·836
	—	3·828	—
Average	3·838	3·830	3·833

A single determination made by measuring directly the decay of a single source for 10 days gave 3·829.

Mean of these 17 results: half-period 3·833 days, corresponding to a decay constant 0·1808 day^{-1}.

It will be seen that the maximum deviation between the averages of the sets is about 1 in 500 of the mean value.

Sources of error.

(1) Errors inherent in the electroscopic method of comparing activities, due to lack of saturation, irregular ionisation in the corners of the electroscope case, etc.

(2) Errors in timing of the electroscope leaf.

(3) Errors in estimating the time intervals between observations.

(4) Errors due to temperature and pressure variation between comparative measurements.

(5) Errors due to inaccuracy in replacing emanation tubes in the slot.

(6) Errors due to movement of radium salt in tube between standardisations.

(7) Variations in the natural leak during measurements.

(8) Errors in estimation of the most probable value of ionisation at a given time, from the ionisation-time curves.

(9) Errors due to the migration of portion of the sources into the glass of the containing tubes, due to recoil.

Of these, (1) and (2) together form the most formidable, and limit the possible accuracy of the method. As already stated, careful measurements, under the best conditions, as obtaining in this method, of comparison of sources give consistency between averages to about 1 in 500.

(9) is a possible error of uncertain magnitude, but owing to the distance of the sources from the electroscope, it is very unlikely to affect the results by as much as 1 in 1000.

The remaining sources of error were all investigated, by experiment or calculation, and found to be of the order of 1 in 2000 or less.

The simplicity of the method leaves very little loophole for the introduction of systematic errors. All the above, except (9) are irregular errors, which should be to a great extent balanced out by the large number of results obtained. It therefore seems highly improbable that the half-period is liable to error in excess of 3 or 4 units in the third decimal place.

Summary.

(1) A method has been devised and applied for measuring the decay constant of radium emanation, the method depending on careful comparisons with a simple electroscope under the most favourable conditions.

(2) A mean value for the constant of radium emanation has been obtained, with an error unlikely to exceed 0·2 per cent.: $\lambda = 0{\cdot}1808\ \text{day}^{-1}$, or half-period 3·833 days.

Acknowledgments.

It gives me great pleasure to acknowledge my indebtedness to Prof. Sir Ernest Rutherford, and to Dr J. Chadwick, the former for suggesting an outline of the problem, and both for their helpful interest during its prosecution.

On the current induced in a wireless telegraph receiving antenna. By Mr E. B. MOULLIN, Downing College. (Communicated by Mr E. V. APPLETON.)

[*Read* 24 November 1924.]

The propagation of electric currents through long cables has often been treated theoretically and the solutions, first obtained by Kelvin and by Heaviside, have moulded the development of transcontinental telegraphy and telephony. These solutions presume that the source of electrical disturbance is situated at one point in the cable and this is the condition appropriate to wired communication, but in the more recent process of wireless communication we have an analogous problem which requires different terminal conditions.

In wireless reception we often employ an elevated system of conductors whose linear dimensions are either equal to or comparable with the length of the incident electromagnetic waves: currents are propagated in the conducting antenna and are urged by electric forces distributed along the whole or a considerable part of the cable system and not by a battery applied at one end.

For example, consider an isolated wire placed parallel to the electric component of an advancing harmonic electromagnetic wave: then at any instant every portion of the wire is situated in the same electric field and the resulting current distribution will not be the same as if an harmonic electric force had been impressed only at the middle of the wire. We must solve the equations of a system having distributed capacity and inductance energised by an harmonic electric force distributed uniformly along it.

Let us consider an electric wave advancing parallel to a plane perfectly conducting earth with the electric component perpendicular to the surface of the ground. Let this wave strike a straight wire of length l placed perpendicular to the ground and with its lower end earthed: let the capacity and inductance per unit length of this wire be supposed constant and equal to C and L respectively, and let its resistance be zero. Let the electric field of the advancing wave be $f = F \sin pt$: then it is required to find the distribution of current along the wire and in particular the current at the earth connection: let i be the current at time t at a height x above the ground and let v be the potential at this point.

Then $$-\frac{\partial v}{\partial x} = L\frac{\partial i}{\partial t} + F\sin pt,$$

and $$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t},$$

$$\therefore\ \frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + pFC\sin pt \qquad \text{......(1)}.$$

We require a solution such that $i = 0$ when $x = l$ and $v = 0$ when $x = 0$: the second condition requires that $\frac{\partial i}{\partial x} = 0$ when $x = 0$.

We will suppose that $i = \phi(x)\cos pt + \psi(x)\sin pt$ is the required solution: then we obtain by substituting in (1)

$$\phi''(x) + LCp^2\phi(x) = pFC \qquad \text{......(2)}.$$

$$\psi''(x) + LCp^2\psi(x) = 0 \qquad \text{......(3)}.$$

$$\therefore\ \phi(x) = A\sin p\sqrt{LC}x + B\cos p\sqrt{LC}x + \frac{F}{Lp} \qquad \text{......(4)}.$$

and $$\psi(x) = D\sin p\sqrt{LC}x + E\cos p\sqrt{LC}x \qquad \text{......(5)}.$$

Since $\frac{\partial i}{\partial x} = 0$ if $x = 0$, therefore $\phi'(0) = \psi'(0) = 0$, and it follows readily that $A = D = 0$.

Also since $\psi(l) = 0$ and $\phi(l) = 0$, we have that

$$B = -\frac{F}{Lp\cos p\sqrt{LC}l},$$

and $$E\cos p\sqrt{LC}l = 0,$$

$$\therefore\ E = 0 \text{ if } p\sqrt{LC}l \neq \frac{n\pi}{2},$$

$$\therefore\ i = \frac{F}{Lp}\left(1 - \frac{\cos p\sqrt{LC}x}{\cos p\sqrt{LC}l}\right)\cos pt \qquad \text{......(6)}.$$

Let I_0 be the maximum value of the current at the ground connection, then

$$I_0 = \frac{F}{pL}\left(1 - \frac{1}{\cos p\sqrt{LC}l}\right)$$

$$= \frac{F}{pL}\left(1 - 1 - \frac{p^2LCl^2}{2} - \frac{5}{24}p^4L^2C^2l^4\right) \text{ approx. if } p\sqrt{LC} << 1$$

$$= -\frac{Fpl^2C}{2}\left(1 + \tfrac{5}{12}p^2l^2LC\right).$$

Now the total capacity $C_s = l \,.\, C$ and the fundamental resonance frequency is given by $p_0\sqrt{LC}l = \frac{\pi}{2}$,

$$\therefore\ I_0 = \frac{pC_s \,.\, F \,.\, l}{2}\left(1 + \frac{5\pi^2}{48}\frac{p^2}{p_0^2}\right) \qquad \ldots\ldots(7).$$

Hence, so long as $p << p_0$, the current at the foot of the antenna is nearly the same as that produced by an E.M.F. of value $\frac{Fl}{2}\sin pt$ acting on a concentrated capacity C_s.

When measuring radiated field strengths by means of an antenna we employ a factor called the "effective height of the antenna" which may be defined as follows. Let a P.D. $E\sin pt$, applied between the foot of the antenna and the ground, produce the same current I_0 there as an advancing wave whose field is $F\sin pt$: then the effective height h is defined by the equation

$$E = Fh.$$

We will now determine the relation between h, l and p, if we apply $E\sin pt$ at the foot of the wire: if L and C remain constant it follows readily that

$$\begin{aligned} I_0 &= -E\sqrt{\frac{L}{C}}\tan p\sqrt{LC}l \\ &= -E\sqrt{\frac{L}{C}}\,p\sqrt{LC}l\left(1 + \frac{p^2LCl^2}{3}\right) \text{ approx. if } p\sqrt{LC}l << 1 \\ &= -pC_sE\left(1 + \frac{\pi^2}{12}\frac{p^2}{p_0^2}\right) \qquad \ldots\ldots(8). \end{aligned}$$

Hence for the same value of I_0, we have

$$E = \frac{Fl}{2}\frac{\left(1 + \frac{5\pi^2}{48}\frac{p^2}{p_0^2}\right)}{\left(1 + \frac{\pi^2}{12}\frac{p^2}{p_0^2}\right)},$$

$$\therefore\ h = \frac{l}{2}\left(1 + \frac{\pi^2}{48}\frac{p^2}{p_0^2}\right) \text{ approx.}$$

Let λ be the length of the incident waves and λ_0 the fundamental wave-length of the antenna: then $\lambda_0 = 4l$, and

$$\begin{aligned} h &= \frac{l}{2}\left(1 + \frac{\pi^2}{48}\frac{\lambda_0^2}{\lambda^2}\right) \\ &= \frac{l}{2}\left(1 + \frac{\pi^2}{4}\frac{l^2}{\lambda^2}\right) \qquad \ldots\ldots(9), \end{aligned}$$

hence in most practical cases h is very nearly equal to $\frac{1}{2}l$.

We cannot use this solution to examine the resonance condition because the antenna had no resistance and consequently the current would be infinite. If the antenna possesses a constant resistance per unit length, the fundamental differential equation is satisfied by a space distribution in hyperbolic functions: the solution does not present any serious difficulties but the reduction for the constants is laborious, and troublesome to reduce to a convenient form.

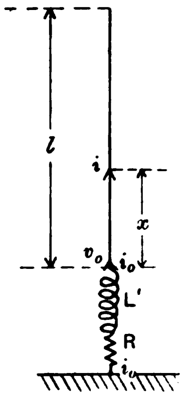

Fig. 1

In general an antenna is used to receive waves of a length much greater than its fundamental and usually it is tuned to the frequency of the incident waves, this may be done by placing a suitable inductance between the foot of the antenna and the ground. We may analyse this condition very simply: let the loading coil have an inductance L' and a resistance R. Now the boundary conditions are that $i = 0$ when $x = l$ and that when $x = 0$

$$v_0 = -\left(L\frac{di_0}{dt} + Ri_0\right).$$

Hence $$\frac{\partial i}{\partial x} = C\left(L\frac{d^2 i_0}{dt^2} + R\frac{di_0}{dt}\right) \text{ when } x = 0.$$

Let $i_0 = I_1 \sin pt + I_2 \cos pt$, then it follows from (4) and (5) that

$$B = I_2 - \frac{F}{Lp} \text{ and } E = I_1.$$

Also it follows after some reduction that

$$p\sqrt{LC}\,A = -L'Cp^2 I_2 + RCpI_1,$$

$$p\sqrt{LC}\,D = -L'Cp^2 I_1 - RCpI_2,$$

and that $$A \sin p\sqrt{LC}\,l + B \cos p\sqrt{LC}\,l + \frac{F}{Lp} = 0,$$

and $$D = -I_1 \cot p\sqrt{LC}\,l.$$

From these it may be found that

$$I_2 = \frac{(p\sqrt{LC}\cot p\sqrt{LC}\,l - p^2L'C)\,I_1}{pRC},$$

and that

$$I_1\{R^2p^2C^2 + (p\sqrt{LC}\cot p\sqrt{LC}\,l - L'Cp^2)^2\} = \frac{RpC\,Fp\sqrt{LC}\,(\cos p\sqrt{LC}\,l - 1)}{Lp \sin p\sqrt{LC}\,l}.$$

It may be found that $(I_1^2 + I_2^2)$ is a maximum with respect to variations of L', when

$$L'Cp^2 = p\sqrt{LC} \cot p\sqrt{LC}\,l,$$

i.e. when $$p = \frac{1}{L'}\sqrt{\frac{L}{C}} \cot p\sqrt{LC}\,l.$$

This is the condition of resonance and is independent of R. This condition makes I_2 zero, which was to be expected.

Then $$I_1 = \frac{F}{p\sqrt{LC}R} \frac{(\cos p\sqrt{LC}\,l - 1)}{\sin p\sqrt{LC}\,l}.$$

Now suppose $p\sqrt{LC}\,l = \frac{\pi}{2}$, then $L' = 0$,

and $$I_1 = \frac{F \cdot \frac{2}{\pi} l}{R}.$$

Hence at the fundamental frequency the effective height for reception is equal to $\frac{2l}{\pi}$.

If $$p\sqrt{LC}\,l << 1,$$

$$I_1 = \frac{F}{p\sqrt{LC}R} \frac{\left(1 - 1 + \frac{p^2LCl^2}{\underline{|2}} - \frac{p^4L^2C^2l^4}{\underline{|4}}\right)\left(1 + \frac{p^2LCl^2}{\underline{|3}}\right)}{p\sqrt{LC}\,l} \text{ approx.}$$

$$= \frac{Fl}{2R}\left(1 + \frac{p^2LCl^2}{12}\right)$$

$$= \frac{Fl}{2R}\left(1 + \frac{\pi^2p^2}{48p_0^2}\right) \text{ since } p_0\sqrt{LC}\,l = \frac{\pi}{2}$$

$$= \frac{Fl}{2R}\left(1 + \frac{\pi^2}{4}\frac{l^2}{\lambda^2}\right).$$

Hence it is obvious that the value of h agrees with that already found in (9) and so the effective height is not altered by tuning.

When the antenna is used for transmitting, again we use the term "effective height," but now it is defined by the equation $h = \frac{\int_0^l i \, . dx}{I_0}$. It is important to enquire whether the two values of h are the same supposing that the distribution of L and C remains the same: for although it seems natural to presume that the effective height is the same both for transmitting and receiving it does not appear that the theorem can be proved from any general considerations of energy.

When transmitting from a resistanceless uniform antenna the current distribution is sinusoidal and hence it follows readily that

$$i = \frac{I_0 \sin a(l-x)}{\sin al} \text{ where } a = p\sqrt{LC},$$

$$\therefore \int_0^l i \, . \, dx = \frac{I_0}{a \sin al}[1 - \cos al]$$

$$= \frac{I_0 l}{2}\left(1 + \frac{\pi^2}{48}\frac{p^2}{p_0{}^2}\right) \text{ approx. if } al << 1,$$

and hence it is seen that the effective height of such an antenna is the same both for transmitting and receiving.

But the current distribution is not the same in the two conditions nor is $\int_0^l i \, . \, dx$ the same in the two conditions: it is approximately equal to $\frac{1}{2}I_0 l$ for transmitting and to $\frac{2}{3}I_0 l$ for receiving.

Consequently for a given current at the ground the energy radiated from the transmitter is less than the energy reradiated or reflected from the receiver. Therefore it would appear that the effective resistance of an antenna depends on whether it is receiving or transmitting: but since most antennas have a low efficiency the difference in the total resistance would be small.

It is usual to provide an antenna with a large horizontal roof of wires situated at the top of the vertical wire: the distributed capacity of the vertical wire is dominated by the capacity of the roof and consequently the effective height is increased and may approach very closely to the value l.

If the wires of the roof are horizontal and if the incident electric field is vertical there is no electric force acting along the roof and therefore its only function is to increase the total capacity of the system. Therefore we could analyse this by imposing the boundary condition that when $x = l$, $i = K\frac{dv_l}{dt}$, where K is the capacity of the roof. But the outstanding features of the problem are illustrated sufficiently clearly by the model circuit depicted in Fig. 2, where the total capacity of the vertical wire is supposed concentrated at half the height of the antenna, and an alternator of E.M.F. $E \sin pt$ is placed in each half of the antenna.

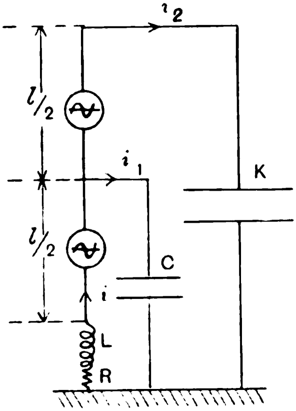

Fig. 2

Then we have

$$pE\sin pt = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i_1}{C} = \frac{i_2}{K} - \frac{i_1}{C},$$

also

$$i = i_1 + i_2.$$

Whence

$$\left(LD^2 + RD + \frac{1}{C+K}\right)i = pE\left\{2 - \frac{C}{(K+C)}\right\}\sin pt;$$

∴ at resonance

$$i = \frac{2E}{R}\frac{\left(K + \frac{C}{2}\right)}{(K+C)}\sin pt$$

$$= \frac{Fl}{R}\left(\frac{K + \frac{C}{2}}{K+C}\right)\sin pt, \text{ since } E = F\frac{l}{2}.$$

Hence the effective height is given by

$$h = \left(\frac{K + \frac{C}{2}}{K+C}\right)l.$$

If the upper alternator is suppressed the system is appropriate to a transmitter and it follows readily that

$$\frac{\int_0^l i\,.\,dx}{I_0} = \frac{\left(K + \frac{C}{2}\right)}{(K+C)}\,l.$$

Hence once more the effective height is the same both for transmitting and receiving: but $\int_0^l i\,.\,dx$ is not the same for transmitting as for receiving and again it appears that the apparent resistance of the antenna is somewhat less for transmitting than for receiving. This difference disappears in the limiting condition when $\frac{C}{K}$ tends to zero.

On the positive flash in vacuum discharge tubes. (Preliminary note.) By Mr R. Whiddington, St John's College.

(With Plate V.)

[*Read* 19 January 1925.]

It has been known for a very long time that, under certain conditions rather difficult to specify, the discharge through an ordinary two-electrode vacuum tube containing gas at a low pressure, even when the potential is maintained by a battery of low resistance storage cells, is intermittent*.

During the course of some experiments I recently carried out on the extension of the Balmer Series spectrum of hydrogen at low pressures, using a glowing cathode, it was observed that when the current density in the tube reached comparatively high values this intermittence, during the cleaning up of the tube, was particularly marked.

Remembering the experiments of Aston and Kikuchi† in which irregular intermittence had been observed with a discharge through the rare gases it was thought worth while to examine carefully the particular case of argon, which gas is particularly easy to obtain in quantity at the present time.

Striking experimental results have been obtained and their interpretation, admittedly incomplete, is very interesting so far as it has been found possible to go at present.

This account is intentionally brief and is intended only as a preliminary notice.

The discharge tube was of quartz, with both anode and cathode well water-cooled. A Pirani gauge to observe any rapid pressure fluctuations was used in conjunction with a McLeod gauge of the usual type. Joints were made with high melting point cement so as to avoid as far as possible vapour impurities. Evacuation was carried out by oil pump, diffusion pump and liquid air traps in the ordinary way.

The cathode was usually a spiral of tungsten wire battery heated, while the discharge through the tube was provided by a battery of lead accumulators giving up to 300 volts.

Currents up to 3 amperes could be passed for short periods through the tube.

With pure argon in the tube the current was nearly always intermittent with a very constant frequency. Observed in a rotating mirror the appearance of Fig. 1* is frequently seen.

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* See e.g. M. Cantor, *Wied. Annal.* 67, 2 (1899), and Rieche, *Ann. d. Phys.* 4, p. 614 (1901).

† *Proc. Roy. Soc.* 98 (1920).

This is an untouched reproduction of an actual photograph, and shows clearly that there are luminous flashes passing through the tube (in this case) from anode to cathode†, all travelling at the same speed throughout their visible journey and succeeding each other at regular intervals of time. These anode flashes travel with a speed dependent on the gas pressure, approximately inversely proportional to the pressure, but the spectrum when photographed through a Fabry and Perot etalon shows no signs of the Döppler effect, suggesting that the radiating centres of luminosity are themselves not in motion down the tube.

Sometimes there is a background of luminosity in the tube through which these flashes move but often under the conditions of the experiment the whole of the current was carried by the positive flashes as was proved by examining the form of the pulses of current by suitably applying a cathode ray oscillograph.

Under such conditions as this, it was calculated that with the higher currents few if any of the atoms of gas present could be escaping ionization.

Sometimes the appearance in the rotating mirror was that of Fig. 2. Here there are clearly two sets of flashes both proceeding in the same direction but with very different velocities, the ratio in all cases so far examined being about 4·2 : 1.

An explanation of this type of discharge, which seems to fit most of the observed facts, supposes that positive ions pass down the tube away from the anode emitting in their passage invisible (probably ultraviolet) radiation, which excites luminous radiation in those atoms over which they pass. This would account for the absence of the Döppler effect. The supply of these ions is kept up by electrons existing in the space between the sheets of positive ions, which acquire ionizing speed in falling from the mid position under the potential set up by the positive space charge. The positive ions themselves move with a speed determined partly by the field set up by the electrodes but mainly by the resultant field produced by neighbouring positive sheets. At first sight this seems highly improbable, for a glance at Fig. 1 for example shows that the flashes are apparently equally spaced down the tube; this being so it might be thought that in the case of a flash in the central part of the tube its neighbours on the anode and cathode side would be equidistant and that their effects would thus balance. It is possible however from a knowledge of the number of flashes in the tube, their speed, and the current passing, to calculate the charge in each flash; assuming now that Langevin's formula for ionic mobility holds, the electric force necessary to produce the observed speed of flash can then readily be estimated. It comes out that a force of this magnitude would be called into existence if the flashes were not exactly equally spaced along the tube but were very slightly nearer together at the anode end. A very close examination of the plates shows this very slight change in spacing of the flashes along the tube of about the right amount.

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* Simultaneous photographs in the mirror of a mercury arc fed with A.C. of known frequency are seen on the right of Fig. 2 and in the centre of Fig. 1.

† Occasionally there is evidence of flashes travelling from the cathode.

There is an interesting possibility in the interpretation of the two sets of flashes shown in Fig. 2, for from Langevin's formula the ratio of the speeds might have been expected to be exactly 4 : 1.

The fact that the ratio is rather greater suggests that the more highly charged ions in the faster flashes behave as though they had a smaller effective diameter, which should be calculable: this point however is not yet fully investigated.

As to the manner in which so remarkably regular a series of flashes can be set up, it seems possible to give a fairly accurate interpretation on the same basis that I have already put forward to explain the presence of ionic oscillations in thermionic valves containing traces of gas*.

In the present instance we must imagine positive argon ions moving away from the anode, low speed electrons being set free from the gas through which they pass. These electrons fall back to the anode and acquire energy depending on the distance traversed: when this distance is great enough for ionization by collision at the surface of the anode to occur, a new flash starts. The new flash repeats the process and so the regular sequence is set up.

It is intended, in a short time, to present a full account of these experiments and their explanation, but it is hoped that this preliminary note may not be without interest.

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* *Radio Review*, 1919.

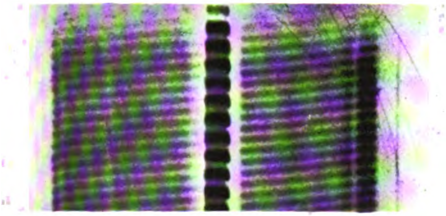

Fig. 1

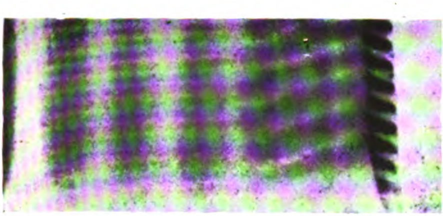

Fig. 2

The structure of radiation. BY DR EDMUND C. STONER, Emmanuel College.

[*Read* 19 January 1925.]

Introduction.

The difficulties in the way of attaining a coherent view which embraces both the structure of the radiation field, and of matter, and allows the appropriate interchanges of energy between them, are well known*. They are indeed such as to seem inherent in the phenomena themselves, and to suggest that only a formal treatment can be given. Interference, and "pure optical" effects generally are such as can be most simply explained by assuming transverse waves to spread out from the source of light. If atoms emit independently, it must be supposed that it is not merely statistical average effects which are observed, but that the light coming from each individual atom can spread spherically, and that interference can occur between resulting secondary wavelets. The effect produced at a point of observation is a function of its distance from the source not along one path only, but along all possible paths, and the maximum path difference for interference gives a measure of the "coherence length" of the train of waves, that is the number of waves emitted by the source without change of phase. To account for the whole range of phenomena typified by the photoelectric effect, on the other hand, some form of light quantum hypothesis which supposes radiation to be propagated by discrete entities, small "bundles of energy," seems absolutely necessary. Neither the wave nor the quantum hypothesis, in its present form, however, seems to offer any means of explaining the phenomena covered by the other; indeed each hypothesis seems definitely, in many cases, to rule out the possibility of the occurrence of effects which are actually observed.

An attempt to overcome some of the difficulties has been made by Jeans and Bohr, by the former in terms of what may be called "waves of probability," and by the latter in terms of "virtual fields of radiation." Neither of these view-points suggests any satisfying physical picture, and both involve the abandonment of the principles of conservation of energy and momentum applied to individual emission and absorption processes. This step at present seems too far-reaching unless definite evidence can be found in its favour, for it is precisely on the basis of those conceptions which lead to the conservation laws that the most striking triumphs of the quantum theory have been founded; and such evidence as there is seems antagonistic. It seems desirable, then, to suppose these laws to hold, and to explore the problem as far as possible, assuming their validity.

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* For a general account of the present status of the problem see Jeans, *Report on Radiation and the Quantum Theory* (2nd ed. 1924), particularly Ch. VIII, and Bohr, *On the Application of the Quantum Theory to Atomic Structure*, Pt I (1924), particularly Ch. III, supplemented by Bohr, Kramers and Slater, *Phil. Mag.* 47, 785 (1924).

It will be shown here that, assuming energy and momentum to be conserved in the individual processes of emission, absorption and scattering, there seems no escape from the conclusion that radiation is propagated by linearly directed quanta. Evidence is brought forward which suggests that these quanta are spatially localised longitudinally as well as laterally, having, that is, a point rather than a line structure.

The properties of the radiation field will depend on the characteristics of the quanta (or "quants" as they may be termed when regarded as discrete physical entities) of which it is constituted; the specification of these involves a more detailed knowledge of the actual transition processes than is at present provided by theory. The evidence for discrete quants, however, seems sufficiently strong to justify a discussion of the question as to whether there is any possibility of associating with them properties which would enable them to produce effects such as are ordinarily explained on the basis of a wave structure of radiation. The conception of the phase of a quant is introduced and it is shown that if a quant has a constant phase at emission, and if further the effect of a quant on atoms which it passes may be supposed to persist, it seems not impossible to obtain a tentative explanation of interference phenomena.

Some of the characteristics of quants may be deduced from the polarisation phenomena observed, particularly under the influence of magnetic and electric fields. These are briefly considered in relation to the angular momentum changes which take place in the atoms on emission and absorption. It seems as if the ultimate key to the problem as to how quants can "imitate" waves is to be sought by investigating phenomena in which the emitting or absorbing systems are restricted as to orientation, and the polarisation of the light is of a definite type.

The Linear Direction of Quanta.

In whatever form radiation exists in free space there can be no doubt that matter can only gain or lose radiant energy in complete quanta. If as a result of a quantum switch an atom loses an amount of energy E, this energy must then be present in free space in a form, and with properties which must be deduced from the behaviour of matter surrounding the emitting source. From this it is concluded that radiant energy travels with a velocity c, and that a quantum emitted may be re-absorbed as a whole. The question at issue is whether quanta emitted from atoms spread spherically, with power to collapse at one point, or travel linearly. The former idea will be considered first.

On the spherical spreading view, which is widely held—Jeans states that "it is now generally accepted that in regions where matter is absent radiation is propagated in accordance with the classical laws of Maxwell"*—the field surrounding an emitting atom may be considered as being such as would result on the classical theory from the vibrations of an electron in an appropriate orbit. In the hands of Bohr this view has not been unfruitful. In the quantum picture of the atomic change resulting in such emission, however, there seems no possibility of introducing an electron vibrating in the required way. The electron is an "effective" one. Moreover, insuperable difficulties arise when absorption is considered; any spreading, and consequently weakening, seems incompatible with the view of absorption which the quantum theory suggests as the simple inverse of emission, in accordance with the wider generalisations of thermodynamics.

Jeans attempts to evade some of the inconsistencies by a re-interpretation of the meaning to be assigned to the symbols X, Y, Z, expressing on the old electrodynamics the electric force components, and the amplitude A of the waves. "We can avoid all contradiction with the conceptions of the quantum theory," he says, "if we suppose that X, Y, Z, or some functions of these quantities, measure, in some way at present unknown, the probabilities of jumps in the velocity and perhaps also in the position of an electron which forms part of an atomic system."† Further, $A^2/h\nu$ may be taken as a measure of the probability of a quantum $h\nu$ of energy being transferred in unit time.

It is here suggested that such a treatment virtually begs the whole question. Consider a single atom which has emitted a quantum of radiation which may be supposed to spread through space as a spherical wave train. At any particular point there is a probability depending on $A^2/h\nu$ of its being absorbed. When it is absorbed, the whole wave system disappears; and the idea of such disappearance seems no more comprehensible for probability waves than electromagnetic ones. "From the point of view of the old mechanics," Jeans says, "[photoelectric] transfer of energy is entirely unintelligible."‡ If the different parts of the wave system do not carry appropriate energy, however, there seems little value in postulating its existence—for our interest in waves depends fundamentally on their energy-carrying properties; if they do, it is necessary to suppose that absorption begins at a particular point on the wave front (or that energy is absorbed from a part of the wave train that has passed) and that the energy all "collapses" on to this point, if absorption is always by quanta; this is equivalent to postulating a particularised point of absorption, which at once undermines the attempted explanation of interference.

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* *L.c.* p. 80.

† *L.c.* p. 81.

‡ *L.c.* p. 82.

Jeans' treatment, then, while of value in bringing out difficulties, is purely formal. It may be briefly summarised by saying that if, to account for interference, it is desired to retain the idea of waves spreading spherically from an emitting atom, it is necessary to reinterpret $A^2/h\nu$ in terms of probability of energy transfer from the whole wave system.

The idea that emission of radiation is completely directed—the "light-dart" hypothesis—is associated with the name of Einstein, who derived arguments in its favour from a consideration of the recoil of atoms from radiation necessary to maintain the correct energy distribution under temperature equilibrium conditions*. The conclusion has not been generally accepted†, and is not a necessary consequence of Planck's law. Breit‡ has in fact shown that while linearly directed quanta are not inconsistent with the known equilibrium conditions, these can be maintained equally well by assuming that emission is non-directed, if account is taken of the fact that finite though small ranges of frequency are associated with quantum transitions.

Unless somewhat fantastic conceptions are adopted, however, a simple consideration of photoelectric absorption seems to provide a very cogent argument in favour of linearly directed quanta. In complete accord with electromagnetic theory, and relativity, experiments on the pressure of radiation have shown it to be statistically true that absorption of radiant energy E is associated with transfer of momentum E/c. According to modern views all absorption of radiant energy takes place by quanta $h\nu$, and it seems difficult to escape the conclusion that this process is associated with transfer of momentum $h\nu/c$. Actually the association is only known experimentally to be true for a large number of switches; if in some switches less momentum is transferred, in others more must be. Now for waves with energy E the total associated momentum varies from zero for spherical waves to a maximum of E/c for plane waves. Unless some artificial means of increasing this maximum is devised, it seems, therefore, that all absorption corresponds to absorption of plane waves—in other words, since absorption takes place only by quanta, that quanta are linearly directed.

The importance of this conclusion may make it worth while to look at the question from a slightly different standpoint, which enables the distinction between the classical and quantum pictures of emission to be seen more clearly. Consider a classical radiator emitting spherical waves. The total momentum associated with the wave system is zero. If absorption occurs over a certain area, momentum is transferred, the momentum of the modified wave system now being no longer zero. The conception of spherical radiation can only be considered as consistent with the energy-momentum relations if absorption can occur over part of the wave front, that is, assuming that a single quantum is associated with the whole wave system, if absorption of fractions of quanta can take place. That quanta are linearly directed seems to follow uniquely from the fact that emission and absorption of radiation can only occur in complete quanta.

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* Einstein, *Phys. Zeit.* 18, 121 (1917).

† Cf. Bohr, *l.c.* p. 40.

‡ Breit, *Phys. Rev.* 22, 313 (1923).

It is more difficult to find conclusive objections to the "virtual field" view than to that of Jeans, partly because of its greater vagueness; but it has a number of unsatisfying features. According to the conceptions put forward by Bohr, Kramers and Slater, there is some interconnection between two atoms one of which emits, and one absorbs, on account of the virtual field surrounding each, but although the probabilities of absorption and emission transitions depend on the character of the fields, the times at which corresponding transitions occur are not related in any definite manner. If one atom undergoes an emission transition, a second atom at a distance l may undergo a corresponding absorption transition, but the second event does not necessarily occur at a time l/c after the first. There seems no reason, in fact, why an absorption switch should not actually occur before the "corresponding" emission switch.

Now in Wilson's photographs* large numbers of definite pairs of long and short range tracks occur (those referred to as class 2), the smaller of which can scarcely be interpreted as due to anything other than an electron ejected from the second atom by the K radiation emitted by the first on its return to a normal state after a fast β particle, represented by the longer range track, has been ejected from it. The mere fact that such tracks are definitely associated in pairs strongly suggests that the second switch is definitely the result of the first switch rather than that it occurs by chance, or probability, on account of the virtual field surrounding the first atom.

On Bohr's view the pairing might be linked up with the changing of the virtual field resulting after a transition switch. Bohr, however, says that "the occurrence of a certain transition in a given atom will depend on the initial stationary state of this atom itself and on the states of the atoms with which it is in communication through the virtual radiation field, but not on the occurrence of transition processes in the latter atoms."* On the present view, the time between the switches should be l/c; but the possibility of experimental verification of this seems remote.

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* Wilson, *Proc. Roy. Soc.* 104, 1 (1923), see p. 18.

As to conservation, however, it may at least be said that it seems unwarranted to assume that it does not hold in individual processes when there is no definite evidence of its breakdown, unless the supposition leads to a much more complete and satisfying explanation of observed phenomena than has hitherto been put forward.

The argument developed above for directed quanta is quite independent of the evidence which is gradually accumulating in connection with mono-electric quantum scattering, particularly in relation to the "fish tracks," due to recoil electrons, observed by Wilson and Bothe.

Skobelzyn† also has recently used the cloud-chamber method to investigate the secondary radiations produced by γ-rays. The greater range of the recoil electrons in this case enables more precise measurements to be made, and the results are definitely in accord with a mono-electronic scattering of linearly directed quanta, as opposed to spherically spreading radiation. Evidence from this direction will not, however, be further considered here, but reference may be made to recent papers on this question by Compton‡.

The Spatial Localisation of Quanta.

Accepting tentatively the conclusion that quanta are linearly directed, and hence that they may be pictured crudely as having a small, though undefined, cross section, the question arises as to their degree of extension in the direction of propagation. To suit different circumstances values varying from zero to infinity have been postulated. Assuming quantum emission, however, the length may be taken as finite, and the difference of degree assumes importance as a difference in kind when an attempt is made to decide whether the length is of the order of one or a million wave lengths.

Stationary waves will first be considered. For the derivation of the correct Planck distribution law for equilibrium temperature radiation it is necessary, as is seen most clearly in Jeans' treatment, to regard the radiant energy as concentrated in integral multiples of quanta $h\nu$ in definite stationary ethereal vibrations set up in the impermeable isothermal enclosure. The energy of a whole quantum must be supposed to be capable of localisation within a volume λ^3. Since stationary waves may be regarded as due to the superposition of progressive waves, although in terms of "quants" the process may be obscure, it seems necessary to suppose that in progressive "waves" also, the energy is similarly concentrated.

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* *Phil. Mag.*, *l.c.* p. 791.

† Skobelzyn, *Zeit. für Phys.* 28, 278 (1924).

‡ Compton, *Phys. Rev.* 23 (1924).

This may be brought at once into relation with the characteristics of quantum transitions. Adopting a straightforward view, the length of a quantum could not be greater than ct, where t is the time between the beginning and end of a transition. These transitions are usually regarded as "sudden" jumps, which is unfortunately somewhat vague. The conception may, however, be rendered more precise by the consideration that the time t occupied by a transition must be indefinitely small compared with the duration of life τ of the atom in a stationary state, for otherwise there would not be that sharp distinction between stationary and intermediate states which forms so essential a feature of the quantum picture of the atom. Now the mean life of a number of atoms in upper stationary states has been estimated by a variety of methods* and the results are in agreement that the time τ is generally of the order 10^{-8} sec. In this time a wave train of some 10^6 wave lengths could be emitted; so that in the time t of a transition, the length of the train which could be emitted must be indefinitely less than this. The usual view of a transition, then, is in accordance with the supposition that the energy of a quantum is localised longitudinally within a length of the order of one wave length.

With a view to a possible explanation of interference, however, it is natural to attempt to retain waves even of the rope type, and to attribute to the train such a length as the observed "coherence" would demand. If an eel-like quantum (having a finite length, with head and tail) is assumed, difficulties of a type previously mentioned immediately arise. If the energy is distributed along the eel, so that the waves have some physical reality, since absorption, being by quanta, corresponds to the disappearance of the whole eel, either absorption of radiation which has passed a point can occur, or absorption necessarily begins at the head of the eel. Both alternatives are unsatisfactory, the former robbing the assumed structure of physical reality, and the latter undermining its power to explain interference effects.

An important experiment, beautifully simple in idea, recently described by G. P. Thomson†, sheds a good deal of light on the question, and supports the above views in that the results seem quite incapable of explanation on the basis of a line-like, as opposed to a point-like, structure of the quantum.

The light emitted by positive rays moving transversely to a slit was examined spectroscopically. The disposition of the slits was such that, for a single moving emitting atom, only the light emitted during a very short determinate time could affect the spectroscope. In a typical experiment quoted, for $H\beta$, this corresponded to 3·5 cm. as the "length of the longest train of waves which could enter the spectroscope from one atom." Now Thomson calculates on the basis of the usual interference theory that 6 cm. is the least distance over which the vibrations of a radiator must extend without change of phase for the width of the line to be as small as it is. This then gives the minimum length of the quantum (on the eel view) corresponding to the classical wave train, and it would be anticipated that if only the energy in a fraction of this length could reach the point of observation, the normal effects of $H\beta$ frequency radiation would be modified. Yet in the experiment no modification in the spectral character of the light was observed.

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* E.g. Tolman, *Phys. Rev.* 23, 693 (1924); Milne, *Phil. Mag.* 47, 209 (1924).

† Thomson, *Proc. Roy. Soc.* 104, 115 (1923).

From this it would seem to follow that the length of a quantum probably bears no relation to the coherence length of the classically corresponding train of waves; the characteristic properties of a quantum, which determine its effect on matter, being localised in a length certainly much less than this.

Thomson puts forward five possible deductions, not necessarily mutually exclusive, of which the most satisfactory (2) appears to be that the energy of the quanta is concentrated in points and not in lines. The others may be briefly criticised: (1) that one emitting particle emits several quanta consecutively in phase with each other, and (3) suggesting a "trigger effect" in absorption, are not in harmony with the quantum picture of the atom; (5) that "an electron can collect energy from a wider area than seems possible on the classical theory" is rendered improbable by the arguments advanced above; (4) that conservation of energy is statistical only, is, as Thomson remarks, difficult to distinguish from (2) experimentally, but, as already discussed in connection with virtual fields, the idea is so far-reaching and disturbing that much more evidence is required before it should be accepted. Deduction (2) may be tentatively accepted, with the obvious reservation that "point" is not to be taken in the strict mathematical sense.

It is certainly going beyond the experiment to suppose that the quanta are indefinitely small. Assuming quantum emission, what the experiment does definitely show is that there are phenomena of the "pure optical" type which it is impossible to explain by an attempted association with a quantum of a classical wave structure, but that these phenomena are consistent with a point-like quantum.

In this connection reference may be made to a series of papers by Ornstein and Burger* in which, by an extension of a method used earlier by de Broglie†, a kinetic theory treatment is applied to the determination of the equilibrium between radiation and electrons, the radiation being considered as concentrated in quants with effective cross sections varying as the square of the wave length. It is shown further that the quant may consistently be supposed to have its energy concentrated within an "effective sphere" of volume proportional to λ^3, the factor, somewhat crudely estimated, being of the order of a tenth.

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* Ornstein and Burger, *Zeit. f. Phys.* 20, 345, 351 and 21, 358 (1924).

† de Broglie, *Journ. de Phys.* VI, iii, 422 (1922).

Any attempt to estimate the "size" of a quant involves the difficulties as to the interpretation of the terms used (a similar difficulty arises in connection with the size of the electron), and there is necessarily considerable vagueness; bearing this in mind, however, there can be little doubt that the evidence so far reviewed, if interpreted in any straightforward manner, leads definitely to the conclusion that radiation is propagated by linearly directed, spatially localised quanta.

Interference.

Although arguments in favour of regarding quanta as "accumulations of energy in certain minute spaces" that have hitherto been put forward have been somewhat indefinite, the simplicity of the view, and its power of co-ordinating so many phenomena, would have made it much more generally acceptable but for the obvious grave objections that can be brought against it.

The main objection is clearly stated by Lorentz*. If quanta are mutually independent, interference and resolving power effects show that each quantum must have considerable length, of the order of some million wave lengths, in the direction of propagation, and also a considerable breadth at right angles to it, of the order of the maximum useful diameter of a telescope objective.

Bohr† further maintains that the light-quant picture "excludes in principle the possibility of a rational definition of the conception of a frequency ν."

While completely adequate answers to these objections certainly cannot at present be given, in view of the strength of the light-quant position, it seems legitimate to inquire what sort of modifications in outlook are necessary, or what additional properties of radiation must be postulated, in order that interference and allied phenomena might be explained on a light-quant basis.

As to Bohr's objection, it does not seem to carry great weight; for if a quant is assumed to have periodic properties, so that it varies periodically in some way as it travels through space, a conception of frequency and also of "wave length" can at once be arrived at. The "wave length" is, of course, a term that can only be used by analogy, in that adoption of the quant involves abandonment of the wave picture. The "wave length" may be defined as the least distance between two points at which the quant has the same characteristics; the period as the time between two returns to the same state, and the frequency the inverse of this. The idea of the "phase of a quant" may also be introduced, a measure of it being given by the time which has elapsed since the quant was in some arbitrarily defined state. As to a more definite specification of these states, it may be tentatively suggested that an interpretation is to be sought in the variation in the magnitudes, absolute and relative, of the electric and magnetic vectors which characterise the quant. These presumably vary jointly in such a way that the total energy associated with a quant remains constant. The specification of the phase of a quant would involve a specification of the magnitudes and directions of characterising electric and magnetic vectors.

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* See Jeans, *l.c.* p. 79.

† Bohr, *l.c.* p. 35.

A wave theory can account admirably for the production of interference fringes; but to explain interference it is necessary to do more than this; it is necessary to correlate the observed intensity at different points of a fringe system with the number of quantum switches induced by the radiation. Maxima and minima of fringe systems must ultimately be regarded as points at which the greatest and least number of quanta are actually absorbed (or absorption switches induced), and not merely as points passed by a greater or lesser number of quanta.

A single quantum could not produce a fringe system, yet the idea that this is possible lies at the root of the gravest objection to light quanta. "The bright and dark fringes to which it [a beam consisting of separate quanta] gives rise," Lorentz says, "could never be sharper than those that would be produced by a single quantum." The picture of a single quantum producing a fringe system over which there is a variation in the total number of whole quanta absorbed is purely fantastic, and perhaps points out a direction of possible escape from difficulties.

An explanation of interference must be sought in the joint action of different quanta, and not in the properties of a single one. Experimentally, interference fringes are observed when radiation reaches the point of observation by different paths from a "single point" source. It is important to notice that interference effects do not depend for their appearance on the light originating from a single emitting source, say a single atom. The phase relations necessary are apparently satisfied if the light can be supposed to emanate from a "geometrical point"; there seems to be no evidence that the physical structure of the point (that is the actual emitting systems) must remain constant even over periods comparable in time with the duration corresponding to the coherence length of the wave trains emitted (as is shown in a special case by Thomson's experiment). Though the reason may thus be obscure, even on rthodox classical consideration, from a wave standpoint, the ngle point source ensures that there shall be some definite phase lation between the interfering wavelets. To account for interrence properties there must be some corresponding relation for ifferent quanta originating at the same point. It seems not unatural or far-fetched to assume that their phase at the point of rigin is the same—that the "phase at emission" for similarly olarised quanta, at least, is constant, and hence that different uanta travelling by two paths arrive at a point of observation in he same phase if the path difference is a whole number of "wave engths," as above defined.

To "explain" interference, however, a further assumption is ecessary. It must be supposed that a quantum in a given phase assing an absorbing system, say an atom, tends to modify it by orientation or otherwise) in such a way as to increase its probability of absorbing a quantum in that phase. It is assumed, in other words, that for absorption to occur, "coupling" must take place between the absorbing system and the quantum; this "coupling" requires a definite relation between the phase of the quantum and the state of the system; and the passage of a quantum gives the system a definite impulse towards the state corresponding to its own phase. Thus the greater the number of quanta in the same phase which pass an absorbing system, the greater the probability of absorption of one of them; the passage of quanta of opposite phase, producing opposite impulses, will diminish the probability of absorption. It must of course be supposed that the effect of the impulse persists, unless the state is modified by external disturbances, for Taylor's classical experiment shows that interference effects still occur even for very weak light.

As to the mechanism involved, it may be tentatively suggested that the impulses are given in the process of scattering. This involves, in general, little transfer of energy, but a transfer of momentum, and probably also changes in direction of the angular momentum associated with the quantum and the absorbing system; this suggests strongly that one of the characteristics of the state which influences the probability of "coupling" and so of absorption is the orientation of the absorbing system with reference to the direction of propagation of the radiation.

"Coupling," of course, it may be mentioned parenthetically, cannot be supposed possible between any quantum and any absorbing system; it need only be supposed to occur in a complete sense when it results in a process of selective or photoelectric absorption; the tendency of a quantum to modify the state of an absorbing system is, however, much more general.

The return to the Newtonian corpuscular theory which the light-quant hypothesis suggests has been frequently noticed; and with it come all the difficulties of accounting adequately for the whole range of diffraction and dispersion phenomena. All these, however, may be regarded as special manifestations of scattering, and in so far as scattering may be explained on a quantum basis, wider developments will cover them also. The view that scattering is explicable on a quantum basis has, it need hardly be mentioned, been inherent in the whole of the present discussion on interference. While the goal is far from having been reached, the work of Compton and others, previously referred to, suggests that it is by no means unattainable.

The above remarks may be summarised as follows. There seems a possibility of explaining interference on a light-quantum basis by attributing phase properties to quanta, and to absorbing systems states corresponding to these phases, and assuming that the general reaction between a system and a quantum is such as to give the system an impulse, whose effect persists, towards the state corresponding to the phase of the quantum. The intensity of radiation at any point, if defined in terms of the number of absorption switches induced, will then depend not only on the number of quanta which pass it, but also on their relative phases. The varying intensity in a fringe system, on this view, does not arise from differences in the numbers of quanta passing, but owing to the phase relations between the quanta arriving at different points being different—in the bright parts tending to bring the atoms into states in which the reaction is a maximum, and in the dark parts a minimum. The nomenclature and general wave theory methods can largely be taken over and applied to the "phase field" surrounding point sources. A more definite picture of the meaning of the "states," and of the nature of the reactions, can only be arrived at by a closer study of the processes of emission and absorption, theoretically and experimentally. Experimental work of the relevant type is as yet in its infancy.

In the following section a few theoretical considerations are brought forward, and some experimental observations briefly noted, which seem to indicate that a more complete and precise explanation of interference on the lines vaguely sketched out above may eventually be possible.

The Connection between Quantum Properties and Transition Characteristics.

Recent successful developments in the application of the quantum theory, and the widening of its scope, have been mainly concerned with the structure of matter, rather than of radiation. Bohr's first postulate, for the fixation of the stationary states of the atom, is, however, intimately bound up with the second postulate, which defines the frequency character of the radiation emitted as a result of a change in the atomic state. While the nature of the processes of emission and absorption are unknown, n so far as a definite picture can be formed of the initial and final states of the atom in which the processes occur, it is possible to form some idea of what must be the properties of the radiation itself, in order that the changes in the radiation field resulting in emission and absorption may be more closely linked up with the changes in the atomic state. Conversely, a knowledge of the nature of the radiation field, and of the changes which take place in it when emission or absorption processes occur, would enable the changes in the atomic state to be deduced.

It is this linking which has been attempted above in reference to energy and momentum. Angular momentum, which can be treated similarly, has not so far been discussed. The correlation of changes in angular momentum of the emitting system with observed states of polarisation has been considered by Rubinowicz, Sommerfeld and Bohr, and developed in considerable detail by these and others. Until the nature of the atomic changes is more definitely known, however—in particular until a more precise meaning can be attached to "inner" and "magnetic" quantum numbers—no conclusive deductions as to radiation can be made. Here it is intended to discuss very briefly the connection which it seems must exist, so that any developments in atomic theory may be utilised for the elucidation of the radiation problem.

On the classical theory the total change in the angular momentum ΔP in the radiation field associated with a vibrating electron is related to the change in energy ΔE due to emission or absorption in a manner which varies according to the mode of vibration of the electron from

$$\Delta P = \frac{\Delta E}{2\pi\nu}$$

for a circularly vibrating electron to

$$\Delta P = 0$$

for a linearly vibrating electron. Carrying this over to the quantum theory, in which ΔE must be put equal to $h\nu$, it is supposed that the angular momentum associated with a quantum may vary from $h/2\pi$ (the unit) to 0. The former is supposed to correspond to circularly, the latter to linearly, polarised light. On the basis of the correspondence which must, in the limit, exist between quantum and classical processes, it has been shown with considerable probability by Bohr that in a normal emission process (that is one uninfluenced by external fields) the azimuthal quantum number k defining the angular momentum of the electron concerned in the atom must change by unity. Whether this condition is unavoidable theoretically or not, it does certainly provide a satisfactory selection principle for the occurrence of spectral lines so that it may be said from an experimental standpoint that in an emission process k changes by unity.

Consider now the simplest possible case of a quantum transition in which the electron switches from one orbit for which $k = n$ to a second orbit in the same plane for which $k = n - 1$. The emitting system loses unit angular momentum, and assuming conservation this momentum must be associated with the quantum emitted. Now Zeeman effect observations show that the type of polarisation of the emitted light depends on the direction of observation, that is on the direction of emission relative to the planes of the two orbits involved. From this it would seem to follow, adopting the localised quantum view, that the polarisation depends not on the absolute magnitude of the angular momentum associated with the quantum, which is possibly always unity, but on its resolved magnitude in, or at right angles to, the direction of propagation.

This conception clears up some of the difficulties which have hitherto beset the consideration of absorption*. An absorption transition of the normal type, the inverse of the emission transition, involves also a change in k by unity; and this can occur, without any breakdown of conservation laws, whatever the polarisation of the quantum, provided the absorbing system is suitably orientated.

The fact that conservation principles, with reference to angular momentum, may be easily retained in a consideration of processes in which the atomic change is comparatively simple, and relatively understood, makes it seem probable that they will hold also in the more complicated transitions which occur in the presence of external fields.

A large number of observed phenomena connected with the Zeeman effect may be correlated by assigning to the terms, as well as electronic azimuthal quantum numbers k, magnetic quantum numbers m which define the atomic angular momentum in the direction of the applied magnetic field. When this quantum number changes by unity (as well as k) the light observed is circularly or linearly polarised according to the direction of observation; but transitions also occur for which m does not change, and the light corresponding to these transitions is linearly polarised, with its electric vector in the direction of the applied magnetic field. Adopting the light-quantum view, this seems to suggest that, somewhat as on the classical theory two different mechanisms—namely, circularly vibrating electrons, observed in a special direction, and linearly vibrating electrons—may give rise to plane polarised light, plane polarised quanta may be of two different types. This dual nature of "plane polarised transitions" is, of course, remarkable whatever view of radiation is adopted, and it emphasises the gaps in the picture of transitions from the atomic side. Particularly it makes it hazardous at present to attempt to define any more precisely the exact relation between the directions of the axis of angular momentum, of propagation, and of the electric and magnetic vectors characterising quanta of a given state of polarisation.

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* Cf. Bohr, *l.c.* p. 41.

That orientation occurs in magnetic fields is shown absolutely definitely by the experiments of Gerlach and Stern, and that the polarisation character of the emitted light depends on orientation necessarily follows from the Zeeman effect. It is desired to emphasise here the importance of orientation in connection with absorption, which becomes clear from an examination of the experiments of Wood and Ellett* and others, on the effect of magnetic fields on the polarisation of resonance radiation. An analysis of their results† shows that certain absorption transitions may be entirely prevented by appropriate orientations of the magnetic fields. The probability of quanta in a definite state of polarisation being absorbed thus depends essentially on the orientation of the absorbing system.

The experiments on mercury vapour suggest even further possibilities. The resonance radiation with this was observed to be polarised even in the complete absence of external magnetic fields to an extent which seems to indicate that the absorbing systems were orientated in a definite way relative to the incident beam. With the experimental results available at present it would be unwise to speculate dogmatically; but the experiments certainly suggest strongly that absorbing systems may be orientated by the influence of a beam of radiation traversing them; the only mode of interaction, moreover, is that involved in scattering, which, in the case considered here, with the incident radiation corresponding to a "natural frequency" of the absorbing system, is of the "enhanced" type.

From the experiments, then, emerges the certainty that absorption depends on orientation, and the possibility that this orientation may be affected by radiation itself.

Considerations of the above type suggest the wide possibilities there are to be explored before hope of explaining interference on a light-quantum basis should be given up, or principles of conservation abandoned. The question of the influence of quanta in different phases—essential for a full explanation of interference—can only be tentatively and speculatively considered. "Coupling" and absorption of a quantum of a definite type, coming from a definite direction, will on the present view only occur, for a definite phase, if the atom is in a definite corresponding state; moreover this state will vary according to the phase of the quantum. The state may depend partly on orientation, but not wholly—Weiner's experiment alone seems to show this—but precisely what are the variables which define it is still obscure.

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* Wood and Ellett, *Proc. Roy. Soc.* 103, 396 (1923); *Phys. Rev.* 24, 243 (1924).

† See Joos, *Phys. Zeit.* 25, 130 (1924).

Orientation, however, by no means exhausts the possibilities. Orientation phenomena seem to be primarily magnetic in character. Now a quant is not characterised simply by a magnetic vector, but by electric and magnetic vectors jointly, whose directions and magnitudes define not only the state of polarisation but also the phase of the quant at any point of its path.

An electron orbit in an atom may also be regarded as characterised by two vectors, one of which—the one so far considered—defines its orientation and magnetic moment in a particular direction. In general, however, an orbit will be characterised by a second vector defining its electric moment, whose axial direction will be given by the line joining the orbit's "electric centre of gravity" to the nucleus of the atom. It seems as if this may be of equal importance with the magnetic vector.

Both the electric and magnetic moment of the orbit-nucleus (or orbit-core) system may be modified by a passing quant, and in order that a quant in a given phase may be absorbed it is supposed that they must have an appropriate value.

The necessity, in connection with interference, for supposing that emission occurred at constant phase, while absorption of quants in any phase was possible, seemed at first sight to run counter to the demands of thermodynamics for the strict reversibility of individual processes. The absorption of quants in any phase by suitably "modified" systems is now, however, seen not to be thermodynamically incompatible with the constancy of phase at emission from "unmodified" systems.

The conception of the electric vector, moreover, strongly suggests a clearer outlook on the difficulty of the dual nature of "plane polarised transitions" referred to above, circularly polarised components (in transverse observation with the electric vector perpendicular to the field) corresponding to sudden transition changes in the magnetic, and plane polarised components (in transverse observation with the electric vector parallel to the field) to changes in the electric vector.

The Stark effect becomes of equal importance with the Zeeman effect as a key to the whole problem, but the working out of the details must be left to the future.

It has been shown here that it seems possible to apply the law of conservation of angular momentum as well as of energy and momentum to individual quantum emission and absorption processes, and that this may enable the properties of radiation to be eventually more precisely defined. The importance of an electric vector defining orbit systems has been emphasised. Some of the factors which would be involved in an explanation of interference in terms of spatially localised linearly directed quanta, to which the previous arguments led, have been considered, vaguely and speculatively. Vague speculation may however be justifiable if the only alternative is theory involving the complete abandonment of fundamental physical generalisations, and the substitution of purely formal concepts.

Summary.

It is shown that, assuming conservation of energy and momentum in individual transitions, radiation must be regarded as propagated by linearly directed quants. Evidence is brought forward in favour of the view that these quants are spatially localised longitudinally as well as laterally. As these conclusions seem unavoidable, an attempt is made to see if there is any possibility of accounting for interference effects in terms of quants possessing periodic properties.

The idea that a single quantum can produce interference effects is untenable. The conception of the phase of a quant is introduced, and it is supposed that atoms may absorb a quant in a given phase provided they are in a "state" corresponding to that phase. Considering radiation emanating from a point, it is shown that a tentative explanation of interference may be obtained if the quants have a constant phase at emission, and if they give atoms at any point an impulse towards the state corresponding to their phase at the point. According as these impulses reinforce or oppose each other, bright or dark parts of a fringe system are obtained. Practically the whole nomenclature and method of the wave theory may be taken over and applied to the quant phase field surrounding an emitting source. Essentially, on this view, an explanation of interference is sought in terms of different quants, and the dark and bright parts of a fringe system are to be explained, not in terms of different numbers of quants arriving at them, but in different phase relations of quants which do arrive.

Quants are characterised by a magnetic and electric vector, whose magnitudes and directions define the state of polarisation and the phase. Electron orbits may also be regarded as characterised by two vectors in terms of which it seems as if the "state" may be specified.

The relation between transition character and type of radiation emitted is finally considered and it is shown that the evidence so far available renders plausible an explanation of interference of the kind put forward. Much further experimental work, particularly

in connection with the Stark and Zeeman effects, is, however, necessary, before sufficient precision can be given to the views put forward to make them satisfying. On the other hand, it seems premature to suppose that only a formal treatment is possible of the physical phenomena involved.

I would like to thank in particular Mr R. H. Fowler and Mr C. T. R. Wilson for helpful criticism and discussion of special points: I would also like generally to express my feeling of indebtedness to these, and many others, in conversation with whom have originated in part some of the ideas put forward.

Note on the heating effect of the γ-Rays from RaB and RaC. By Dr C. D. Ellis, Trinity College, and W. A. Wooster, B.A., Peterhouse.

[*Read* 2 February 1925.]

Now that the wave-lengths of the main γ-rays of radium B and C are known it has become a matter of some importance to determine their total energy, because from such measurements it is possible to deduce the average number of quanta emitted by each disintegrating atom. If in addition the relative intensities of the different γ-rays could be determined it would be possible to give a complete description of the γ-ray emission.

We have recently made an attempt at the first part of this problem, that of measuring the heating effect of the γ-rays. The chief difficulty lies in the fact that the γ-ray heating has to be measured in the presence of the heating effect of the α-rays of radium C. The latter is ten times greater than the former and in practice this ratio may become one hundred times if, as is probable in any experimental arrangement, only one-tenth of the γ-rays are absorbed. A method by which the γ-ray heating is obtained by subtraction of the α-ray heating is bound to be inaccurate, so we have worked out a method of measuring the γ-ray heating by itself, independently of the α-ray heating.

The method is briefly as follows.

A composite hollow circular cylinder was built up of four equal sectors. Two opposing sectors were of lead and aluminium respectively, insulated from one another by intervening Balsa wood sectors. Down the axis of this cylinder ran a copper rod. This was supported by an outer case and did not touch any of the sectors. A hole was bored throughout the length of the copper rod and a small tube containing radium emanation in transient equilibrium with its products radium A, B, C, was placed in this hole at the mid-point of the axis. The aluminium sector had holes bored in it so as to bring it to the same total heat capacity as the lead sector. The entire heating effect of the α- and β-particles of radium A, B and C is produced inside the copper rod and during the whole process of flow to the outer case is symmetrically disposed to the lead and aluminium sectors. Since they are of equal heat capacity they heat up to the same extent. The γ-rays from radium B and C, however, are incompletely absorbed in either the aluminium or the lead, but are more absorbed in the lead. The lead sector therefore heats up more than the aluminium sector and the resulting temperature difference is entirely due to the γ-rays.

This temperature difference was measured by a system of nineteen silver-constantan thermocouples in series, one set of junctions being on the lead sector and the other on the aluminium. The heat corresponding to this temperature difference was determined by calibration measurements with suitably disposed heating coils. The heating effect is very small, but since the entire effect is due to the γ-rays it is possible to use a sensitive galvanometer. This would not be feasible if the small γ-ray effect had to be measured when superimposed on the large α-ray effect.

Knowing the absorption of the γ-rays in the lead and aluminium sectors it is possible to calculate the total heating effect of the γ-rays. Our final result is that the heating effect of the γ-rays from the amount of radium B and radium C in equilibrium with one gram of radium is 8·1 gram-calories an hour. This should be correct to 5 per cent.

This figure is somewhat higher than the previous measurements of Rutherford and Robinson*, and of Eve†, but the former experiments were concerned chiefly with the α-particle heating, and the γ-ray effect was determined only by the subtraction method. Our result is in good agreement with the β-ray evidence and Kovarik's‡ recent determination of the number of γ-rays emitted per atom disintegrating.

A fuller account will be published later.

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* *Phil. Mag.* (6), 25, p. 312, 1913.

† *Phil. Mag.* (6), 27, p. 394, 1914.

‡ *Phys. Rev.* 23, p. 559, 1924.

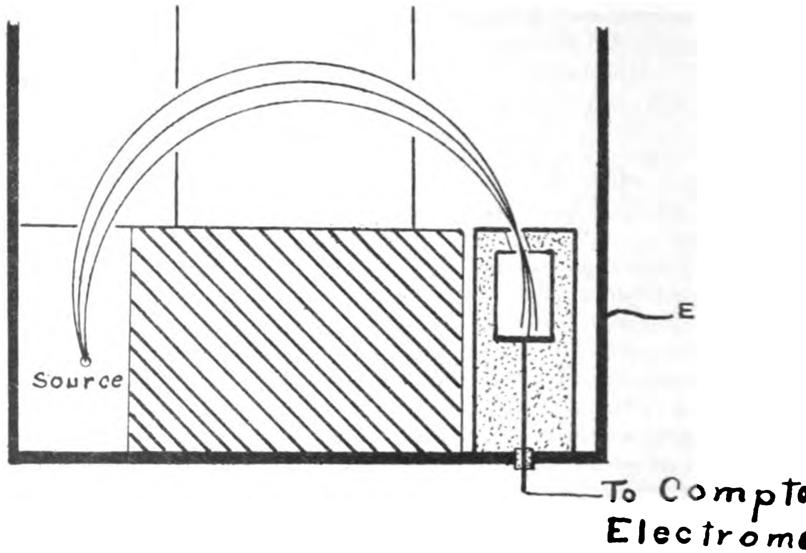
A preliminary note on a direct determination of the distribution of intensity in the natural β-ray Spectrum of RaB and RaC. By Dr L. F. CURTISS. (Communicated by Professor Sir E. RUTHERFORD.)

[*Read* 2 February 1925.]

Many investigations have shown the presence of a "continuous" β-ray spectrum accompanying the β-ray lines in the natural spectrum of Ra B and Ra C. The existence of such a spectrum is not easily accounted for if one is to regard the nucleus as existing in sharply quantised states. It has been pointed out that this spectrum may be a spurious effect, arising from scattering from the walls of the apparatus and not from within the atom itself. Therefore it is of importance to obtain accurate information regarding the continuous spectrum and its relation to the line spectrum.

Most of the previous experiments have been performed in such a way that the γ-radiation and other disturbing effects contributed considerably to the observed readings which were corrected for these effects. Since there is uncertainty in such circumstances it seemed desirable to seek a method which would measure directly the number of β-particles ejected per second with a given $\Delta H\rho$ for the range of the spectrum. There are two possible methods. The Geiger counter has been used. In common with the photographic plate and the ionisation chamber, it suffers from the disability that it is not reliable for comparing numbers of β-particles varying greatly in velocity. The other obvious method is to collect the particles in a Faraday cylinder and measure the rate of accumulation of charge. In principle this method of measurement is quite independent of the γ-radiation and has the great advantage that it gives equal weight to every particle whatever its velocity. The only serious question is that of a detecting instrument of sufficient sensitivity. A rough calculation indicated that 1/1000 of the spectrum of a 20 mg. source would include on the average about 4×10^4 particles per second under the usual conditions of measurement. Therefore an electrometer with a sensitivity of 10,000 mm./volt connected to a system of 50 cm. capacity should show a drift of the needle of about 60 div./min.

Accordingly the arrangement indicated in Fig. 1 was adopted. The particles after being bent by a magnetic field in a semi-circle delimited by suitable screens enter the collecting cylinder and are measured by a Compton electrometer. This instrument was maintained at a sensitivity of 10,000 to 15,000 mm./volt throughout these preliminary measurements. It was found desirable to sur-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Electrom

Fig. 1

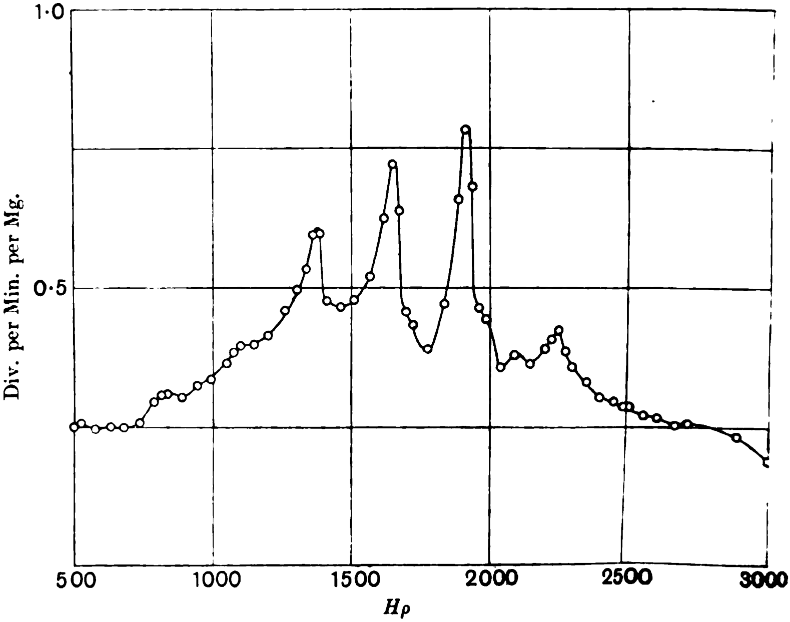
1·0
0·5
Div. per Min. per Mg.
500
1000
1500
2000
2500
3000
Hρ

Fig. 2

ound the cylinder with sulphur to eliminate disturbances arising from ionisation. This arrangement was entirely free from any disturbance attributable to γ-radiation. On reversing the magnetic field a weak negative effect could be detected, but since this could be entirely cut off by a piece of aluminium foil of one centimetre equivalent stopping power for α-particles placed over the slit of the cylinder, this effect was probably due to a soft delta-radiation. The apparatus was placed between the poles of an electromagnet and evacuated with a Gaede mercury pump with a liquid air trap to exclude mercury vapour. The pressure was kept below 10^{-3} mm. of mercury in all experiments. This was necessary to avoid great trouble caused by the formation of positive ions in the neighbourhood of the slit of the Faraday cylinder. At pressures of about a tenth of a millimetre of mercury the positive ions entirely mask the effect of the β-particles.

In order to test the apparatus and to make sure that it was adequately sensitive a thin-walled emanation tube (2·1 cm. equivalent stopping power for α-particles) was filled with emanation and used as a source. The curve shown in Fig. 2 gives the results obtained under these conditions. As can be seen, the lines of low velocity are entirely absent, having been absorbed in the walls of the tube. The line of $H\rho = 1410$ is the first to appear at all and even then with much reduced intensity. The other strong lines higher up in the spectrum come out clearly but their sharpness is limited by the low resolving power of the apparatus ($\rho = 3$ cm.). The defining slits were about 3 mm. wide. The scattering from the walls of the emanation tube, the low resolving power, and the wide slits of course all contribute to the background as shown in this figure. Measurements were extended to the extreme upper region of the spectrum to obtain a rough idea of its extent. These observations were made at considerable intervals of $H\rho$ and no attempt was made to detect lines. They indicate that the spectrum ends at about $H\rho = 12{,}000$ in agreement with other observers.

As soon as it had been demonstrated that the method was reliable work was begun with bare sources consisting of the active deposit of Ra B plus Ra C on a brass plate. Owing to the rapid decay of the Ra B in such a source it was necessary to add a diffusion pump to increase the speed of pumping so that observations might be started as soon as possible after removing the source from the emanation. This work was started at the low velocity end of the spectrum and the results are shown in Fig. 3. They indicate the presence of a number of β-particles of slow speed below the line $H\rho = 661$. There is also evidence of a diminution of the background compared with the observations made with the emanation tubes. However, these measurements have not yet been carried far enough for much of a conclusive nature to be deduced from them.

This report, as has been stated, is only of a preliminary nature. In order actually to get at the problem in hand a systematic series of measurements must be made under conditions which permit each factor which may influence the background to be studied independently. In this way it should be possible to ascertain how much of the continuous spectrum originates within the atom.

In conclusion I wish to express my thanks to Sir E. Rutherford for his advice and interest as well as for providing the necessary apparatus, and to Mr G. R. Crowe for preparing the courses. I also wish to acknowledge aid from the National Research Council (U.S.A.), whose support in the form of a fellowship made it possible for me to work at the Cavendish Laboratory.

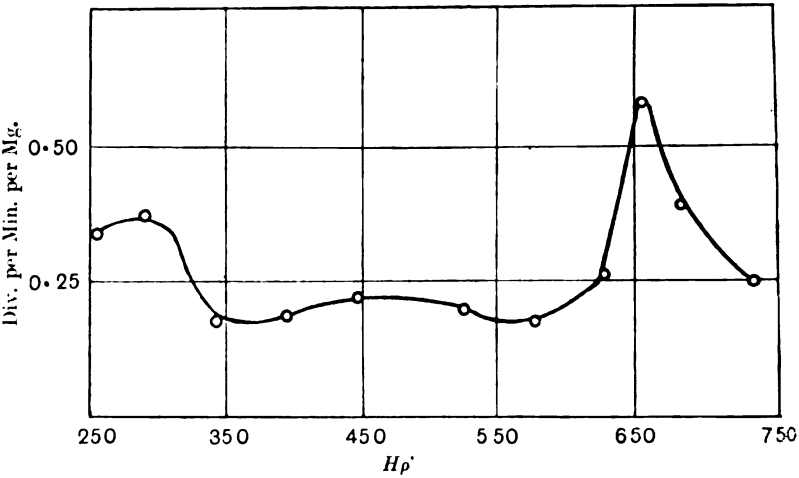

Fig. 3

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CONTENTS.

PAGE

On the vector algebra of eight associated points of three quadric surfaces. By Mr H. W. TURNBULL, Trinity College 481

Why wireless electric rays can bend round the earth (Abstract). By Sir JOSEPH LARMOR, F.R.S., Lucasian Professor 488

Note on the chemical constant of chlorine. By R. R. S. COX, B.A., Christ's College. (Communicated by Mr R. H. FOWLER) . . 491

Dissociative equilibrium in an external field of force. By Mr E. A. MILNE, Trinity College 493

Integrals developable about a singular point of a Hamiltonian system of differential equations. Part II. By Dr T. M. CHERRY, Trinity College. (With One Text-figure) 510

The acceleration of β-particles in strong electric fields such as those of thunderclouds. By Professor C. T. R. WILSON, Sidney Sussex College 534

The determination of the frequency of an alternating current supply by the vibrations of rods. By Dr G. F. C. SEARLE, Peterhouse. (With Three Text-figures) 539

Photographic plates for the detection of mass rays. By Dr F. W. ASTON, Trinity College. (With Plate IV) 548

The kinetic theory of metallic conduction. By E. L. DAVIES and Mr G. H. LIVENS 555

The decay constant of radium emanation. By L. BASTINGS. (Communicated by Professor Sir E. RUTHERFORD) 562

On the current induced in a wireless telegraph receiving antenna. By Mr E. B. MOULLIN, Downing College. (Communicated by Mr E. V. APPLETON.) (With Two Text-figures) 567

On the positive flash in vacuum discharge tubes. (Preliminary note.) By Mr R. WHIDDINGTON, St John's College. (With Plate V) . 574

The structure of radiation. By Dr EDMUND C. STONER, Emmanuel College 577

Note on the heating effect of the γ-rays from RaB and RaC. By Dr C. D. ELLIS, Trinity College, and W. A. WOOSTER, B.A., Peterhouse . 595

A preliminary note on a direct determination of the distribution of intensity in the natural β-ray spectrum of RaB and RaC. By Dr L. F. CURTISS. (Communicated by Professor Sir E. RUTHERFORD.) (With Three Text-figures) 597

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Cambridge Philosophical Society.

A Generalization of Feuerbach's Theorem. By Mr J. P. GABBATT, Peterhouse.

[*Received* 17 October 1922; *read* 5 February 1923.]

Introduction. It is well known that in non-euclidean geometry a plane triangle has four circumcircles, and that each of these circles touches each of four other circles. The latter theorem, which is an extension of that of Feuerbach, is essentially due to Hart.

In a previous paper* the writer has indicated the place of Hart's Theorem in the theory of the plane cubic. The principal object of the present paper is to exhibit a more general theorem, related to the same theory, of which Hart's Theorem is a particular case.

The theorem in question is briefly as follows. Associated with a given line and a syzygetic range of class-cubics there are ∞^2 systems of sixteen conics. Each system may be regarded as composed of four groups of four conics each; these may be termed the first four groups of conics. Associated with each of the first four groups of conics is a conic which has double contact with each of the four conics of that group; we have thus a fifth group of four conics. Again, associated with each of the first four groups of conics is a conic which has simple contact with each of the four conics of that group and double contact with the associated conic of the fifth group; we have thus a sixth group of four conics. Associated with each of the last two groups of conics is a conic which has double contact with each of the four conics of that group; we have thus two further conics. These two conics have double contact on the given line.

In three cases the original sixteen conics reduce to four. Two of these cases are of the same type, and either of the two may be identified with that of Hart.

1. *Notation and Assumptions.* The following is a summary of the necessary parts of the former paper, to the paragraphs of which the starred numbers refer.

1.1. The nine points of inflexion of any plane order-cubic determine a syzygetic pencil $[T_n]$ of order-cubics; this pencil will be termed the *T-pencil.* (1.11)

The Cayleyans of the order-cubics of the T-pencil constitute a syzygetic range $[\Theta_n]$ of class-cubics; this range will be termed the Θ-*range.* (1.12)

---

* *Proceedings Camb. Phil. Soc.* 21 (1923), pp. 297–362.

Given any plane order-cubic T_n; then there is one and only one class-cubic Θ_n such that the polar conic, q.* T_n, of every point in the plane is outpolar to the pole conic, q. Θ_n, of every line in the plane. The order-cubic T_n and the class-cubic Θ_n are termed *apolar*. The Hessian of T_n is the Cayleyan of Θ_n, and the Cayleyan of T_n is the Hessian of Θ_n. If T_n be an order-cubic of the T-pencil, then Θ_n is a class-cubic of the Θ-range. (1.13)

Three order-cubics (T_1, T_2, T_3) of the T-pencil have the same Hessian (T_0); thus T_0 is the common Cayleyan of Θ_1, Θ_2, Θ_3, the class-cubics apolar to T_1, T_2, T_3 respectively. The Hessian of Θ_n will be denoted by Δ_n, and the other two order-cubics which have the same Cayleyan as T_n by T_n', T_n'' respectively ($n = 1, 2, 3$). (1.14)

1.2. The polar conic, q. T_n, of any point X will be denoted by $S_n(X)$, and the pure polar (-line), q. T_n, T_n', T_n'', of X by $p_n(X)$, $p_n'(X)$, $p_n''(X)$ respectively ($n = 1, 2, 3$). (1.21)

The pole conic, q. Θ_n, of any line x will be denoted by $\Sigma_n(x)$, the pure pole (-point), q. Θ_n, of x by $\varpi_n(x)$, and the mixed pole (-point), q. Θ_n, of two lines x, y by $\varpi_n(x, y)$ ($n = 1, 2, 3$). (1.22)

$S_n(X)$ degenerates into two lines if and only if X is a point of T_0; the meet of the two lines is the *conjugate pole*, q. T_n, of X, and is the *n-correspondent* on T_0 of X. The two lines are *corresponding tangents* to Δ_n ($n = 1, 2, 3$). (1.23)

$\Sigma_n(x)$ degenerates into two points if and only if x is a tangent to Δ_n; the join of the two points is the *conjugate polar*, q. Θ_n, of x, and is the corresponding tangent (to Δ_n) of x. The two points are n-correspondents on T_0 ($n = 1, 2, 3$). (1.24)

1.3. The poloconic (*10.1), q. T_n, of any order-conic s will be denoted by $C_n(s)$; and in particular the mixed poloconic, q. T_n, of any pair of lines x, y by $C_n(x, y)$, and the pure poloconic, q. T_n, of x by $C_n(x)$. The six points common to T_0 and $C_n(s)$ are the n-correspondents of the six points common to T_0 and s; in particular, $C_n(x)$ has triple contact *of species n* with T_0 ($n = 1, 2, 3$). (1.31)

The poloconic, q. Θ_n, of any class-conic σ will be denoted by $\Gamma_n(\sigma)$; and in particular the mixed poloconic, q. Θ_n, of any pair of points X, Y by $\Gamma_n(X, Y)$, and the pure poloconic, q. Θ_n, of any point X by $\Gamma_n(X)$. The six tangents common to Δ_n and $\Gamma_n(\sigma)$ are corresponding tangents to those common to Δ_n and σ; in particular, $\Gamma_n(X)$ has triple contact with Δ_n ($n = 1, 2, 3$). (1.32)

Whatever point may be denoted by X; $\Gamma_l(X)$ is (*11.62) identical with $C_l\{p_m(X), p_n(X)\}$ ($l, m, n = 1, 2, 3$). (1.33)

Each of the ∞^2 conics $\Sigma_m(x)$, $\Sigma_n(x)$ is (*11.31) its own poloconic q. Θ_l, and there are no other *autopoloconics* q. Θ_l. In general, a class-conic σ and its poloconic $\Gamma_l(\sigma)$ determine (*11.33) an involution range of class-conics, of which one of the double conics is a pole conic q. Θ_m and the other a pole conic q. Θ_n; in particular, if σ be $\Sigma_l(x)$, then the double conics of the range are $\Sigma_m(x)$ and $\Sigma_n(x)$ $(l, m, n = 1, 2, 3)$. (1.34)

---

* The symbol q. signifies *with respect to*.

1.4. The four poles, q. T_n, of any line determine a quadrangle of which opposite sides are corresponding tangents to Δ_n; such a quadrangle is termed a *circumscribed quadrangle* of Δ_n $(n = 1, 2, 3)$. (1.41)

The four polars, q. Θ_n, of any point determine a quadrilateral of which opposite vertices are n-correspondents on T_0; such a quadrilateral is termed an *inscribed quadrilateral, of species n*, of T_0 $(n = 1, 2, 3)$. (1.42)

The polars of a given point q. all the order-cubics of the T-pencil constitute a pencil of lines projective with the T-pencil. Whatever point may be denoted by X; the lines $p_l'(X)$, $p_l''(X)$ are (*14.2) harmonic q. $p_0(X)$, $p_l(X)$ and q. $p_m(X)$, $p_n(X)$; and therefore q. $p_m'(X)$, $p_m''(X)$ and q. $p_n'(X)$, $p_n''(X)$ $(l, m, n = 1, 2, 3)$. (1.43)

The poles of a given line q. all the class-cubics of the Θ-range constitute a range of points projective with the Θ-range. Whatever line may be denoted by x; the points $\varpi_m(x)$, $\varpi_n(x)$ are harmonic q. $\varpi_l(x)$ and the pole of x q. Δ_l $(l, m, n = 1, 2, 3)$. (1.44)

1.5. From any point of T_0 there are four tangents to T_0, other than that at the given point. Let A_0, B_0, C_0 denote any three collinear points of T_0, and let the tangents (other than those at A_0, B_0, C_0) to T_0 from A_0, B_0, C_0 respectively touch T_0 at L_0, L_1, L_2, L_3; M_0, M_1, M_2, M_3; N_0, N_1, N_2, N_3. Then (*5.4) the twelve points $[L, M, N]$ determine a configuration of symbol $\boxed{\begin{matrix}12 & 4\\ 3 & 16\end{matrix}}$, and the suffixes may be so arranged that L_0, M_0, N_0 are in line; L_0, M_n, N_n; L_n, M_0, N_n; L_n, M_n, N_0 are in line $(n = 1, 2, 3)$; and L_l, M_m, N_n are in line $(l, m, n = 1, 2, 3)$. (1.51)

Let the four poles of the line $A_0B_0C_0$ q. T_1; T_2; T_3 be denoted by $I_0, I_\alpha, I_\beta, I_\gamma$; $O_0, O_\alpha, O_\beta, O_\gamma$; $U_0, U_\alpha, U_\beta, U_\gamma$ respectively. Then (*5.5, *6.5) the twelve points $[I, O, U]$ determine a configuration of symbol $\boxed{\begin{matrix}12 & 4\\ 3 & 16\end{matrix}}$, and the suffixes may be so arranged that I_0, O_0, U_0 are in line; I_0, O_ν, U_ν; I_ν, O_0, U_ν; I_ν, O_ν, U_0 are in line $(\nu = \alpha, \beta, \gamma)$; and I_λ, O_μ, U_ν are in line $(\lambda, \mu, \nu = \alpha, \beta, \gamma)$. (1.52)

Consistently with (1.51, 2), the meets of I_0I_α, $I_\beta I_\gamma$; I_0I_β, $I_\gamma I_\alpha$; I_0I_γ, $I_\alpha I_\beta$ are respectively A_1, B_1, C_1, the 1-correspondents (on T_0) of A_0, B_0, C_0 respectively. (1.53)

Let the meet (1.43) of the polars, q. the order-cubics of the T-pencil, of I_λ be denoted by J_λ $(\lambda = 0, \alpha, \beta, \gamma)$. Since $A_0B_0C_0$ is (1.52) the common polar, q. T_1, of the four points $[I_\lambda]$; therefore the four points $[J_\lambda]$ are in the line $A_0B_0C_0$. (1.54)

The sixteen lines $[LMN]$ may (*6.3) be separated in three distinct ways into four groups of four lines each, the four lines of each group being the polars, q. Θ_1 (or Θ_2 or Θ_3), of one of the four points I_ν (or O_ν or U_ν respectively). Consistently with (1.51, 2), the suffixes (0, 1, 2, 3, α, β, γ) may be so arranged that the lines $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ (say $l_{0,0}$, $l_{0,\alpha}$, $l_{0,\beta}$, $l_{0,\gamma}$) are the polars, q. Θ_1, of I_0, and that the poles, q. Θ_2, of $l_{0,0}$, $l_{0,\alpha}$, $l_{0,\beta}$, $l_{0,\gamma}$ are O_0, O_α, O_β, O_γ respectively, and their poles q. Θ_3 are U_0, U_α, U_β, U_γ respectively; each of the sixteen lines $[IOU]$ being (1.44) the locus of the poles, q. the class-cubics of the Θ-range, of one of the sixteen lines $[LMN]$. (1.55)

1.6. The pencil $[C_0]$ of order-conics determined by the four points $\varpi_1(I_0O_\mu)$ as base-points ($\mu = 0, \alpha, \beta, \gamma$) is constituted (*19.71) of the mixed poloconics, q. T_1, of the line-pairs meeting at J_0 and harmonically conjugate q. $p_1'(I_0)$, $p_1''(I_0)$; similarly when the suffix 0 is replaced by α or β or γ. (1.61)

The four conics $\Gamma_1(O_\mu)$ (*11.61) circumscribe $A_1B_1C_1$ and have (*11.63. *19.91) double contact with $\Sigma_3(A_0B_0C_0)$. The conic $C_1\{p_3'(I_\lambda),\ p_3''(I_\lambda)\}$ of the pencil $[C_\lambda]$ has simple contact, at $\varpi_1(I_\lambda O_\mu)$, with the four conics $\Gamma_1(O_\mu)$ respectively, and double contact, on $p_3(I_\lambda)$, with $\Sigma_3(A_0B_0C_0)$. The four conics $\Gamma_1(U_\nu)$ circumscribe $A_1B_1C_1$ and have double contact with $\Sigma_2(A_0B_0C_0)$. The conic $C_1\{p_2'(I_\lambda),\ p_2''(I_\lambda)\}$ of the pencil $[C_\lambda]$ has simple contact, at $\varpi_1(I_\lambda U_\nu)$, with the four conics $\Gamma_1(U_\nu)$ respectively, and double contact, on $p_2(I_\lambda)$, with $\Sigma_2(A_0B_0C_0)$ ($\lambda = 0, \alpha, \beta, \gamma$; $\mu = 0, \alpha, \beta, \gamma$; $\nu = 0, \alpha, \beta, \gamma$). (1.62)

2. *Fundamental Equations.* 2.1. Consider the equation

$$\Theta_1 \equiv \frac{l^3}{a^2} + \frac{m^3}{b^2} + \frac{n^3}{c^2} + \frac{p^3}{d^2} = 0\text{*}. \qquad (2.11)$$

If (l, m, n, p) denote line-coordinates connected by the relation

$$l + m + n + p = 0, \qquad (2.12)$$

then (2.11) determines a plane class-cubic.

Conjugate polars (1.24) q. Θ_1 are well known to be determined by the equations

$$\frac{ll'}{a^2} = \frac{mm'}{b^2} = \frac{nn'}{c^2} = \frac{pp'}{d^2}, \qquad (2.13)$$

where (2.12) holds, and

$$l' + m' + n' + p' = 0. \qquad (2.14)$$

The (l', m', n', p') eliminant of (2.13, 4) may be written

$$\Delta_1 \equiv a^2mnp + b^2nlp + c^2lmp + d^2lmn = 0, \qquad (2.15)$$

and is the equation of the Hessian of Θ_1.

---

* The curve determined by the equation $f=0$ will often be referred to as *the curve f*.

2.2. From the form of (2.15), the quadrangle of reference $(l = 0, m = 0, n = 0, p = 0)$ is circumscribed (1.41) to Δ_1; the points determined by $l = 0$, $m = 0$, $n = 0$, $p = 0$ may therefore (1.52) be identified with I_α, I_β, I_γ, I_0 respectively. (2.21)

Since (1.53) $I_0 I_\alpha$, $I_\beta I_\gamma$; $I_0 I_\beta$, $I_\gamma I_\alpha$; $I_0 I_\gamma$, $I_\alpha I_\beta$ meet at A_1, B_1, C_1 respectively; therefore the equations

$$\begin{aligned} m + n &= -(l + p) = 0,\\ n + l &= -(m + p) = 0,\\ l + m &= -(n + p) = 0 \end{aligned}$$

determine A_1, B_1, C_1 respectively. (2.22)

From (2.11), the coordinates (l, m, n) of the four lines of which I_0 is the common pole, q. Θ_1, are $(\pm a, \pm b, \pm c)$; let (a, b, c), $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$ be the coordinates of $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ respectively (cf. 1.55). (2.23)

2.3. By means of (2.12), the equation

$$(A, B, C, F, G, H \between l, m, n)^2 = 0$$

of any class-conic may be reduced to the form

$$\sigma \equiv fmn + gnl + hlm + ulp + vmp + wnp = 0, \quad (2.31)$$

where f, g, h, u, v, w denote constants.

Substituting for l, m, n, p in terms of l', m', n', p' from (2.13), and remembering (1.32), we see that

$$\Gamma_1(\sigma) \equiv ua^2d^2mn + vb^2d^2nl + wc^2d^2lm + fb^2c^2lp + gc^2a^2mp + ha^2b^2np = 0 \quad (2.32)$$

determines the poloconic, q. Θ_1, of σ. Each of the quadratic forms in (2.31, 2) may therefore be termed (to a multiplicative constant) the *poloform* of the other.

There are two and only two types of forms, viz.

$$\begin{aligned} \Sigma_2 &\equiv \alpha\,(admn - bclp) + \beta\,(bdnl - camp) + \gamma\,(cdlm - abnp),\\ \Sigma_3 &\equiv \alpha'\,(admn + bclp) + \beta'\,(bdnl + camp) + \gamma'\,(cdlm + abnp), \end{aligned} \quad (2.33)$$

which are (to a multiplicative constant) identical with their respective poloforms. Thus $\Sigma_2 = 0$, $\Sigma_3 = 0$ determine (1.34) the ∞^2 pole conics of the lines of the plane q. Θ_2, Θ_3 (1.14); the association of the negative sign with Θ_2 being, of course, arbitrary. In particular [multiplying (2.31) by $abcd$, and subtracting or adding (2.32)] the equations of the conics of type $\Sigma_2 = 0$, $\Sigma_3 = 0$ which belong to the range determined (1.34) by σ and $\Gamma_1(\sigma)$ are

$$(uad \mp fbc)(admn \mp bclp) + (vbd \mp gca)(bdnl \mp camp) + (wcd \mp hab)(cdlm \mp abnp) = 0. \quad (2.34)$$

Thus (1.34) if g denote any line, and if the equation (2.31) determine $\Sigma_1(g)$; then the equations (2.34) determine $\Sigma_2(g)$, $\Sigma_3(g)$. Hence and from (2.23) we may obtain the equations of $\Sigma_2(L_0M_0N_0)$, $\Sigma_3(L_0M_0N_0)$; and since (1.55) O_0, U_0 are the poles of $L_0M_0N_0$ q. the last two conics respectively, we may deduce the equations of the points O_0, U_0, which may be written in the forms

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}-\frac{p}{d}=0,$$

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{p}{d}=0$$

respectively. Introducing a fifth coordinate q by means of the equation

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{q}{d}=0,$$

we may write $\quad O_0\equiv p+q=0,\quad U_0\equiv p-q=0.$ (2.35)

Since (2.21) $I_0\equiv p=0$; therefore the equation $q=0$ determines the point (I_0', say) which is the harmonic conjugate of I_0 q. O_0, U_0, and is thus (1.44, 1.55) the pole of $L_0M_0N_0$ q. Δ_1.

(2.36)

From (2.35) we obtain $[a(b-c), b(c-a), c(a-b)]$ as the line-coordinates (l, m, n) of $I_0O_0U_0$, and thence and from (2.11)

$$(b-c)^2l+(c-a)^2m+(a-b)^2n=0 \qquad (2.37)$$

as the equation of $\varpi_1(I_0O_0U_0)$.

It follows from (2.23, 2.34) that the equations of the points O_ν, U_ν may be obtained from those of O_0, U_0 by changing the sign of the constant d and of one of the constants a, b, c ($\nu=\alpha, \beta, \gamma$). It is convenient to associate O_α, U_α; O_β, U_β; O_γ, U_γ with the equations obtained by changing the sign of d and of a, b, c respectively; a course which is clearly consistent with (1.52). Thus

$$O_\alpha\equiv-\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{p}{d}=0, \qquad (2.381)$$

$$U_\alpha\equiv-\frac{l}{a}+\frac{m}{b}+\frac{n}{c}-\frac{p}{d}=0, \qquad (2.382)$$

$$I_0O_\alpha U_\alpha\equiv[-a(b-c),\quad b(c+a),\quad -c(a+b),\quad 0], \qquad (2.383)$$

$$\varpi_1(I_0O_\alpha U_\alpha)\equiv(b-c)^2l+(c+a)^2m+(a+b)^2n=0, \qquad (2.384)$$

etc.

3. *The Sympoloconic.* 3.1. From (1.44, 2.35), the equation

$$X_{0,k}\equiv kp+q=0 \qquad (3.11)$$

determines the pole ($X_{0,k}$) of $L_0M_0N_0$ q. some class-cubic (Θ_k, say) of the Θ-range. Now, eliminating squares of l, m, n, p by means of (2.12), we have

$$-\frac{1}{d^2}(kp+q)^2 \equiv \left(\frac{1}{b}-\frac{1}{c}\right)^2 mn + \left(\frac{1}{c}-\frac{1}{a}\right)^2 nl + \left(\frac{1}{a}-\frac{1}{b}\right)^2 lm$$
$$+ \left(\frac{k}{d}+\frac{1}{a}\right)^2 lp + \left(\frac{k}{d}+\frac{1}{b}\right)^2 mp + \left(\frac{k}{d}+\frac{1}{c}\right)^2 np,$$

whence, and from (2.32),

$$\Gamma_1(X_{0,k}) \equiv \Gamma_1\{(kp+q)^2\} \equiv (ka+d)^2 mn + (kb+d)^2 nl$$
$$+ (kc+d)^2 lm + (b-c)^2 lp + (c-a)^2 mp + (a-b)^2 np = 0. \qquad (3.12)$$

Since $X_{0,k}$ is a point of $I_0O_0U_0$, therefore $\Gamma_1(X_{0,k})$ contains the point

$$\varpi_1(I_0O_0U_0) \equiv (b-c)^2 l + (c-a)^2 m + (a-b)^2 n = 0.$$

Thus, from (3.12), the conic determined by

$$(ka+d)^2 mn + (kb+d)^2 nl + (kc+d)^2 lm = 0 \qquad (3.13)$$

also contains $\varpi_1(I_0O_0U_0)$, and has the same tangent at that point as $\Gamma_1(X_{0,k})$; and, since the point of contact of the conic of (3.13) with any tangent (l', m', n') is determined by

$$\frac{(ka+d)^2}{l'^2} l + \frac{(kb+d)^2}{m'^2} m + \frac{(kc+d)^2}{n'^2} n = 0,$$

therefore the coordinates (l', m', n') of the tangent at $\varpi_1(I_0O_0U_0)$ to either of the conics in question are

$$\left(\frac{ka+d}{b-c}, \frac{kb+d}{c-a}, \frac{kc+d}{a-b}\right). \qquad (3.14)$$

3.2. Re-writing (3.12), we have

$$\Gamma_1(X_{0,k}) \equiv (k^2a^2+d^2) mn + (k^2b^2+d^2) nl + (k^2c^2+d^2) nl$$
$$+ (b^2+c^2) lp + (c^2+a^2) mp + (a^2+b^2) np$$
$$- kabcd\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}+\frac{p^2}{k^2d^2}\right) + \frac{abc}{kd}(p+kq)^2 = 0. \qquad (3.21)$$

Thus $\Gamma_1(X_{0,k})$ has double contact with the conic

$$\mathrm{K}_{0,k} \equiv (k^2a^2+d^2) mn + (k^2b^2+d^2) nl + (k^2c^2+d^2) lm$$
$$+ (b^2+c^2) lp + (c^2+a^2) mp + (a^2+b^2) np$$
$$- kabcd\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}+\frac{p^2}{k^2d^2}\right) = 0, \qquad (3.22)$$

the pole of double contact of the two conics being

$$X_{0,k^{-1}} \equiv k^{-1}p + q = 0,$$

and thus being the harmonic conjugate, q. O_0, U_0 of $X_{0,k}$. (3.23)

Now $\mathrm{K}_{0,k}$ is unaltered when the sign of d and of any one of a, b, c are changed; and (cf. 2.38), when we change these signs in (3.11), we obtain, instead of $X_{0,k}$, the poles $(X_{\alpha,k}, X_{\beta,k}, X_{\gamma,k})$, q. Θ_k, of $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ respectively. Hence the theorem: *The pure poloconics, q.* Θ_1, *of the poles of* $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ *q. any class-cubic* Θ_k *of the* Θ*-range all have double contact with a conic* $\mathrm{K}_{0,k}$. *The poles of double contact are respectively the poles of* $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ *q. a class-cubic* $\Theta_{k^{-1}}$ *of the* Θ*-range; the class-cubics* $\Theta_k, \Theta_{k^{-1}}$ *being harmonically conjugate in the range, q.* Θ_2, Θ_3.

We shall term the conic $\mathrm{K}_{0,k}$ the 1-*sympoloconic*, q. Θ_k, of I_0. (3.24)

If Θ_2 be taken for Θ_k, then (1.55) the points $X_{0,k}, X_{\alpha,k}, X_{\beta,k}, X_{\gamma,k}$ become $O_0, O_\alpha, O_\beta, O_\gamma$ respectively, and therefore (1.62) the conic $\mathrm{K}_{0,k}$ becomes $\Sigma_3\,(A_0B_0C_0)$. If Θ_3 be taken for Θ_k, then the points $X_{0,k}, X_{\alpha,k}, X_{\beta,k}, X_{\gamma,k}$ become $U_0, U_\alpha, U_\beta, U_\gamma$ respectively, and the conic $\mathrm{K}_{0,k}$ becomes $\Sigma_2\,(A_0B_0C_0)$. (3.25)

3.3. We may re-write the equation (3.22) in the form

$$\begin{aligned}\mathrm{K}_{0,k} \equiv \frac{mn}{bc}(kca+bd)(kab+cd) &+ \frac{nl}{ca}(kab+cd)(kbc+ad)\\ + \frac{lm}{ab}(kbc+ad)(kca+bd) &+ k^{-1}\left\{\frac{lp}{ad}(ca+kbd)(ab+kcd)\right.\\ &\left.+ \frac{mp}{bd}(ab+kcd)(bc+kad) + \frac{np}{cd}(bc+kad)(ca+kbd)\right\} = 0.\end{aligned} \tag{3.31}$$

Applying (2.32), we find that $\Gamma_1(\mathrm{K}_{0,k})$ differs only by a multiplicative constant from

$$\begin{aligned}\mathrm{K}_{0,k^{-1}} \equiv \frac{mn}{bc}(k^{-1}ca+bd)(k^{-1}ab+cd) &+ \frac{nl}{ca}(k^{-1}ab+cd)(k^{-1}bc+ad)\\ + \frac{lm}{ab}(k^{-1}bc+ad)(k^{-1}ca+bd) &+ k\left\{\frac{lp}{ad}(ca+k^{-1}bd)(ab+k^{-1}cd)\right.\\ &\left.+ \frac{mp}{bd}(ab+k^{-1}cd)(bc+k^{-1}ad) + \frac{np}{cd}(bc+k^{-1}ad)(ca+k^{-1}bd)\right\} = 0.\end{aligned} \tag{3.32}$$

Thus: *If* Θ_k, $\Theta_{k^{-1}}$ *denote any two class-cubics of the* Θ*-range harmonically conjugate q.* Θ_2, Θ_3, *then the* 1*-sympoloconics of* I_0 *q.* Θ_k, $\Theta_{k^{-1}}$ *are mutually poloconics q.* Θ_1. (3.33)

3.4. Since the Hessian of any class-cubic of the Θ-range is also a class-cubic of that range, therefore the equation of any class-cubic of the range may be written in the form

$$kabcd\Theta_1 + 3\Delta_1 = 0, \tag{3.41}$$

where k is a parameter; and since the equation of the pole of $L_0M_0N_0$ q. the class-cubic of (3.41) is easily $kp + q = 0$, therefore, consistently with (3.11), we may write

$$\Theta_k \equiv kabcd\Theta_1 + 3\Delta_1 = 0,$$

where (cf. (2.35)) Θ_1, Θ_2, Θ_3, Δ_1 correspond to $k = \infty, 1, -1, 0$ respectively. (3.42)

3.5. Now it is well known that the line-equation of the poloconic of the point (X, Y, Z) q. the class-cubic $f(l, m, n) = 0$ is

$$\begin{vmatrix} f_{ll} & f_{lm} & f_{ln} & X \\ f_{ml} & f_{mm} & f_{mn} & Y \\ f_{nl} & f_{nm} & f_{nn} & Z \\ X & Y & Z & 0 \end{vmatrix} = 0,$$

where f_{ll}, f_{mn} respectively denote $\dfrac{d^2f}{dl^2}, \dfrac{d^2f}{dm.dn}$, etc.

Hence the equation of the poloconic of the point determined by

$$-p = l + m + n = 0$$

q. the class-cubic determined by

$$f(l, m, n, p) = 0$$

(where l, m, n, p satisfy the equation

$$l + m + n + p = 0)$$

may be written

$$\begin{vmatrix} f_{ll} & f_{lm} & f_{ln} & 1 \\ f_{ml} & f_{mm} & f_{mn} & 1 \\ f_{nl} & f_{nm} & f_{nn} & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0,$$

where in the differentiation p is regarded as constant.

Applying this to Θ_k, a little algebra gives, for the equation of the poloconic, q. Θ_k, of I_0

$$\begin{aligned}\Gamma_k(I_0) \equiv \Gamma_k(p^2) \equiv 4d^2.\mathrm{K}_{0,k} \\ - p^2\{a^4 + b^4 + c^4 + d^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 - 2a^2d^2 - 2b^2d^2 \\ - 2c^2d^2 - 4(k + k^{-1})\,abcd\} = 0.\end{aligned} \quad (3.43)$$

Thus: *The poloconic and the* 1-*sympoloconic, q.* Θ_k, *of* I_0 *have double contact, the pole of double contact being* I_0. (3.44)

It may further be proved that the two conics are identical if and only if Θ_k be *either* Θ_1 *or* either of the other two class-cubics with the same Hessian as Θ_1.

4. *Further Theorems on the Double Contact of Class- and Order-Conics.* 4.1. We obtained the theorem (3.24) by considering $\Gamma_1\{(kp+q)^2\}$ and the three related quadratic forms got by changing, in $\Gamma_1\{(kp+q)^2\}$, the signs of a and d, of b and d, and of c and d respectively. If Σ denote any quadratic form in l, m, n, p which

remains invariable for the three such changes of two signs, if $\Sigma_0 \equiv \Sigma + (kp + q)^2$, and if the three changes of two signs convert Σ_0 into Σ_α, Σ_β, Σ_γ respectively; then from the consideration of the four forms $\Gamma_1(\Sigma_\mu)$ we obtain the more general theorem: *If* Θ_k, $\Theta_{k^{-1}}$ *denote class-cubics of the* Θ*-range harmonically conjugate q.* Θ_2, Θ_3; *if* I_0', I_α', I_β', I_γ'; $X_{0,k}$, $X_{\alpha,k}$, $X_{\beta,k}$, $X_{\gamma,k}$; $X_{0,k^{-1}}$, $X_{\alpha,k^{-1}}$, $X_{\beta,k^{-1}}$, $X_{\gamma,k^{-1}}$, *denote the four poles of* $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ *respectively q.* Δ_1, Θ_k, $\Theta_{k^{-1}}$ *respectively, if the conics* Σ_0, Σ_α, Σ_β, Σ_γ *have double contact, poles* $X_{0,k}$, $X_{\alpha,k}$, $X_{\beta,k}$, $X_{\gamma,k}$ *respectively with a given conic* Σ, *and are such that one of the vertices of the common circumscribed quadrilateral of any two of the conics* Σ_μ *is contained by the join of the corresponding two points* I_μ', *and if* Σ_0', Σ_α', Σ_β', Σ_γ' *denote the poloconics, q.* Θ_1, *of* Σ_0, Σ_α, Σ_β, Σ_γ *respectively; then the conics* Σ_0', Σ_α', Σ_β', Σ_γ' *have double contact, poles* $X_{0,k^{-1}}$, $X_{\alpha,k^{-1}}$, $X_{\beta,k^{-1}}$, $X_{\gamma,k^{-1}}$ *with a certain conic* Σ', *and are such that one of the vertices of the common circumscribed quadrilateral of any two of the conics* Σ_μ' *is contained by the join of the corresponding two points* I_μ' $(\mu = 0, \alpha, \beta, \gamma)$; *i.e.*, we may write $\Sigma_0' = \Sigma' + (k^{-1}p + q)^2$, etc. (4.11)

A case of special interest arises when the form Σ is a (constant) multiple of p^2. Then $\Sigma + (kp + q)^2$ may be expressed in the form $(k_1p + q)(k_2p + q)$. Hence: *If* X_{0,k_1}, X_{α,k_1}, X_{β,k_1}, X_{γ,k_1}; X_{0,k_2}, X_{α,k_2}, X_{β,k_2}, X_{γ,k_2} *denote the four poles of* $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ *q. two class-cubics* Θ_{k_1}, Θ_{k_2} *respectively of the* Θ*-range; then the mixed poloconics, q.* Θ_1, *of the four point-pairs* X_{μ,k_1}, X_{μ,k_2} *have double contact with a certain conic. The pole of double contact of that conic with* $\Gamma_1(X_{\mu,k_1}, X_{\mu,k_2})$ *may be obtained as follows: Let* Y_μ *denote the harmonic conjugate of* I_0 *q.* X_{μ,k_1}, X_{μ,k_2}, *and* Z_μ *the harmonic conjugate of* Y_μ *q.* O_μ, U_μ; *then* Z_μ *is the required pole of double contact* $(\mu = 0, \alpha, \beta, \gamma)$.

Z_μ is therefore (1.44) the pole of $l_{0,\mu}$ (1.55) q. some class-cubic of the Θ-range. (4.12)

4.2. The dual of the theorem of (4.12) (when instead of the class-cubics of the Θ-range we consider the order-cubics of the T-pencil) is of course also true. A case with which we are specially concerned results from the consideration of the line $A_0B_0C_0$ and its four poles I_0, I_α, I_β, I_γ q. T_1, instead of the point I_0 and its four polars $l_{0,0}$, $l_{0,\alpha}$, $l_{0,\beta}$, $l_{0,\gamma}$ q. Θ_1. Thus: *If* x_{λ,k_1}, x_{λ,k_2} *denote the polars of* I_λ *q. any two order-cubics of the* T*-pencil; then the mixed poloconics, q.* T_1, *of the four line-pairs* x_{λ,k_1}, x_{λ,k_2} *have double contact with a certain conic. The chord of double contact of* $C_1(x_{\lambda,k_1}, x_{\lambda,k_2})$ *with that conic may be determined as follows: Let* y_λ *denote the harmonic conjugate of* $A_0B_0C_0$ *q.* x_{λ,k_1}, x_{λ,k_2}, *and* z_λ *the harmonic conjugate of* y_λ *q.* $p_1'(I_\lambda)$, $p_1''(I_\lambda)$\*; *then* z_λ *is the required chord of double contact* $(\lambda = 0, \alpha, \beta, \gamma)$.

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* See (1.21).

z_λ is therefore the polar of I_λ q. some order-cubic of the T-pencil. (4.21)

5. *The Pencil* $[C_0]$ *of Mixed Poloconics q.* T_1. 5.1. Let us now consider the two conics respectively determined by the line-equations

$$a^2mn + b^2nl + c^2lm = 0, \tag{5.11}$$

$$mn + nl + lm = 0. \tag{5.12}$$

Of these the first is (2.32) $\Gamma_1(I_0)$, and is therefore (1.33) $C_1\{p_2(I_0), p_3(I_0)\}$. The second contains the three points severally determined by the line-equations

$$m + n = 0, \quad n + l = 0, \quad l + m = 0,$$

viz. (2.22) the points A_1, B_1, C_1; and is therefore (1.31) the mixed poloconic, q. T_1, of $A_0B_0C_0$ and some other line.

Writing the point-equations of the two conics in the respective forms

$$(b^2y + c^2z - a^2x)^2 - 4b^2c^2yz = 0, \tag{5.13}$$

$$(y + z - x)^2 - 4yz = 0, \tag{5.14}$$

we see that both conics contain the point of which the coordinates (x, y, z) are $(b-c)^2$, $(c-a)^2$, $(a-b)^2$, viz. (2.37) the point $\varpi_1(I_0O_0)$. Similarly both conics contain the three points $\varpi_1(I_0O_\mu)$ $(\mu = \alpha, \beta, \gamma)$. Thus (1.61) the two conics (5.11, 2), regarded as order-conics, determine the pencil $[C_0]$. In particular, $mn + nl + lm = 0$ is (cf. 1.43) the line-equation of $C_1\{p_0(I_0), p_1(I_0)\}$.

From (5.13, 4) the point-equation of any conic of the pencil $[C_0]$ may be written in the form

$$C_{0,k} \equiv abcd\,(x^2 + y^2 + z^2 - 2yz - 2zx - 2xy)$$
$$+ k\,(a^4x^2 + b^4y^2 + c^4z^2 - 2b^2c^2yz - 2c^2a^2zx - 2a^2b^2xy) = 0, \tag{5.15}$$

where k is a parameter; and easily the equation (5.15) determines that conic of the pencil $[C_0]$ which touches the line of which the coordinates (l, m, n) are

$$\left(\frac{ka+d}{b-c}, \ \frac{kb+d}{c-a}, \ \frac{kc+d}{a-b}\right)$$

at $\varpi_1(I_0O_0)$, and therefore (3.14) touches $\Gamma_1(X_{0,k})$ at that point.

Since $C_{0,k}$ is unaltered when the signs of d and of either a or b or c are changed, therefore the conic of (5.15) also touches $\Gamma_1(X_{\mu,k})$ at $\varpi_1(I_0O_\mu)$ $(\mu = \alpha, \beta, \gamma)$. Thus: *The four poloconics* $\Gamma_1(X_{\mu,k})$ *respectively have simple contact, at* $\varpi_1(I_0O_\mu)$, *with a certain conic* $C_{0,k}$ *of the* $[C_0]$ *pencil.* (5.16)

5.2. The line-equation reciprocal to (5.15) may be written

$$\Gamma_{0,k} \equiv d^2(mn+nl+lm) + k\,.\,\frac{d}{4abc}\{(b^2-c^2)^2 l^2 + (c^2-a^2)^2 m^2$$
$$+ (a^2-b^2)^2 n^2 + 2(c^2+a^2)(a^2+b^2)\,mn + 2(a^2+b^2)(b^2+c^2)\,nl$$
$$+ 2(b^2+c^2)(c^2+a^2)\,lm\} + k^2(a^2mn+b^2nl+c^2lm) = 0, \quad (5.21)$$

or again

$$\Gamma_{0,k} \equiv \frac{mn}{bc}(kca+bd)(kab+cd) + \frac{nl}{ca}(kab+cd)(kbc+ad)$$
$$+ \frac{lm}{ab}(kbc+ad)(kca+bd) - \frac{kd}{4abc}\{(b^2-c^2)^2(mn+lp)$$
$$+ (c^2-a^2)^2(nl+mp) + (a^2-b^2)^2(lm+np)\} = 0.$$

Thus, from (3·31),

$$4kabcd\,(\mathrm{K}_{0,k}-\Gamma_{0,k}) = k^2d^2\{(b^2-c^2)^2mn + (c^2-a^2)^2nl + (a^2-b^2)^2lm\}$$
$$+ \{2abc + kd\,(b^2+c^2)\}^2\,lp + \{2abc + kd\,(c^2+a^2)\}^2\,mp$$
$$+ \{2abc + kd\,(a^2+b^2)\}^2\,np$$
$$= -\{2abc\,(l+m+n)$$
$$+ kd\,[(b^2+c^2)\,l + (c^2+a^2)\,m + (a^2+b^2)\,n]\}^2, \quad (5.22)$$

and therefore the conics $\mathrm{K}_{0,k}$, $\Gamma_{0,k}$ have double contact, the locus (for k variable) of the pole of double contact being a straight line, which may be proved to be the join of I_0 and the pole, q. Δ_1, of $A_0B_0C_0$.

From (5.15), the equation of the chord of double contact of the conics $\mathrm{K}_{0,k}$, $\Gamma_{0,k}$ is found to be

$$a^2(b^2+c^2+d^2-a^2)\,x + b^2(c^2+a^2+d^2-b^2)\,y$$
$$+ c^2(a^2+b^2+d^2-c^2)\,z + abcd\,(k+k^{-1})\,(x+y+z) = 0. \quad (5.23)$$

Since this equation is linear in the (variable) parameter $k+k^{-1}$, therefore the line which it determines contains a fixed point; from (1.62, 3.25) that point must be J_0, and the line is therefore (1.54) the polar of I_0 q. a cubic of the T-pencil. Remembering that the conics $C_{0,k}$, $\Gamma_{0,k}$ are identical, we have, with the notation of (4.11): *The conic $C_{0,k}$ has double contact with that 1-sympoloconic $\mathrm{K}_{0,k}$ which doubly touches the four poloconics $\Gamma_1(X_{\mu,k})$; and the chord of double contact of the conics $\mathrm{K}_{0,k}$, $C_{0,k}$ is the polar of I_0 q. one of the order-cubics of the T-pencil* $(\mu = 0, \alpha, \beta, \gamma)$. (5.24)

6. *Groups of Four Sympoloconics.* 6.1. The point I_0 is one of four (strictly indistinguishable) points, viz. the four poles, q. T_1, of $A_0B_0C_0$. Each of the theorems (3.24, 3.33, 3.44, 4.11, 4.12, 5.16, 5.24) therefore represents a group of four theorems, one of which is associated with each of the four points I_λ $(\lambda = 0, \alpha, \beta, \gamma)$.

For example, the 1-sympoloconic $\mathrm{K}_{\alpha,k}$, q. Θ_k, of I_α may be specified as follows. Consider the four polars, q. Θ_1, of I_α, and the four poles $(X'_{\mu,k})$ q. Θ_k of those polars. Then the four conics $\Gamma_1\,(X'_{\mu,k})$ have double contact with a certain conic, the 1-*sympoloconic*, q. Θ_k, of I_α. Following (3.44), $\mathrm{K}_{\alpha,k}$ has double contact with the poloconic, q. Θ_k, of I_α; the pole of double contact being I_α. Again, the four points $\varpi_1\,(I_\alpha O_\mu)$ $(\mu = 0, \alpha, \beta, \gamma)$ determine a pencil $[C'_\alpha]$ of order-conics which bear the same relation to I_α as is borne to I_0 by the pencil $[C_0]$; and theorems corresponding to (5.16, 5.24) hold for the conics $C_{\alpha,k}$ of the pencil $[C_\alpha]$. Similarly for (β, γ).

6.2. We shall now consider certain relations between the group of four conics $\mathrm{K}_{\lambda,k}$ and the group of four conics $C_{\lambda,k}$ which correspond to the same value of the parameter k $(\lambda = 0, \alpha, \beta, \gamma)$.

Since the choice of a particular one of the four points I_λ is independent of the choice of a particular value of the parameter k, therefore (cf. 1.61) the conics $C_{0,k}$, $C_{\alpha,k}$, $C_{\beta,k}$, $C_{\gamma,k}$ are mixed poloconics, q. T_1, of the polars of I_0, I_α, I_β, I_γ respectively q. *the same* pair of order-cubics of the T-pencil. It then follows immediately from (4.21) that: *The four conics $C_{\lambda,k}$ all have double contact with a certain conic c_k, the chord of double contact of $C_{\lambda,k}$, c_k being the polar of I_λ q. one of the order-cubics of the T-pencil* $(\lambda = 0, \alpha, \beta, \gamma)$. (6.21)

It is to be noted that the chord of double contact of the conics $C_{\lambda,k}$ and c_k is not, in general, the same as that of $C_{\lambda,k}$ and $\mathrm{K}_{\lambda,k}$. For (5.15) the point-equation of $C_{\lambda,k}$ is linear in k; hence (cf. 2.32) the point-equation of the line-pair which (1.61) constitutes the poloconic, q. T_1, of $C_{\lambda,k}$ is also linear in k. It follows from (4.21) that the equation of the chord of double contact of $C_{\lambda,k}$ and c_k is linear in k, while (5.23) that of the chord of double contact of $C_{\lambda,k}$ and $\mathrm{K}_{\lambda,k}$ is quadratic in k. (6.22)

6.3. The consideration that the chords of double contact of $C_{\lambda,k}$ with c_k and with $\mathrm{K}_{\lambda,k}$ are both polars of I_λ q. order-cubics (generally distinct) of the T-pencil is vital in the proof of the two theorems (6.42, 6.51) which follow.

Let k_n denote a value of the parameter k. Let the point-equation reciprocal to $\mathrm{K}_{0,k_1} = 0$ be $K_{0,k_1} = 0$. Let the chord of double contact of C_{0,k_1} and c_{k_1} be determined by $p_{0,k_1} = 0$; and let k_2 be such that the chord of double contact of C_{0,k_1} and K_{0,k_1} is the chord of double contact (determined by $p_{0,k_2} = 0$) of C_{0,k_2} and c_{k_2}. Then for some values of the constants A_1, A_2, a_2, b_2, α_1, we have

$$A_1\,(C_{0,k_1} - \alpha_1 K_{0,k_1}) = -\,b_2 A_2\,(p_{0,k_2})^2 = -\,A_2\,(C_{0,k_2} - a_2 c_{k_2}),$$

whence
$$A_1 C_{0,k_1} + A_2 C_{0,k_2} = \alpha_1 A_1 K_{0,k_1} + a_2 A_2 c_{k_2}. \qquad (6.31)$$

But (for k variable) the conics $C_{0,k}$ constitute a pencil; and there is a conic (determined by $C_{0,k_3} = 0$) of that pencil such that

$$A_1 C_{0,k_1} + A_2 C_{0,k_2} + A_3 C_{0,k_3} = 0, \qquad (6.32)$$

where A_3 is a constant. Also (6.21)

$$C_{0,k_3} = a_3 c_{k_3} + b_3 (p_{0,k_3})^2, \tag{6.33}$$

where a_3, b_3 are constants and the equation $p_{0,k_3} = 0$ determines the polar of I_0 q. some order-cubic of the T-pencil.

Thus from (6.31, 2, 3)

$$a_1 A_1 K_{0,k_1} + a_2 A_2 c_{k_2} + a_3 A_3 c_{k_3} + b_3 A_3 (p_{0,k_3})^2 = 0,$$

and therefore the conic K_{0,k_1} has double contact with the conic determined by

$$\kappa_{k_1} \equiv a_2 A_2 c_{k_2} + a_3 A_3 c_{k_3} = 0, \tag{6.34}$$

the chord of double contact being determined by $p_{0,k_1} = 0$.

6.4. Either of the suffixes α, β, γ may be substituted for 0 throughout the paragraph (6.3) without modification of the forms c_k or of the constants; any such change of suffix being analytically equivalent to a different choice, in the forms $C_{\lambda,k}$, $K_{\lambda,k}$ of a group of three ambiguous signs (of which one may be regarded as fixed) which do not affect the forms c_k*. Hence: *The* 1-*sympoloconics* $\mathrm{K}_{\lambda,k}$, *q.* Θ_k, *of the four points* I_λ *all have double contact with a certain conic* κ_k, *the chord of double contact of* $\mathrm{K}_{\lambda,k}$, κ_k *being the polar of* I_λ *q. one of the order-cubics of the T-pencil* $(\lambda = 0, \alpha, \beta, \gamma)$. (6.41)

Thus each 1-sympoloconic $\mathrm{K}_{\lambda,k}$ has double contact with seven notable conics; viz. (3.24) four poloconics q. Θ_1, (3.44) the poloconic of I_λ q. Θ_k, (5.24) the conic $C_{\lambda,k}$ and (6.41) the conic κ_k.

6.5. Now, consistently with (6.3), we may write

$$C_{\lambda,k_1} = a_1 c_{k_1} + b_1 (p_{0,k_1})^2,$$
$$C_{\lambda,k_2} = a_2 c_{k_2} + b_2 (p_{0,k_2})^2,$$
$$C_{\lambda,k_3} = a_3 c_{k_3} + b_3 (p_{0,k_3})^2,$$

where a_1, b_1 are constants and

$$A_1 C_{\lambda,k_1} + A_2 C_{\lambda,k_2} + A_3 C_{\lambda,k_3} = 0,$$

and therefore

$$b_1 A_1 (p_{\lambda,k_1})^2 + b_2 A_2 (p_{\lambda,k_2})^2 + b_3 A_3 (p_{\lambda,k_3})^2$$
$$= -(a_1 A_1 c_{k_1} + a_2 A_2 c_{k_2} + a_3 A_3 c_{k_3}). \tag{6.51}$$

The left-hand member of (6.51) (which from 6.22 does not, in general, vanish identically) has thus the same value, whichever of the suffixes 0, α, β, γ may be identified with λ. But, for all values of k, the equation $p_{\lambda,k} = 0$ determines the polar of I_λ q. one of

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* The equation of the order-cubic T_1 may (cf. 2.1) be written in the form

$$\frac{\xi^2}{\alpha^2} + \frac{\eta^2}{\beta^2} + \frac{\zeta^2}{\gamma^2} + \frac{\upsilon^2}{\delta^2} = 0,$$

where
$$\xi + \eta + \zeta + \upsilon = 0,$$

and the sides $(\xi = 0,\ \eta = 0,\ \zeta = 0,\ \upsilon = 0)$ of the quadrilateral of reference are respectively B_1C_1, C_1A_1, A_1B_1, $A_0B_0C_0$. The coordinates (ξ, η, ζ) of the four points I_λ are then $(\pm\alpha, \pm\beta, \pm\gamma)$ and the ambiguous signs involved are those referred to in the text; while the forms c_k involve only even powers of the numbers α, β, γ, as is of course implied in the theorem (6.21).

the order-cubics of the T-pencil; i.e. (1.54) determines a line containing J_λ. Every equation of the form $\Sigma B_n (p_{\lambda, k_n})^2 = 0$ (where the coefficients B_n are constant, and there are any number of terms in the sum) therefore determines a pair of lines meeting at J_λ $(\lambda = 0, \alpha, \beta, \gamma)$. The equation

$$b_1A_1(p_{\lambda, k_1})^2 + b_2A_2(p_{\lambda, k_2})^2 + b_3A_3(p_{\lambda, k_3})^2 = 0$$

thus determines a pair of lines of which the meet is at once J_0, J_α, J_β, and J_γ; and the only such pair of lines is (1.54) the pair of coincident lines $A_0B_0C_0$.

Hence, if $A_0B_0C_0$ be determined by $p_1 = 0$; then, to a multiplicative constant

$$b_1A_1(p_{\lambda, k_1})^2 + b_2A_2(p_{\lambda, k_2})^2 + b_3A_3(p_{\lambda, k_3})^2 = p_1^2,$$

and thus from (6.51)

$$a_1A_1c_{k_1} + a_2A_2c_{k_2} + a_3A_3c_{k_3} + p_1^2 = 0;$$

or, what is (6.34) the same thing

$$\kappa_{k_1} + a_1A_1c_{k_1} + p_1^2 = 0.$$

Therefore: *The conic c_k (which has double contact with a group of four conics $C_{\lambda, k}$) and the conic κ_k (which has double contact with the associated* group of four conics $K_{\lambda, k}$ themselves have double contact, the line $A_0B_0C_0$ being the chord of double contact* $(\lambda = 0, \alpha, \beta, \gamma)$.

(6.52)

It may further be proved that both the conics c_k, κ_k have double contact, on $A_0B_0C_0$, with the polar conic, q. T_1, of the pole, q. Θ_k, of $A_0B_0C_0$.

7. *Recapitulation*. The theorem referred to in the introduction has now been completely proved. It may be stated in full as follows: Consider any plane order-cubic (T_1)† and any line $(A_0B_0C_0)$ in the plane. $A_0B_0C_0$ has four poles $(I_0, I_\alpha, I_\beta, I_\gamma)$ q. T_1, and each of the points $[I]$ has four polars q. the class-cubic (Θ_1) apolar to T_1. These four groups of four polars are the sixteen lines (say the lines $[l]$) of a configuration of symbol $\boxed{\begin{matrix}12 & 4\\ 3 & 16\end{matrix}}$, $A_0B_0C_0$ being the tangential of each of the lines $[l]$ q. the Hessian of T_1.

Each of the lines $[l]$ has a single pole q. any class-cubic (Θ_k) of the range (the Θ-range) determined by Θ_1 and its Hessian; we have thus four groups of four points each (say the points $[X]$). Each of the points $[X]$ has a pure poloconic q. Θ_1; we have thus four groups of four conics each (say the conics $[\Gamma]$).

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* As in (5.24), $C_{\lambda, k}$ having double contact with $K_{\lambda, k}$ $(\lambda = 0, \alpha, \beta, \gamma)$.

† Any plane order-cubic, T_i, of the T-pencil, and the apolar class-cubic, Θ_i, of the Θ-range, may be substituted for T_1. Θ_1 respectively throughout the specification. Since i, k are independent, ∞^2 systems of sixteen conics $[\Gamma]$ may thus be obtained.

The locus of the poles, q. the class-cubics of the Θ-range, of each of the lines $[l]$ is another line; we have thus four new groups of four lines (say the lines $[r]$), which are the lines of a second configuration of symbol $\boxed{\begin{matrix}12 & 4\\ 3 & 16\end{matrix}}$. Each of the lines $[r]$ has a single pole q. Θ_1. We have thus four new groups of four points each (say the points $[\varpi]$).

Associated with each group of four conics $[\Gamma]$ is a conic which (3.24) has double contact with each of the four conics of that group (the four poles of double contact being points of four lines $[r]$ which meet at one of the points I); we have thus a new group of four conics (say the conics $[K]$). Again, associated with each group of four conics $[\Gamma]$ is a conic which (5.16) has simple contact with each of the four conics of that group (the four points of contact being four of the points $[\varpi]$), and (5.24) has double contact with the conic K associated with that group; we have thus a further group of four conics (say the conics $[C]$).

There is a conic (κ) which (6.41) has double contact with each of the four conics $[K]$; there is a conic (c) which (6.21) has double contact with each of the four conics $[C]$; and (6.52) the conics κ, c have double contact, $A_0B_0C_0$ being the chord of double contact.

8. *Some Special Cases.* 8.1. If $k = 1$, then (3.42) Θ_k becomes Θ_2, and each of the four groups of four points $[X]$ involves the same four points, viz. (1.55) the points O_0, O_α, O_β, O_γ; the sixteen points X thus reduce to the four points O_0, O_α, O_β, O_γ. The four conics $[K]$ reduce to one, viz. (1.62) the conic $\Sigma_3\,(A_0B_0C'_0)$, with which also the conics κ, c coalesce. The four conics $[C]$, which remain distinct, therefore all have double contact with $\Sigma_3\,(A_0B_0C'_0)$. (8.11)

If $\Theta = -1$, then the sixteen points $[X]$ similarly reduce to four, viz. U_0, U_α, U_β, U_γ; the four conics $[K]$ and the conics κ, c all coalesce with $\Sigma_2\,(A_0B_0C_0)$. (8.12)

The case of (8.11) or (8.12) is identical with Hart's Theorem according as $\Sigma_3\,(A_0B_0C'_0)$ or $\Sigma_2\,(A_0B_0C_0)$ is identified with the Absolute of non-euclidean geometry.

8.2. If k be infinite, then (3.42) Θ_k becomes Θ_1, and (1.55) the four points $[X]$ of each group coalesce into a single point. The sixteen points $[X]$ therefore again reduce to four, viz. I_0, I_α, I_β, I_γ. It follows that the four conics $[\Gamma]$ of a group, and the conic K and the conic C associated with that group, all coalesce; thus the twenty-four conics $[\Gamma]$, K, C reduce to four. A fifth conic, resulting from the coalescence of κ and c, has double contact with these four.

8.3. If $k = 0$, then (3.42) Θ_k becomes Δ_1, and the group of four points $[X]$ associated with I_0 becomes the group of four points $[I_\mu']$, the harmonic conjugates of I_0 q. the point-pair O_μ,

U_μ ($\mu = 0, \alpha, \beta, \gamma$). The conic $K_{\lambda, 0}$, regarded as a class-conic, degenerates (*v.* 3.22 for $K_{0,0}$) into the point I_λ counted twice. $K_{\lambda, 0}$, regarded as an order-conic, therefore degenerates into a pair of lines meeting at I_λ. Thus: *If* $k = 0$, *then the four conics* $[\Gamma]$ *of each group have a pair of tangents common to all four, the meet of the tangents common to a group being the appropriate point* I_λ ($\lambda = 0, \alpha, \beta, \gamma$). (8.31)

From (3.12), the line-coordinates (l, m, n, p) of the pair of tangents (8.31) from I_0 are $(1, \omega, \omega^2, 0)$, $(1, \omega^2, \omega, 0)$, where ω, ω^2 denote the unreal cube roots of unity. Let this pair of tangents be denoted by $[t_0]$, and let the pairs of tangents $[t_\lambda]$ be similarly specified q. I_λ ($\lambda = \alpha, \beta, \gamma$). (8.32)

Now from (5.21), the line-equation of the conic $C_{0,0}$ is

$$mn + nl + lm = \tfrac{1}{2}(p^2 - l^2 - m^2 - n^2) = 0, \qquad (8.33)$$

and more generally, the line-equation of each of the conics $C_{\lambda,0}$ ($\lambda = 0, \alpha, \beta, \gamma$) is of the form $\pm\, l^2 \pm m^2 \pm n^2 \pm p^2 = 0$, where one and only one of the ambiguous signs is positive. Hence the line-equation of the conic c_0 is

$$l^2 + m^2 + n^2 + p^2 = 0. \qquad (8.34)$$

Thus (8.32) each line of the four pairs $[t_\lambda]$ touches the conic c_0, which therefore (6.41) coincides with κ_0.

It is a case of (5.24), and is obvious from (8.32, 3), that both the lines $[t_0]$ touch the conic $C_{0,0}$. Remembering that the pure poloconic, q. Θ_1, of any point has triple contact of a certain species with Δ_1, we have the following: The four conics $[\Gamma_1(I_\mu')]$ which have triple contact of a certain species with Δ_1 and touch both the lines $[t_0]$, all have simple contact with a certain conic ($C_{0,0}$) which also touches both the lines $[t_0]$. This theorem is also a case of the dual (for class-cubics) of the theorem*: The four conics which have triple contact of a certain species with a given order-cubic, and contain two given points, all have simple contact with a certain conic which also contains the two given points.

9. *A Further Generalization.* 9.1. We have from (3.21, 2)

$$4a^2b^2c^2\left\{(l + m + n) + kd\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right)\right\}^2 = 4kabcd\,\{\Gamma_1(X_{0,k}) - \mathrm{K}_{0,k}\},$$

and from (5.22)

$$[2abc\,(l + m + n) + kd\,\{l\,(b^2 + c^2) + m\,(c^2 + a^2) + n\,(a^2 + b^2)\}]^2$$
$$= 4kabcd\,(\Gamma_{0,k} - \mathrm{K}_{0,k}).$$

Subtracting, we see that

$$\frac{\{l\,(b - c)^2 + m\,(c - a)^2 + n\,(a - b)^2\}\,[4abc\,(l + m + n) + kd\,\{l\,(b + c)^2 + m\,(c + a)^2 + n\,(a + b)^2\}]}{4abc}$$
$$= \Gamma_{0,k} - \Gamma_1(X_{0,k}).$$

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* Malgouzou, *Nouv. Annales* (4), 19 (1919), p. 210.

Now, if Σ' denote any quadratic form in l, m, n, p, and if

$$\frac{kabc}{d}.\Sigma_0' = \frac{kabc}{d}.\Sigma' + \Gamma_1(X_{0,k}) - \mathrm{K}_{0,k}$$
$$= \frac{kabc}{d}\left[\Sigma' + \left\{k^{-1}(l+m+n) + d\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)\right\}^2\right].$$

and

$$\frac{kabc}{d}.\Xi_0' = \frac{kabc}{d}.\Sigma' + \Gamma_{0,k} - \mathrm{K}_{0,k}$$
$$= \frac{kabc}{d}\left[\Sigma' + \left(\frac{2abc(l+m+n) + kd\{l(b^2+c^2) + m(c^2+a^2) + n(a^2+b^2)\}}{2kabc}\right)^2\right]$$

then

$$\frac{4ka^2b^2c^2}{d}(\Xi_0' - \Sigma_0') = \{l(b-c)^2 + m(c-a)^2 + n(a-b)^2\}$$
$$[4abc(l+m+n) + kd\{l(b+c)^2 + m(c+a)^2 + n(a+b)^2\}].$$

Remembering that

$$\varpi_1(I_0O_0) \equiv l(b-c)^2 + m(c-a)^2 + n(a-b)^2 = 0,$$

we see that the conic determined by $\Xi_0' = 0$ touches the tangents from $\varpi_1(I_0O_0)$ to the conic determined by $\Sigma_0' = 0$; similarly the conic determined by $\Xi_0' = 0$ touches the tangents from $\varpi_1(I_0O_\mu)$ to the conic determined by $\Sigma_\mu' = 0$, where

$$\frac{kabc}{d}.\Sigma_\mu' = \frac{kabc}{d}\Sigma' + \Gamma_1(X_{\mu,k}) - \mathrm{K}_{\mu,k} \quad (\mu = \alpha, \beta, \gamma).$$

But since $\Sigma_0' = \Sigma' + (k^{-1}p + q)^2$ etc., we have the theorem: *If* Σ', Σ_0', Σ_α', Σ_β', Σ_γ' *be specified as in* (4.11), *then there is a conic* Ξ_0' *which touches the pairs of tangents from* $\varpi_1(I_0O_\mu)$ *to* Σ_μ', *and has double contact with* Σ' $(\mu = 0, \alpha, \beta, \gamma)$. Clearly the theorem remains true when the accents are dropped. (9.11)

Following (4.12), we may write $\Gamma_1(X_{\mu,k_1}, X_{\mu,k_2})$ for Σ_μ' $(\mu = 0, \alpha, \beta, \gamma)$. Now, whatever points may be denoted by X, X', the conics $\Gamma_1(X)$, $\Gamma_1(X')$ contain the point $\varpi_1(XX')$; and the tangents at $\varpi_1(XX')$ to $\Gamma_1(X)$, $\Gamma_1(X')$ touch $\Gamma_1(X, X')$. In particular, since (1.44) X_{μ,k_1}, X_{μ,k_2} are points of I_0O_μ, therefore the tangents at $\varpi_1(I_0O_\mu)$ to $\Gamma_1(X_{\mu,k_1})$, $\Gamma_1(X_{\mu,k_2})$ touch $\Gamma_1(X_{\mu,k_1}, X_{\mu,k_2})$. Also (5.16) these tangents touch, at $\varpi_1(I_0O_\mu)$, the conics C_{0,k_1}, C_{0,k_2} respectively $(\mu = 0, \alpha, \beta, \gamma)$. The pairs of tangents from the points $\varpi_1(I_0O_\mu)$ to the conics $\Gamma_1(X_{\mu,k_1}, X_{\mu,k_1})$ $(\mu = 0, \alpha, \beta, \gamma)$ are thus the tangents to the conics C_{0,k_1}, C_{0,k_2} at their four common points, and therefore touch the Φ-conic of C_{0,k_1}, C_{0,k_2}. Thus if $\Gamma_1(X_{\mu,k_1}, X_{\mu,k_2})$ be written for Σ_μ', then Ξ_0' becomes the Φ-conic of C_{0,k_1} and C_{0,k_2}; *i.e.*: *The conic which has double contact with the four conics* $\Gamma_1(X_{\mu,k_1}, X_{\mu,\kappa_2})$ *of* (4.12) *has double contact also with the* Φ*-conic of* C_{0,k_1}, C_{0,k_2}. (9.12)

When the values k_1, k_2 of the parameter are equal, this theorem of course reduces to that of (5.24).

9.2. The dual (cf. 4.2) of the theorem (9.12) in its general form s also true. In the special case of (4.21) it is to be noted that the four points J_λ (each of which is contained by the polar of one of the four points I_λ q. every order-cubic of the T-pencil) replace the four lines I_0O_μ (each of which contains the pole of one of the four ines $l_{0,\mu}$ q. every class-cubic of the Θ-range); the lines $p_1(J_\lambda)$ therefore replacing the points $\varpi_1(I_0O_\mu)$. Thus: *With the notation of* (4.21), *the conic which has double contact with the four conics* $C_1(x_{\lambda,k_1}, x_{\lambda,k_2})$ *has double contact also with a conic containing the meets of each of four conics with the corresponding line* $p_1(J_\lambda)$ $(\lambda = 0, \alpha, \beta, \gamma)$. (9.21)

In particular, we may write $p_n'(I_\lambda)$, $p_n''(I_\lambda)$ for x_{λ,k_1}, x_{λ,k_2} $(n = 2, 3)$. We have then (*v.* 8.11, 2): $\Sigma_n(A_0B_0C_0)$ *has double contact with a conic* (F, *say*) *which contains the meets of each of the four conics* $C_1\{p_n'(I_\lambda), p_n''(I_\lambda)\}$ *with the corresponding line* $p_1(J_\lambda)$ $(n = 2, 3; \lambda = 0, \alpha, \beta, \gamma)$. (9.22)

If $\Sigma_n(A_0B_0C_0)$ be identified with the Absolute of non-euclidean geometry, then the four conics $C_1\{p_n'(I_\lambda), p_n''(I_\lambda)\}$ and the conic F become circles. We have thus obtained a single circle F (not one of a group of four) which in the euclidean case becomes the circumcircle of the triangle $A_1B_1C_1$. For it is easy to show that, if the Absolute $\Sigma_n(A_0B_0C_0)$ degenerates into a point-pair, then each of the four lines $p_1(J_\lambda)$ touches the corresponding circle $C_1\{p_n'(I_\lambda), p_n''(I_\lambda)\}$; and the theorem (9.22) therefore reduces to Feuerbach's Theorem for the triangle of which the sides respectively contain the vertices, and are parallel to the opposite sides, of the triangle $A_1B_1C_1$.

[*Note added April* 1925. It is clear, from the dependence of the analysis on the equation (2.15), that Hart's Theorem, and the generalization here attempted, are intimately related to the theory of Steiner's quartic surface. The relation may conveniently be expressed in geometrical form; the theorems leading to that of Hart being for the most part well known, and the remainder being capable of very simple proof.

Any Steiner quartic surface (R) has four *singular planes* (σ_0, σ_α, σ_β, σ_γ), each of which touches the surface R along a conic (C_0, C_α, C_β, C_γ respectively), here termed a *singular conic* of R. The four conics C_ν lie on a quadric (Q), the *singular quadric* of R. The surface R has three nodal lines, which meet at a point (P).

Let R' denote a second Steiner surface *consingular* with R, *i.e.*, having the same four singular planes σ_ν; and let C_ν', Q' denote the singular conics and quadric of R'. The intersection of R, R'

consists of eight conics, $[G]$, each of which touches all the four planes σ_ν. Of the ∞^1 quartic surfaces which contain the eight conics $[G]$, one breaks up into two quadrics (S_2, S_3), the *twin-quadrics* of the pair of surfaces R, R'. Four of the conics $[G]$ are common to R, R', S_2, and the remaining four to R, R', S_3. The quadrics S_2, S_3 belong to the ∞^1 linear system of quadrics, $[Q]$, containing the common points of Q, Q'; and the section of S_2 (or S_3) by the plane σ_ν is therefore a member of the ∞^1 linear system of conics containing the common points of C_ν, C_ν' $(\nu = 0, \alpha, \beta, \gamma)$.

Projecting from the point P on any plane ω, we get, together, the theorems (8.11, 2); the four conics $[\Gamma]$ (see § 7) of those theorems being the projections of the four conics $[G]$ contained by S_n; the four conics $[C]$ being the projections of the sections of S_n by σ_ν; and the conic $\Sigma_n (A_0B_0C_0)$ being the section, by ω, of the tangent cone from P to S_n $(n = 2, 3;\ \nu = 0, \alpha, \beta, \gamma)$.

For the generalization, let $\bar{Q}$ denote any quadric of the system $[Q]$; and let $\bar{C}_\nu$ denote the section of $\bar{Q}$ by σ_ν. Then there is one and only one Steiner surface $(\bar{R}_\nu)$, consingular with R, and such that one $(\bar{S}_\nu)$ of the twin-quadrics of R', $\bar{R}_\nu$ contains the conic $\bar{C}_\nu$. The four quadrics $\bar{S}_\nu$, and the quadric $\bar{Q}$, all have plane contact with another quadric $(\bar{T})$ $(\nu = 0, \alpha, \beta, \gamma)$.

Projecting from P on ω, we get the theorems of § 7. Each of the four quadrics $\bar{S}_\nu$ intersects the surface R' in four conics, of which the projections constitute one of the four groups of four conics $[\Gamma]$; the associated conic K being the section, by ω, of the tangent cone from P to $\bar{S}_\nu$; and the associated conic C being the projection of $\bar{C}_\nu$. The conics κ, c are the sections, by ω, of the tangent cones from P to $\bar{T}$, $\bar{Q}$ respectively.]

Characteristics of complexes of conics in space of four dimensions. By Mr C. G. F. JAMES, Trinity College.

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In this note we obtain a system of characteristics on which depend the main enumerative properties of complexes, or systems ∞^3, of conics in space of four dimensions. The method used is applicable to a large number of problems of this nature, and we select this illustration as being analogous to congruences in ordinary space. In particular a finite number of conics pass through an arbitrary point, thereby defining the order, n, of the system. Linear complexes are those for which this order is unity. The conics of a complex satisfy eight simple conditions, in general conditions of contact with a fixed form or forms*, but they may include conditions of incidence with a surface, each such counting once, with a curve, each such counting twice, or with fixed points each such counting thrice. In particular only incidence conditions can occur when the system is linear, for otherwise more than one conic would pass through certain ∞^3 points of space. Points through which more than a finite number of conics pass are termed singular, as well as their loci. Directrix constructs are necessarily singular, but they do not necessarily exhaust the singular system, for the complex may possess ∞^2 curves lying on a surface. In the linear case through a point of a singular curve pass ∞^2 conics, and through a point of a singular surface ∞^1 conics. The possibilities in the non-linear cases are too numerous to be detailed. Similarly the system of planes of the conics may possess a singular curve, through which ∞^2 of the planes pass.

§ 1. The remaining characteristics which are introduced are as follows:

The space-order, m, or the number of conics in an arbitrary space. If the system is projected from a point on to a space, we obtain in that space a complex of conics of order m in the sense that m of its conics lie in a given plane†.

The first class, θ_1, or the number of conics meeting a line and a plane through it elsewhere. It is also the class‡ of the projected complex.

The second class, θ_2, or the number of conics meeting a line and a plane in arbitrary position.

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* Constructs of points of dimension three; linear forms we call spaces.

† Montesano, "Una estensione del problema...," *Ann. di Mat.* ser. 3, t. I, p. 321 (1898).

‡ Again as defined by Montesano (*loc. cit.*).

The third class, θ_3, or the number of conics meeting three planes.

The rank, N, or the class of the envelope of chords in a plane, multiply counted in the exceptional case when more than one conic lies in each plane of the system of planes. It also equals the degree of the complex which is the section of the system of planes by an arbitrary space.

It is convenient to have a symbol, r, for the number of conics touching a plane, and by applying the correspondence principle to the ∞^1 pairs cut by conics on an arbitrary plane it is seen that r equals $2(\theta_1 - N)$.

§ 2. *General analytical processes.* It is convenient to consider first the special case when the complex is defined by relations

$$f_1(\alpha, x) = f_2(\alpha, x) = g(\alpha, x^2) = 0, \qquad \ldots\ldots(1)$$

of degrees 1, 1, 2 in the coordinates $(x_1 \ldots x_5)$, and of degrees l_1, l_2, l_3 in the homogeneous parameters $(\alpha_1 \ldots \alpha_4)$. $g(\alpha, x, y)$ denotes the polar form of $g(\alpha, x^2)$ for (y), and (α_{ix}) denote the results of solving these equations for the (α) in terms of the (x). These are rational functions in linear cases (since a single-valued algebraic function is rational), and in these cases only.

The conics whose planes pass through a point (ξ) are given by

$$f_1(\alpha, \xi) = f_2(\alpha, \xi) = 0,$$

and the conics generate the surface

$$f_1(\alpha_{ix}, \xi) = f_2(\alpha_{ix}, \xi) = 0,$$

although this is in rational form for linear complexes only. The conics meeting a line are given by

$$\left.\begin{array}{r} f_1(\alpha, \xi) f_2(\alpha, \eta) - f_1(\alpha, \eta) f_2(\alpha, \xi) = 0 \\ g(\alpha, \xi^2) f_1^2(\alpha, \eta) - 2g(\alpha, \xi, \eta) f_1(\alpha, \eta) f_1(\alpha, \xi) + g(\alpha, \eta^2) f_1^2(\alpha, \xi) = 0 \end{array}\right\},$$

(ξ), (η) being points on the line. The surfaces, etc. in this and subsequent cases are given by replacing the (α_i) by the corresponding (α_{ix}). To find the conics meeting a plane joining (ξ), (η), (ζ), we write in (1)

$$x = \kappa_1 \xi + \kappa_2 \eta + \kappa_3 \zeta.$$

Then

$$\kappa_1 : \kappa_2 : \kappa_3 = \left\| \begin{array}{ccc} f_1(\alpha, \xi) & f_1(\alpha, \eta) & f_1(\alpha, \zeta) \\ f_2(\alpha, \xi) & f_2(\alpha, \eta) & f_2(\alpha, \zeta) \end{array} \right\| \qquad \ldots\ldots(2)$$

and the condition to be satisfied by the parameters is given by substituting in

$$\kappa_1^2 g(\alpha, \xi^2) + 2\kappa_1\kappa_2 g(\alpha, \xi, \eta) + \text{etc.} = 0.$$

Further, the matrix in (2) equated to zero represents the system of conics meeting the plane twice.

The degenerate conics are given by the condition that

$$f_1(\alpha, x) = f_2(\alpha, x) = 0$$

shall touch $g(\alpha, x^2) = 0$, and the vertex locus, a surface, by elimination of (α) and λ_1, λ_2 from the set of equations

$$g(\alpha, x, \xi) \equiv \lambda_1 f_1(\alpha, x) + \lambda_2 f_2(\alpha, x),$$
$$f_1(\alpha, \xi) = f_2(\alpha, \xi) = 0,$$

the result being regarded as a set of equations in (ξ). The conics touching a space $\omega_x = 0$ are given by the condition that the line $f_1(\alpha, x) = f_2(\alpha, x) = \omega_x = 0$ shall touch $g(\alpha, x^2) = 0$. Both this and the last condition can be immediately written down.

The locus of points of contact of tangents passing through a point (y) is given by the equation, with (y) as current coordinates, of the tangents at (x) to the conics through (x), namely

$$f_1(\alpha_{ix}, y) = f_2(\alpha_{ix}, y) = g(\alpha_{ix}, x, y) = 0.$$

Similarly the locus for tangents meeting a line (ξ), (η) is

$$\left\| \begin{matrix} f_1(\alpha_{ix}, \xi) & f_2(\alpha_{ix}, \xi) & g(\alpha_{ix}, x, \xi) \\ f_1(\alpha_{ix}, \eta) & f_2(\alpha_{ix}, \eta) & g(\alpha_{ix}, x, \eta) \end{matrix} \right\| = 0;$$

and that for tangents meeting a plane is

$$\left| f_1(\alpha_{ix}, \rho) \quad f_2(\alpha_{ix}, \rho) \quad g(\alpha_{ix}, x, \rho) \right| = 0,$$

where $\rho = \xi, \eta, \zeta$ in the three rows. These are obtained by expressing that one of the tangents at a point (x) to the conics through it contains a point

$$y = \kappa_1 \xi + \kappa_2 \eta,$$
$$y = \kappa_1 \xi + \kappa_2 \eta + \kappa_3 \zeta,$$

in the two cases.

Lastly, the focal form[*] of any system of ∞^2 planes, the conics in which are given by a relation $\phi(\alpha) = 0$, is obtained by expressing that $f_1(\alpha, x) = f_2(\alpha, x) = 0$ represent, as equations in (α), a curve in the space $S_{(a)}$ of three dimensions, which touches the surface $\phi(\alpha) = 0$.

§ 3. *Representation of the complex on a space.* In the present case the parameters (α) can be taken as the coordinates in a space $S_{(a)}$ of three dimensions, and the system is thus represented on $S_{(a)}$. It is now assumed that this representation is birational, since it is so 'in general,' and since the methods and results hold with slight modification when it is uni-rational merely. In particular the geometrical interpretations of the formulae are unaltered. There exist in general principal elements in the representation. Thus we have in $S_{(a)}$ points to which correspond ∞^1 or ∞^2 conics, the former points lying on a curve or curves, the latter being isolated and finite in number. Inversely there may exist conics represented by points of a continuum in $S_{(a)}$, though this latter case is not of such frequent occurrence in cases which naturally present themselves. Following the method, which we have used to deal with complexes of conics in ordinary space*, we proceed to define the following representative constructs for the various systems of conics satisfying the conditions in the left column:

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[*] Touched by each plane along a conic. See Segre, "Sui fochi di 2° ordine...," *Rend. Accad. Lincei* (5), v. 30 (1921), and references there given.

To meet a line.	Curve of order n_1, being the order of the form of conics for a linear relation in the (a). It is rational in the linear case.
To meet an arbitrary plane.	Surface of order n_2, being the order of the surface of conics for two linear relations in the (a). Rational in the linear case.
To meet an arbitrary plane twice.	Curve of order n_3, in general irreducible. It is nodal on the previous surface of order n_2.
Its plane to pass through an arbitrary point.	Curve of order n_4, equal to that of the system of planes for a linear relation in the (a): in general irreducible.
Its plane to meet a line.	Surface of order n_5.
To degenerate.	Surface of order d.
To touch an arbitrary space.	Surface of order t.
To pass through a point of a singular surface.	Curve of order m_1.
To pass through a point of a singular curve.	Surface of order m_2.
Its plane to pass through a point of a singular curve for the system of planes.	Surface of order m_3.

So also for the conics satisfying two independent conditions we have, in a notation to be explained immediately:

To meet two planes.	Curve of order $(n_2)^2$.
To meet a plane and touch a space.	Curve of order $(n_2)\,(t)$.
Etc.	

When more than one singular curve or surface exists, the corresponding m_1, m_2, m_3 are denoted with bracketed upper suffixes. Expressions such as $(n_1)\,(t)$, $(n_3)\,(d)\,(t)$, $(m_1)\,(t)^2$ denote the intersections of the separate constructs of the brackets residual to the principal elements in $S_{(a)}$.

§ 4. *Expression of the characteristics of the system in terms of the representative constructs.* In the first place

$$N = (n_4)\,(n_5) = \tfrac{1}{2}\,\{(n_2)\,(n_4) - n\}, \qquad \dots\dots(3)$$

the second expression being given by consideration of the conics which meet a line, while their planes pass through a point of the line. The space order m equals the number of planes meeting a plane ϖ in lines, and meeting a line meeting ϖ, less the order of the cone of planes through a point, or

$$m = (n_3)\,(n_5) - N = (n_3)\,(n_5) - (n_4)\,(n_5). \qquad \dots\dots(4)$$

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* "On some complexes...," *Proc. Lond. Math. Soc.*, vol. 23, p. 115 (1924).

To find the first class we may consider the number of conics meeting a plane, and meeting an incident* plane twice; or we may consider the number meeting a line, while their planes meet a second line, incident with the first. These give respectively

$$\theta_1 = (n_2)(n_3) - 2m = (n_1)(n_5) - n. \quad \ldots\ldots(5),\ (6)$$

No comment is required on the formulae

$$\theta_2 = (n_1)(n_2) : \theta_3 = (n_2)^3 : x = \tfrac{1}{2}(n_1)(t), \ldots(7)\text{-}(9)$$

where x denotes the order of the surface locus of points of contact of conics touching a space.

§ 5. *Introduction of the auxiliary numbers l_1, l_2, l_3, and fundamental relations for the representative system.* We now introduce into the representation the surfaces of orders l_1, l_2, l_3 given as equations in (α) by the respective equations in (1). Using § 2 it is found without difficulty that

$$\left.\begin{aligned} n &= (l_1)(l_2)(l_3) \\ n_4 &= (l_1)(l_2) : n_5 = l_1 + l_2 \\ n_3 &= (l_1)^2 + (l_1)(l_2) + (l_2)^2 \\ n_1 &= (l_1 + l_2)(l_3) + 2(l_1)(l_2) \\ n_2 &= l_3 + 2l_1 + 2l_2 \\ d &= 3l_3 + 2l_1 + 2l_2 \\ t &= 2(l_1 + l_2 + l_3) \end{aligned}\right\}. \quad \ldots\ldots(10)$$

These and all such subsequent relations connect in the first place orders of constructs, but relations of exactly the same linear form hold between the multiplicities of the principal constructs of the representation, and between the number of intersections with each principal curve (as the case may require). It follows from this that it is possible to operate with all the elementary operators save division. Conversely

$$l_3 = d - t, \quad 2(l_1 + l_2) = 3t - 2d = 2n_5.$$

Substituting we obtain the complete set of *Fundamental Relations*, namely, relations in the representative system, in which, in our notation, the principal and singular constructs do not occur explicitly:

$$(n_5)^2 = n_3 + n_4, \quad \ldots\ldots\mathrm{I}$$

$$(n_2)(n_5) = (2n_3 + n_1), \quad \ldots\ldots\mathrm{II}$$

$$2t = (n_2 + d), \quad \ldots\ldots\mathrm{III}$$

$$t = 2(n_2 - n_5). \quad \ldots\ldots\mathrm{IV}$$

The order of the system may be given in various ways, in particular

$$n = (n_4)(t - n_2).$$

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* Namely a plane meeting the former in a line.

§ 5·1. The relations involved in (3) and (5) are algebraic deductions of the above set of four equations, and so also are any similar relations which may occur. The equation III has also been given by Schubert[*], as well as the relation $(n_3)(t) = r + 2m$, which can be found directly by taking the plane for (n_3) in the space for (t). It is of interest to determine the remainder directly without the use of the auxiliary constructs: namely, for I by taking incident the two lines to which the symbolic factors refer, and then the conics in question are those whose planes are incident with the plane of the two lines, and those whose planes pass through their point of intersection. This is essentially a result for the system of planes. For II we take the line to which (n_5) refers in the plane to which (n_2) refers. Finally, IV, which holds restrictively for systems containing ∞^1 double lines at most[†], is obtained as follows. Consider the surface of conics represented by a curve (a) of $S_{(4)}$. There exists in general a correspondence[‡] (1, 1) on the curve of section by any space. The curve is of order $(a)(n_2)$ and the number of united points is $(a)(t)$, while the scroll of joining lines is of order $(a)(n_5)$. Hence by a well-known formula

$$(a)(t) = 2(a)(n_2) - 2(a)(n_5),$$

and, since this holds for any curve (a), IV must be true.

Repeating for convenience some previous results, we can draw up the following table by simple algebraic manipulation. No further simplification is possible, since from the numerical values of the characteristics in special cases it can be shown that *no linear relation between the characteristics exists* though the possibility of a higher relation is not excluded.

Formulae (11):

$$(n_1)(n_2) = \theta_2 : (n_1)(n_5) = \theta_1 + n : (n_1)(d) = 3\theta_2 - 4\theta_1 - 4n,$$
$$(n_1)(t) = 2\theta_2 - 2\theta_1 - 2n, \ (= 2x).$$

$$(n_3)(n_2) = \theta_1 + 2m \quad : (n_3)(n_5) = m + N,$$
$$(n_3)(d) = 3\theta_1 + 2m - 4N : (n_3)(t) = 2\theta_1 + 2m - 2N.$$

$$(n_4)(n_2) = 2N + n \quad : (n_4)(n_5) = N,$$
$$(n_4)(d) = 2N + 3n \quad : (n_4)(t) = 2N + 2n.$$

$$(n_2)^2(n_5) = 4m + \theta_2 + 2\theta_1 : (n_5)^2(n_2) = 2m + 2N + \theta_1 + n,$$
$$(n_5)^3 = m + 2N \quad : (n_5)^2(d) = 3\theta_1 + 2m - 2N + 3n.$$

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[*] "Allgemeine Anzahlfunctionen für Kegelschnitte...," *Mat. Ann.*, Bd. 45, p. 153.

[†] I.e. for those whose conics are considered as loci of points.

[‡] Not in general linear.

$$(n_5)^2\,(t) = 2\,(\theta_1 + m + n) : (n_5)\,(t)^2 = 4\,(\theta_2 + m - 2N - 2n),$$
$$(n_5)\,(d)\,(t) = 6\theta_2 - 2\theta_1 + 4m - 12N - 14n,$$
$$(n_2)\,(n_5)\,(d) = 3\theta_2 + 2\theta_1 + 4m - 8N - 4n,$$
$$(n_2)\,(n_5)\,(t) = 2\,(\theta_2 + \theta_1 + 2m - 2N - n).$$

$$(n_2)\,(d)\,(t) = 8n - 40m + 16N - 20\theta_1 - 14\theta_2 + 6\theta_3,$$
$$(n_2)^2\,(d) = 3\theta_3 - 16m - 8\theta_1 - 4\theta_2,$$
$$(n_2)^2\,(t) = 2\theta_3 - 8m - 4\theta_1 - 2\theta_2.$$

$$(t)^2\,(n_2) = 4n - 24m + 8N - 12\theta_1 - 8\theta_2 + 4\theta_3,$$
$$(t)^2\,(d) = 44n - 88m + 56N - 36\theta_1 - 40\theta_2 + 12\theta_3,$$
$$(t)^3 = 24n - 56m + 32N - 24\theta_1 - 24\theta_2 + 8\theta_3.$$

§ 6. A set of relations of a different type are some connecting genera, etc. of the constructs in the four-dimensional space in terms of the representative system. Consider first any curve (a) in the representative space $S_{(4)}$ and let its genus be $p\,(a)$. The corresponding conics describe a surface, which cuts any space in a curve of order $(a)\,(n_2)$, on which the conics give a correspondence (1, 1), not in general linear. Each pair of this correspondence can be associated to a single point on (a), and the coincidence pairs are $(a)\,(t)$ in number. Hence by a well-known theorem of Zeuthen*

$$(a)\,(t) = 2\,(p - 1) - 4\,\{p\,(a) - 1\},$$

where p is the genus of the curve of section. In other words the genus of the space sections of the surface of conics is†

$$p = \tfrac{1}{2}\,(a)\,(t) - 1 + 2p\,(a).$$

Consider also the curve of intersection with a plane of those conics which meet it twice, a curve of order θ_1. The conics cut on the curve a correspondence, which is (1, 1) in general, and necessarily in the linear case, and to each of the pairs is associated a point of the curve (n_3) for the plane. The united points are at the r or $2\,(\theta_1 - N)$ points, where a conic touches the plane. Assuming the curve to be irreducible and of genus q, an analogous argument gives for the genus of the curve of intersection the value $(\theta_1 - N - 2q - 1)$. Lastly, the conics define on the curve locus of points of contact of tangents passing through a fixed point, P, a correspondence of the same character as before. This curve‡ is of order $\{(t)\,(n_4) + n\}$, as can be seen by taking the point to which (n_4) refers in the space to which (t) refers, and passes n times through P. Its genus by an analogous argument is $(N + 2n + 2q' - 1)$, where q' is the genus of the curve (n_4), since the united points of the (1, 1) correspondence are at the n branches of the curve through P, and at the vertices of the degenerate conics whose planes pass through P.

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* "Nouvelle démonstration...," *Mat. Ann.*, Bd. 3, p. 150 (1871).

† These genera can be calculated in the majority of linear cases directly using the formulae in my paper "On the intersection...," *Proc. Camb. Phil. Soc.*, vol. 21, p. 435 (1923), principally by formulae X, XI, XIII. Notice that in Form. IX of this paper a term rsh is omitted.

‡ The surface locus for tangents meeting a line is similarly of order $(x + r + n)$, and the form, locus for those meeting a plane, of order $(x + n)$.

In all these cases when the representative curve is rational, the corresponding curve is hyperelliptic.

§ 7. Some further relations exist in the linear case, and these we may say are of *mixed type*, since they involve explicitly the multiplicities singular constructs. We denote by μ_i $(i = 1, 2, ...)$ the orders of the singular surfaces in [4], and by σ_i the order of multiplicity of a point of $V_2^{\mu_i}$ on the surface of conics through it. The following examples are illustrative of this type of relation.

If (α), (ρ) stand for any two of the surfaces of the systems (n_2), (n_5), (d), (t), then the corresponding forms in four dimensions cut in the singular surfaces, and in the surface of conics represented by points of the curve $(\alpha)(\rho)$. Since $V_2^{\mu_i}$ is of multiplicity $(\alpha)(m_1^{(i)})$ on the form for (α), (in general), etc., this gives

$$(\alpha)(\beta)(n_2) = [(\alpha)(n_1)][(\beta)(n_1)] - \sum_i [(\alpha)(m_1^{(i)})][(\beta)(m_1^{(i)})],$$

and in all cases when the representation is without principal elements this is simply

$$n_2 = n_1^2 - \sum_i m_1^{(i)} \mu_i .$$

In each the effect of the singular surfaces may exceptionally be greater than that indicated.

§ 8. Returning to the general case, we have now sufficient data to write down the chief features of the constructs associated with the complex. The only case which requires further development is that of *the system of degenerate conics*, generating a form of order $(d)(n_1)$, while their planes form a system of order* $(d)(n_4)$. In the first place, considering the conics, which meet three planes, which meet in a point, and have line intersections in pairs, it is found that

$$w = (n_2)^3 - 8m - 3\theta_2 + 2n = 2n - 8m - 3\theta_2 + \theta_3, \quad \text{...(12)}$$

w being the number of lines of degenerate conics in a space, counting multiply when each such line belongs to more than one conic. Further, v, the order of the surface locus of vertices of degenerate conics, is obtained by considering the degenerate conics touching a space in the point of a plane. The conics touching a space in this way are represented in $S_{(\alpha)}$ by the points of a curve $\frac{1}{2}(n_2)(t)$, for it is seen without difficulty that the surface (t) is the envelope of the surfaces (n_2) for the planes of the associated space, and is touched along a curve of this order by each of them. Considering the intersection of this curve and the surface, (d) gives*

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* Namely such is the number of planes passing through a point.

$$v = \tfrac{1}{2}\,\{\tfrac{1}{2}\,(d)\,(t)\,(n_2) - w\} = n - 6m + 4N - 5\theta_1 - 2\theta_2 + \theta_3. \quad (13)$$

The ∞^1 *degenerate conics meeting a plane* ϖ generate two scrolls of total order $(n_2)^2\,(d)$; while taking the section by a plane incident with the given plane, it is seen that those lines meeting the plane give a scroll of order equal to the number of degenerate conics meeting a line, together with the number of lines of degenerate conics in a space, or $\{w + (n_1)\,(d)\}$, which by equations (11) and (12) can be expressed in terms of the characteristics. The surface cuts ϖ in a curve of order $(n_1)\,(d)$. The residual lines must describe a scroll of order $\{(d)\,(n_2)^2 - (d)\,(n_1) - w\}$, meeting the former scroll in the singular system, in the $(n_3)\,(d)$ degenerate conics meeting the plane twice, and in the curve locus of vertices, a curve of order $\{\tfrac{1}{2}\,(n_2)\,(d)\,(t)\}$, for it meets ϖ in v double points, and a space through ϖ contains w of its points other than those on ϖ. The residual scroll cuts ϖ in two points for each of the $(n_3)\,(d)$ conics, in the v points, and in singular points only.

§ 9. Consider now the most general complex, represented by the equations (1), with however more than four homogeneous parameters, connected by a number of relations. If j is the number of relations, the system is thereby represented on a construct of three dimensions in space of $(3 + j)$ dimensions. The single relations represent forms (A_1), (A_2), ... (A_j) in the space, and the notation previously employed can be again used for constructs representing conics satisfying given conditions, all these constructs sharing certain principal loci. The equations (10) still hold if the factors (A_1), ... (A_j) are added to the right-hand sides in each case, and it follows that all the subsequent relations remain applicable.

We conclude then that the results we have obtained are universally applicable.

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* From (11) we have $4\,(v + r) = (t)^2\,(n_2)$, and the general form of this result can be verified by considering the intersections of the curves representing the conics which touch two spaces in the points of their common plane, ϖ. These represent either conics touching the plane, or degenerate conics whose vertices lie on the plane. The numerical coefficients however are not easily seen directly. We can deduce a new relation of mixed type by considering the intersection of the x-tic surfaces (9) for the spaces, consisting in fact of the sets of r and v points and the singular points in ϖ, whence

$$x^2 = r + v + \sum_i \{\sigma_i\}^2\,\mu_i \text{ (in general).}$$

Note on the Lorentz group. By Mr J. Brill, St John's College.

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The Lorentz group is defined by the equations

$$X_r = a_{1r}x_1 + a_{2r}x_2 + a_{3r}x_3 + a_{4r}x_4 \quad (r = 1, 2, 3, 4),$$

where the coefficients are subject to the conditions

$$a_{r1}^2 + a_{r2}^2 + a_{r3}^2 - a_{r4}^2 = 1 \; (r = 1, 2, 3), \quad a_{41}^2 + a_{42}^2 + a_{43}^2 - a_{44}^2 = -1,$$

$$a_{r1}a_{s1} + a_{r2}a_{s2} + a_{r3}a_{s3} - a_{r4}a_{s4} = 0 \; (r, s = 1, 2, 3, 4;\; r \neq s).$$

It is an intransitive group possessing the single invariant

$$x_4^2 - x_1^2 - x_2^2 - x_2^2.$$

By an adaptation of the work of Hermite and Cayley the said coefficients may be realized on writing $a_{rs} = u_{rs}/v$, where

$$v = 1 + a^2 + b^2 + c^2 - f^2 - g^2 - h^2 - k^2,$$
$$u_{11} = 1 + a^2 - b^2 - c^2 + f^2 - g^2 - h^2 + k^2,$$
$$u_{21} = 2c + 2(ab + fg) - 2hk,$$
$$u_{31} = -2b + 2(ca + hf) + 2gk,$$
$$u_{41} = -2f + 2(bh - cg) - 2ak,$$
$$u_{12} = -2c + 2(ab + fg) + 2hk,$$
$$u_{22} = 1 + b^2 - c^2 - a^2 + g^2 - h^2 - f^2 + k^2,$$
$$u_{32} = 2a + 2(bc + gh) - 2fk,$$
$$u_{42} = -2g + 2(cf - ah) - 2bk,$$
$$u_{13} = 2b + 2(ca + hf) - 2gk,$$
$$u_{23} = -2a + 2(bc + gh) + 2fk,$$
$$u_{33} = 1 + c^2 - a^2 - b^2 + h^2 - f^2 - g^2 + k^2,$$
$$u_{43} = -2h + 2(ag - bf) - 2ck,$$
$$u_{14} = -2f + 2(cg - bh) - 2ak,$$
$$u_{24} = -2g + 2(ah - cf) - 2bk,$$
$$u_{34} = -2h + 2(bf - ag) - 2ck,$$
$$u_{44} = 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + k^2,$$

where, for brevity, we have written

$$k = af + bg + ch.$$

The group has thus six parameters a, b, c, f, g, h. If we apply the transformation, whose parameters are A, B, C, F, G, H, to that whose parameters are a, b, c, f, g, h, and the resulting parameters are $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}$, we have

$$\begin{aligned}
\mathrm{a} &= (A + a + Cb - Bc + Gh - Hg + Fk + Kf)/U, \\
\mathrm{b} &= (B + b + Ac - Ca + Hf - Fh + Gk + Kg)/U, \\
\mathrm{c} &= (C + c + Ba - Ab + Fg - Gf + Hk + Kh)/U, \\
\mathrm{f} &= (F + f + Cg - Gc + Hb - Bh - Ak - Ka)/U, \\
\mathrm{g} &= (G + g + Ah - Ha + Fc - Cf - Bk - Kb)/U, \\
\mathrm{h} &= (H + h + Bf - Fb + Ga - Ag - Ck - Kc)/U,
\end{aligned}$$

where $U = 1 - Aa - Bb - Cc + Ff + Gg + Hh - Kk,$

and $$K = AF + BG + CH.$$

Further, we find that

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = (K + k + Af + Fa + Bg + Gb + Ch + Hc)/U.$$

From the above formulae we can obtain the equations of each of the parameter groups, having first substituted for K and k their values.

The six operators

$$\begin{aligned}
& y_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_3} + y_6\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_6}, \quad y_1\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_1} + y_4\frac{\partial}{\partial y_6} - y_6\frac{\partial}{\partial y_4}, \\
& y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} + y_5\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_5}, \quad y_3\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_3} - y_2\frac{\partial}{\partial y_6} - y_6\frac{\partial}{\partial y_2}, \\
& y_1\frac{\partial}{\partial y_6} + y_6\frac{\partial}{\partial y_1} - y_3\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_3}, \quad y_2\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_2} - y_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_1},
\end{aligned}$$

define a group having the same structure as the Lorentz group. This group possesses the two invariants

$$y_1y_4 + y_2y_5 + y_3y_6 \text{ and } y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 - y_6^2,$$

and the covariant

$$\begin{aligned}
&(x_1y_1 + x_2y_2 + x_3y_3)^2 - (x_3y_5 - x_2y_6 + x_4y_1)^2 \\
&\qquad - (x_1y_6 - x_3y_4 + x_4y_2)^2 - (x_2y_4 - x_1y_5 + x_4y_3)^2.
\end{aligned}$$

Taking into account the second of the above invariants together with the invariant of the Lorentz group, viz.

$$x_4^2 - x_1^2 - x_2^2 - x_3^2,$$

we deduce another covariant

$$\begin{aligned}
&(x_1y_4 + x_2y_5 + x_3y_6)^2 - (x_2y_3 - x_3y_2 + x_4y_4)^2 \\
&\qquad - (x_3y_1 - x_1y_3 + x_4y_5)^2 - (x_1y_2 - x_2y_1 + x_4y_6)^2
\end{aligned}$$

from the above.

If we take the two expressions

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 - (x_3y_5 - x_2y_6 + x_4y_1)^2 - (x_1y_6 - x_3y_4 + x_4y_2)^2 - (x_2y_4 - x_1y_5 + x_4y_3)^2$$

and

$$(x_1z_1 + x_2z_2 + x_3z_3)^2 - (x_3z_5 - x_2z_6 + x_4z_1)^2 - (x_1z_6 - x_3z_4 + x_4z_2)^2 - (x_2z_4 - x_1z_5 + x_4z_3)^2$$

conjointly; then, in addition to the invariants involving the y's, and those involving the z's, we obtain the two mixed invariants

$$y_1z_4 + y_4z_1 + y_2z_5 + y_5z_2 + y_3z_6 + y_6z_3$$

and

$$y_1z_1 + y_2z_2 + y_3z_3 - y_4z_4 - y_5z_5 - y_6z_6.$$

We have also the covariant

$$(x_1y_1 + x_2y_2 + x_3y_3)(x_1z_1 + x_2z_2 + x_3z_3) - (x_3y_5 - x_2y_6 + x_4y_1)(x_3z_5 - x_2z_6 + x_4z_1) - (x_1y_6 - x_3y_4 + x_4y_2)(x_1z_6 - x_3z_4 + x_4z_2) - (x_2y_4 - x_1y_5 + x_4y_3)(x_2z_4 - x_1z_5 + x_4z_3).$$

The Symmetric Functions of which the General Determinant is a particular case. By Major P. A. MacMahon, Sc.D., F.R.S., St John's College.

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I consider functions of numerical magnitudes a_{st}, where s, t may each assume the values $1, 2, \dots n$.

Symmetric functions are those which are unaltered by the whole of the substitutions of the n integers, when they are applied simultaneously to both suffixes.

I have recently established the theorem that the general n-line determinant

$$| a_{1n} |$$

is a particular case of such a symmetric function*.

In general the symmetric functions of magnitudes $a_{st\dots}$, where the suffix is of multiplicity σ, come under consideration. Here each of the σ integers which constitute the suffix may assume each of the values $1, 2, \dots n$ and the symmetric functions are those which are unaltered by the whole of the substitutions of the n integers when applied simultaneously to the σ integers which compose the suffix.

In this paper I do little more than broach some of the many questions which present themselves. The principal ultimate object is to throw light upon the position which Determinant Theory occupies in the theory of these symmetric functions.

1. An element $a_{st\dots}$ is regarded as being of degree unity and weight σ; a product of θ elements as having a degree θ and weight $\theta\sigma$. A homogeneous and isobaric symmetric function is of degree θ and weight $\theta\sigma$ when the representative product is of degree θ and weight $\theta\sigma$. Taking $\sigma = 2$, we have

1·1. *Degree* 1. Two symmetric functions

$$\sum_1^n a_{ss}, \quad \sum_1^n \sum_1^n a_{st}, \text{ when } s \neq t.$$

We may take as the representative terms

$$a_{11}, \quad a_{12}, \qquad \theta = 1, \quad \sigma = 2,$$

each of degree 1 and weight 2. These terms exhibit, the one, two integers of the same value; the other, two different integers. They are thus associated with the two partitions (2), (1^2) of the number 2 respectively, because 2 is here the weight.

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1·2. *Degree* 2. We find the functions whose representative terms are

$$\begin{array}{ll}
(4) & a_{11}^2 \\
(31) & a_{11}a_{12},\ a_{11}a_{21} \\
(2^2) & a_{11}a_{22},\ a_{12}^2,\ a_{12}a_{21} \\
(21^2) & a_{11}a_{23},\ a_{12}a_{13},\ a_{12}a_{31},\ a_{21}a_{31} \\
(1^4) & a_{12}a_{34}.
\end{array}$$

1·21. For a general value of n, we have n quantities a_{ss} and $2\binom{n}{2}$ quantities a_{st} so that these symmetric functions should account for all combinations two together of n^2 quantities.

These are in number

$$\binom{n^2+1}{2}.$$

As a verification

$$\begin{array}{rcll}
 & a_{11}^2 \text{ accounts for } n \text{ terms} & & \\
a_{12}^2,\ a_{11}a_{12},\ a_{11}a_{21} & ,, & n^2-n & \text{terms each} \\
a_{11}a_{22},\ a_{12}a_{21} & ,, & \tfrac{1}{2}(n^2-n) & ,, \\
a_{11}a_{23},\ a_{12}a_{31} & ,, & n(n-1)(n-2) & ,, \\
a_{12}a_{13},\ a_{21}a_{31} & ,, & \tfrac{1}{2}n(n-1)(n-2) & ,, \\
a_{12}a_{34} & ,, & \tfrac{1}{2}n(n-1)(n-2)(n-3) & ,,
\end{array}$$

and
$$n+3(n^2-n)+2.\tfrac{1}{2}(n^2-n)+2(n^3-3n^2+2n)$$
$$+2.\tfrac{1}{2}(n^3-3n^2+2n)+\tfrac{1}{2}(n^4-6n^3+11n^2-6n)=\tfrac{1}{2}n^2(n^2+1).$$

When $\sigma=1$, the operations of the various substitutions of the numbers $1, 2, \dots n$ are representable as products of circular substitution affecting the singly suffixed elements.

2. We now examine the expression of the substitutions by means of circular substitutions affecting the doubly suffixed elements.

2·1. For $n=2$, we have

$$\begin{array}{ccccc}
\text{Substitutions} & & & & \\
1 & a_{11} & a_{22} & a_{12} & a_{21} \\
(12) & a_{22} & a_{11} & a_{21} & a_{12}.
\end{array}$$

So that
$$(12) \equiv (a_{11}a_{22})(a_{12}a_{21}),$$

viz. the substitution (12) of order 2 is equivalent to the successive performance of two circular substitutions each of which is of the order two. Its type is (2^2).

2·2. For $n = 3$, we have

Substitutions

$$\begin{aligned}
1 \quad &\equiv (a_{11})\,(a_{22})\,(a_{33})\,(a_{23})\,(a_{32})\,(a_{21})\,(a_{13})\,(a_{12})\,(a_{31})\\
(23) \quad &\equiv (a_{22}a_{33})\,(a_{11})\,(a_{23}a_{32})\,(a_{21}a_{31})\,(a_{13}a_{12})\\
(31) \quad &\equiv (a_{33}a_{11})\,(a_{22})\,(a_{21}a_{23})\,(a_{13}a_{31})\,(a_{32}a_{12})\\
(12) \quad &\equiv (a_{11}a_{22})\,(a_{33})\,(a_{13}a_{23})\,(a_{32}a_{31})\,(a_{21}a_{12})\\
(123) \quad &\equiv (a_{11}a_{22}a_{33})\,(a_{23}a_{31}a_{12})\,(a_{32}a_{13}a_{21})\\
(132) \quad &\equiv (a_{11}a_{33}a_{22})\,(a_{23}a_{12}a_{31})\,(a_{32}a_{21}a_{13}).
\end{aligned}$$

The six substitutions yield circular substitutions of types

$$(1^9), \quad (2^4 1), \quad (2^4 1), \quad (2^4 1), \quad (3^3), \quad (3^3)$$

respectively.

2·3. When $n = 4$ we find similarly

from type (1^4) one substitution of type (1^{16})
(21^2) six ,, $(2^6 1^4)$
(2^2) three ,, (2^8)
(31) eight ,, $(3^5 1)$
(4) six ,, (4^4).

2·4. For general values of n and σ we may proceed as follows:

Consider a circular substitution of order c in the n integers $1, 2, \dots n$. There are n^σ elements under transformation. It is easy to see that

$$n^\sigma - (n-c)^\sigma \text{ suffixes of multiplicity } \sigma$$

taken from the integers $1, 2, \dots n$ involve one or more of the integers $1, 2, \dots c$, and it follows that

$$n^\sigma - (n-c)^\sigma \text{ elements}$$

are affected by a circular substitution which affects the whole of the c integers $1, 2, \dots c$.

Moreover $(n-c)^\sigma$

of the elements are unaffected so that the type of the transformed substitution is

$$c^x 1^{(n-c)^\sigma}$$

where $$xc + (n-c)^\sigma = n^\sigma,$$

giving $$x = \frac{1}{c}\{n^\sigma - (n-c)^\sigma\}.$$

The type, we seek, is thus

$$\left(c^{\frac{1}{c}\{n^\sigma - (n-c)^\sigma\}}\, 1^{(n-c)^\sigma}\right).$$

Now, without difficulty, the transform of the type

$$c_1 c_2 \dots c_s\, 1^{n-c_1-c_2-\dots-c_s}$$

is seen to be

$$\Big(c_1^{\frac{1}{c_1}\{(n^\sigma-(n-c_1)^\sigma\}}$$
$$c_2^{\frac{1}{c_2}\{(n-c_1)^\sigma-(n-c_1-c_2)^\tau\}}$$
$$c_3^{\frac{1}{c_3}\{(n-c_1-c_2)^\sigma-(n-c_1-c_2-c_3)^\sigma\}}$$
$$\vdots$$
$$c_s^{\frac{1}{c_s}\{(n-c_1-c_2-\dots-c_{s-1})^\sigma-(n-c_1-c_2-\dots-c_s)^\sigma\}}$$
$$1^{(n-c_1-c_2-\dots-c_s)^\sigma}\Big).$$

2·41. Note that when $\sigma = 1$, the case of ordinary algebra, the expression becomes

$$(c_1 c_2 \dots c_s\, 1^{n-c_1-c_2-\dots-c_s}).$$

When $\sigma = 2$, the case that is particularly before us, we obtain

$$\Big(c_1^{2n-c_1}\, c_2^{2n-2c_1-c_2}\, c_3^{2n-2c_1-2c_2-c_3} \dots$$
$$c_s^{2n-2c_1-2c_2-\dots-2c_{s-1}-c_s}\, 1^{(n-c_1-c_2-\dots-c_s)^2}\Big).$$

2·5. In the general σ type the number of distinct circular orders is

$$\frac{1}{c_1} n^\sigma - \left(\frac{1}{c_1} - \frac{1}{c_2}\right)(n-c_1)^\sigma - \dots$$
$$- \left(\frac{1}{c_{s-1}} - \frac{1}{c_s}\right)(n - c_1 - c_2 - \dots - c_{s-1})^\sigma$$
$$+ \left(1 - \frac{1}{c_s}\right)(n - c_1 - c_2 - \dots - c_s)^\sigma.$$

and the number of substitutions of the type is identical with that for the case $\sigma = 1$, which is well known.

3. In the ordinary theory of symmetric functions, for which $\sigma = 1$, there is no difficulty in specifying and enumerating the different symmetric functions which present themselves of given degree and weight. The degree, as here defined, is in that case equal to the weight and the exponents of the elements in the representative term constitute a partition of the weight. When σ is greater than unity there is still an association with the partitions of unipartite numbers but a group of symmetric functions is now associated with each partition.

3·1. Moreover from each such partition a number of partitions of the multipartite number

$$\overline{(\theta\theta\theta \dots)}$$

(where the integer θ is repeated σ times) can be derived; and a group of symmetric functions can be associated with each of these multipartite partitions.

4. I find that it is convenient, for the object at present in view, to disuse the notation

$$a_{11},\quad a_{12},\quad a_{21},\quad a_{22},\ \dots$$

in favour of

$$\begin{matrix}\alpha & \alpha & \beta & \beta\\ \alpha, & \beta, & \alpha, & \beta, \dots\end{matrix}$$

The annexed Table proceeds as far as the functions of degree 4 and weight 8.

In the left-hand columns are placed the Greek-letter partitions. Thus for the degree 2, weight 4 we list the two representative terms

$$\begin{matrix}\alpha\ \alpha\\ \alpha\cdot\beta,\end{matrix}\quad \begin{matrix}\alpha\ \beta\\ \alpha\cdot\alpha\end{matrix}$$

in connexion with the partition (31) because each of these terms involves α three times and β once.

The symmetric functions are

$$\Sigma\begin{matrix}\alpha\ \alpha\\ \alpha\cdot\beta,\end{matrix}\quad \Sigma\begin{matrix}\alpha\ \beta\\ \alpha\cdot\alpha\end{matrix}$$

the summations being in respect of the first n letters of the Greek alphabet.

4·1. The letter k will be employed to denote the number of *different* Greek letters which appear in a representative term of the function. $k = 2$ in each of the above written functions.

4·2. The n^2 elements

$$\begin{matrix}\alpha & \alpha\\ \alpha, & \beta, \dots\end{matrix}$$

are considered in alphabetical order.

This order, for $n = 4$, is

$$\begin{matrix}\alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \gamma & \gamma & \gamma & \gamma & \delta & \delta & \delta & \delta\\ \alpha, & \beta, & \gamma, & \delta, & \alpha, & \beta, & \gamma, & \delta, & \alpha, & \beta, & \gamma, & \delta, & \alpha, & \beta, & \gamma, & \delta\end{matrix}$$

The factors in a representative term are written in alphabetical order. When we consider the representative terms of a group of symmetric functions, they are arranged in alphabetical order.

Thus

$$\begin{matrix}\alpha & \alpha\\ \alpha & \beta\end{matrix}$$

is placed before

$$\begin{matrix}\alpha & \beta\\ \alpha & \alpha\end{matrix}.$$

4·3. For a weight 2θ, k takes all values from 1 to 2θ and also denotes the number of parts in the corresponding Greek-letter partition.

5. I am not able, at present, to give the generating function which enumerates the symmetric functions for a given value of θ. It is therefore necessary to provide one or more checks upon the tabulations.

The functions of weight 2θ all arise when the power

$$\left(\Sigma \overset{a}{\alpha} + \Sigma \overset{a}{\beta}\right)^{\theta}$$

is expanded. The representative terms of these functions are divisible into three classes:

(i) those which involve factors of type $\overset{a}{\alpha}$ exclusively,

(ii) " " " " $\overset{a}{\beta}$ "

(iii) the remainder.

Those of Class (i) follow exactly the ordinary symmetric functions for which $\sigma = 1$. They are enumerated by the ordinary theory of the partitions of unipartite numbers. Just one corresponds to every partition of the Greek-letters which has even parts exclusively.

5·1. The number of homogeneous products of the elements

$$\overset{a}{\alpha},\ \overset{a}{\beta},\ \ldots$$

taken θ together is

$$\binom{n^2 + \theta - 1}{\theta}.$$

A certain number of the symmetric functions of degree θ and weight 2θ which appertain to the integer k can be each fully expressed by a number of terms which is a numerical sub-multiple of

$$n(n-1)\ldots(n-k+1).$$

Hence if we take all these functions together the number of terms involved in the full expressions is a numerical multiple of

$$n(n-1)\ldots(n-k+1).$$

This numerical multiple must be A_k, where

$$\binom{n^2+\theta-1}{\theta} = A_1 n + A_2 n(n-1) + \ldots + A_{2\theta} n(n-1)\ldots(n-2\theta+1).$$

Thence we derive

$$k!\,A_k = \binom{\theta + k^2 - 1}{\theta} - \binom{k}{1}\binom{\theta+(k-1)^2-1}{\theta} + \binom{k}{2}\binom{\theta+(k-2)^2-1}{\theta} - \ldots + (-)^{k+1}k.$$

This relation has been employed to verify the assignment of the representative terms, of weight 2θ, which involve k letters.

5·11. Ex. gr. To verify the four representative products which appertain to the partition (21^2) and to the value 3 of k

$$\genfrac{}{}{0pt}{}{\alpha\beta}{\alpha\gamma},\quad \genfrac{}{}{0pt}{}{\alpha\alpha}{\beta\gamma},\quad \genfrac{}{}{0pt}{}{\alpha\gamma}{\beta\alpha},\quad \genfrac{}{}{0pt}{}{\alpha\gamma}{\beta\beta}.$$

The functions $\Sigma \genfrac{}{}{0pt}{}{\alpha\beta}{\alpha\gamma},\ \Sigma \genfrac{}{}{0pt}{}{\alpha\alpha}{\beta\gamma},\ \Sigma \genfrac{}{}{0pt}{}{\alpha\gamma}{\beta\alpha},\ \Sigma \genfrac{}{}{0pt}{}{\alpha\gamma}{\beta\beta}$ have, in their full expressions

$$n(n-1)(n-2),\quad \tfrac{1}{2}n(n-1)(n-2),$$
$$n(n-1)(n-2),\quad \tfrac{1}{2}n(n-1)(n-2)$$

terms respectively. The sum of these is

$$3n(n-1)(n-2).$$

Moreover by the formula

$$6A_3 = \binom{10}{2} - 3\binom{5}{2} + 3 = 45 - 30 + 3 = 18$$

a verification.

We conclude that the four representative products have been correctly assigned.

5·2. When we restrict ourselves to the functions of Class (ii) $n(n-1)$ elements are involved, so that we write

$$\binom{n^2 - n + \theta - 1}{\theta} = B_2 n(n-1) + B_3 n(n-1)(n-2) + \dots$$
$$+ B_{2\theta} n(n-1)\dots(n-2\theta+1)$$

leading to

$$k!\,B_k = \binom{\theta + k(k-1) - 1}{\theta} - \binom{k}{1}\binom{\theta + (k-1)(k-2) - 1}{\theta}$$
$$+ \dots + (-)^k \binom{k}{k-2}\binom{\theta+1}{\theta}.$$

5·21. Ex. gr. To verify the six representative products

$$\genfrac{}{}{0pt}{}{\alpha\alpha\alpha}{\beta\beta\gamma},\quad \genfrac{}{}{0pt}{}{\alpha\alpha\beta}{\beta\beta\gamma},\quad \genfrac{}{}{0pt}{}{\alpha\alpha\gamma}{\beta\beta\alpha},\quad \genfrac{}{}{0pt}{}{\alpha\alpha\gamma}{\beta\beta\beta},\quad \genfrac{}{}{0pt}{}{\alpha\beta\alpha}{\beta\alpha\gamma},\quad \genfrac{}{}{0pt}{}{\alpha\beta\gamma}{\beta\alpha\alpha}$$

appertaining to the partition (321), and the two

$$\genfrac{}{}{0pt}{}{\alpha\beta\alpha}{\beta\gamma\gamma},\quad \genfrac{}{}{0pt}{}{\alpha\beta\gamma}{\beta\gamma\alpha}$$

to the partition (2^3) we note that each of the first written seven products connotes $n(n-1)(n-2)$ terms and the last

$$\tfrac{1}{3}n(n-1)(n-2).$$

In the full expression we have $\frac{22}{3}n(n-1)(n-2)$ terms.

Moreover $$6B_3 = \binom{8}{3} - 3\binom{4}{3} = 44$$

a verification.

6. Exceptionally the functions for which $k = 2$ can be readily enumerated.

If θ be of the form $2\theta'$, the whole number of terms which present themselves in the full expressions of the functions for which $k = 2$, or $k = 1$ is for the degree $2\theta'$

$$\binom{2\theta' + 3}{3}$$

and for the degree $2\theta' + 1$, if $\theta = 2\theta' + 1$,

$$\binom{2\theta' + 4}{3}.$$

Moreover, each representative term gives rise, *when* $n = 2$, to two terms inclusive of itself; with the exception that when the degree is $2\theta'$, $\theta' + 1$ of the representative terms (appertaining to the Greek letter partition $2\theta'$, $2\theta'$) give rise each to only one term, viz. the term itself.

Ex. gr. With $n = 2, \theta' = 3$, and partition (66) the representative products

$$\begin{matrix} \alpha\alpha\alpha\beta\beta\beta & \alpha\alpha\alpha\beta\beta\beta & \alpha\alpha\alpha\beta\beta\beta & \alpha\alpha\alpha\beta\beta\beta \\ \alpha\alpha\alpha\beta\beta\beta, & \alpha\alpha\beta\alpha\beta\beta, & \alpha\beta\beta\alpha\alpha\beta, & \beta\beta\beta\alpha\alpha\alpha \end{matrix}$$

$\theta' + 1$ in number are themselves complete symmetric functions. Hence if $f(\theta)$ denotes the number of symmetric functions of degree θ in two Greek letters or fewer

$$2f(2\theta') = \binom{2\theta' + 3}{3} + \theta' + 1 \quad \text{if } \theta = 2\theta',$$

$$2f(2\theta' + 1) = \binom{2\theta' + 4}{3} \qquad \text{if } \theta = 2\theta' + 1.$$

6·1. By the same method the analogous result can be obtained when the multiplicity of the index is denoted by σ.

There are 2^σ elements which involve two Greek letters α, β and $2^{\sigma-1}$ symmetrical combinations

$$\begin{matrix} \alpha & \beta \\ \beta & \alpha \\ \beta & \alpha \\ \alpha & \beta \\ \vdots & \vdots \end{matrix}$$

which are unchanged by the substitution $(\alpha\beta)$.

If we take products of powers of these combinations with exponents $a_1, a_2, \dots a_{2^{\sigma-1}}$. The degree of such products is

$$2(a_1 + a_2 + \dots + a_{2^{\sigma-1}})$$

and the weight $$2\sigma(a_1 + a_2 + \dots + a_{2^{\sigma-1}}).$$

We have therefore to discuss the representations of the weight $\sigma\theta$ in this form so as to disclose the number of the solutions of the Diophantine equation

$$2(a_1 + a_2 + \dots + a_{2^{\sigma-1}}) = \theta,$$

or if $\theta = 2\theta'$ $$a_1 + a_2 + \dots + a_{2^{\sigma-1}} = \theta'.$$

The number of solutions is equal to the number of compositions of the integer θ' into $2^{\sigma-1}$ parts—zero being admissible as a part.

This number is known to be

$$\binom{\theta' + 2^{\sigma-1} - 1}{\theta'}.$$

This number then denotes the number of one-valued representative products which involve two Greek letters α, β.

Now the number of homogeneous products, of degree $2\theta'$, of the 2^σ elements is

$$\binom{2^\sigma + 2\theta' - 1}{2\theta'}.$$

Let the number of representative products of degree $2\theta'$ be denoted by

$$f_\sigma(2\theta').$$

These are composed of

$$\binom{\theta' + 2^{\sigma-1} - 1}{\theta'}$$

products which are one-valued and of

$$f_\sigma(2\theta') - \binom{\theta' + 2^{\sigma-1} - 1}{\theta'}$$

products which are two-valued for the substitution

$$(\alpha\beta).$$

Hence

$$\binom{\theta' + 2^{\sigma-1} - 1}{\theta'} + 2\left\{f_\sigma(2\theta') - \binom{\theta' + 2^{\sigma-1} - 1}{\theta'}\right\} = \binom{2^\sigma + 2\theta' - 1}{2\theta'},$$

or $$2f_\sigma(2\theta') = \binom{2\theta' + 2^\sigma - 1}{2\theta'} + \binom{\theta' + 2^{\sigma-1} - 1}{\theta'}.$$

But if θ be of the form $2\theta' + 1$, every representative product is two-valued, and the number of different representative products is equal to one-half of the corresponding number of homogeneous products.

Hence $$2f_\sigma(2\theta'+1) = \binom{2\theta'+2^\sigma}{2\theta'+1}.$$

6·11. From these general results we verify those already obtained for $\sigma = 2$.

When $\sigma = 1$, the case of ordinary algebra, we find

$$f_1(2\theta') = \theta' + 1,$$
$$f_1(2\theta'+1) = \theta' + 1,$$

which are obviously correct.

The Tabulation of the Symmetric Functions.

7. As an illustration of the plan which has been adopted I will consider in detail the case of weight 8, for which $\theta = 4$, $\sigma = 2$. Only the representative terms of the functions are tabulated.

A representative term is composed of Greek letters $\alpha, \beta, \gamma, \ldots$, these letters occurring $t_1, t_2, t_3, \ldots$ times respectively where

$$t_1 \geqslant t_2 \geqslant t_3 \geqslant \ldots.$$

Then $$(t_1 t_2 t_3 \ldots)$$

is a partition of the unipartite number $\sigma\theta$.

Every such partition presents itself.

The first arrangement of the functions is according to these partitions which arise from the representative terms.

A group of representative terms is associated with each of the partitions.

When $\sigma = 1$, only one representative term is associated with a partition. We are therefore so far in agreement with the usual practice.

When the partition involves k different parts each of which may or may not be repeated the associated representative terms involve k of the Greek letters, repeated or not.

These letters appear in two rows, θ in the upper, θ in the lower row respectively. In the general case θ appear in each of σ rows.

The Greek letters which appear

$$t_1, t_2, \ldots t_k \text{ times respectively}$$

present themselves $$t_{11}, t_{21}, \ldots k_{k1}$$

times in the first or upper row and

$$t_{12}, t_{22}, t_{32}, \ldots t_{k2}$$

times in the second or lower row respectively.

Moreover $(t_{11}t_{12}),\ (t_{21}t_{22}) \dots (t_{k1}t_{k2})$
are compositions of $t_1,\ t_2, \dots t_k$
respectively.

The representative term is thus associated with a two-row matrix

$$\begin{array}{c} t_{11}t_{21} \dots t_{k1} \\ t_{12}t_{22} \dots t_{k2} \end{array}$$

having k columns.

In general the matrix is

$$\begin{array}{c} t_{11}t_{21} \dots t_{k1} \\ t_{12}t_{22} \dots t_{k2} \\ \cdot \quad \cdot \ \dots \ \cdot \\ t_{1\sigma}t_{2\sigma} \dots t_{k\sigma}, \end{array}$$

and has σ rows and k columns.

The weight of the jth column is

$$t_j = t_{j1} + t_{j2},$$

and in general $t_j = t_{j1} + t_{j2} + \dots + t_{j\sigma}$.

All representative terms whose matrices can be made identical by *permutation of columns of equal weight* are regarded as having the *same matrix*.

We thus observe that the representative terms which appertain to a given partition of the unipartite number $\sigma\theta$, are further associated with a number of matrices, which are derived from the given partition.

To further elucidate this matter I will take as a typical case, from the table of weight 8, the ten matrices which are associated with the partition (321^3).

These are (see the Table):

$$\begin{array}{lllll} 30100 & 31000 & 22000 & 21100 & 20110 \\ 02011, & 02111, & 10111, & 11011, & 12001, \\[1ex] 12001 & 11011 & 10111 & 01111 & 02011 \\ 20110, & 21100, & 22000, & 31000, & 30100, \end{array}$$

and it is to be noticed that the 6th, 7th, 8th, 9th and 10th are obtained from the 5th, 4th, 3rd, 2nd and 1st by interchange of rows.

In general the matrices present themselves in conjugate pairs.

Exceptionally a matrix is self-conjugate.

Thus for the partition $(2^2 1^4)$ the matrix

$$\begin{array}{c} 201100 \\ 020011 \end{array}$$

when its rows are interchanged and columns of equal weights permuted (see a previous remark) remains unchanged.

It is thus self-conjugate.

The matrix

$$\begin{matrix}111100\\110011\end{matrix}$$

which appertains to the same partition is also seen to be self-conjugate.

The property of the matrices before us for the partition (321^3) is that

(i) the sums of the column numbers are successively 3, 2, 1, 1,

(ii) the sums of the row numbers are successively 4, 4.

In general

(i) the sums of the column numbers are successively $t_1, t_2, \dots t_k$,

(ii) the sums of the row numbers are successively $\theta, \theta, \dots \sigma$ times.

Matrices of this nature were considered by me in "Combinatory Analysis*."

If we allow permutations between columns, of the matrix, of equal weight the number of matrices is enlarged; thus ex. gr. the matrix

$$\begin{matrix}30100\\02011\end{matrix}$$

gives rise to three matrices

$$\begin{matrix}30100 & 30010 & 30001\\02011 & 02101 & 02110.\end{matrix}$$

There are thus 22 matrices in the enlarged system. The members of this enlarged system were shown to be enumerated by either of the expressions

$$D_{t_1} D_{t_2} \dots D_{t_k} h_\sigma^\theta$$

$$D_\theta^\sigma h_{t_1} h_{t_2} \dots h_{t_k},$$

where the notation is that of the ordinary theory for which $\sigma = 1$.

Thus $h_1, h_2, \dots$ are the successive homogeneous product sums or Aleph functions (of Wronski).

$D_1, D_2, \dots$ are the well-known differential operators of the theory.

In the particular case we verify the enumerating number 22 by

$$\begin{aligned}&D_3 D_2 D_1^2 h_4^2\\= &2D_3 D_2 D_1^2 h_4 h_3\\= &2D_3 D_2 D_1 (h_4 h_2 + h_3^2)\\= &2D_3 D_2 (h_4 h_1 + 3h_3 h_2)\\= &2D_3 (4h_3 + 7h_2 h_1)\\= &22.\end{aligned}$$

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* *Comb. Anal.* vol. 1, p. 234.

This process may be regarded as a verification of the assignment of the matrices which appertain to a given partition. The actual enumeration of the unenlarged system, which has presented itself, can be carried out without difficulty. An inspection of the various matrices exhibited in the tables shows that as regards the whole of a table of weight 2θ the complete number of matrices is equal to the number of the partitions of the multipartite number $(\overline{\theta\theta})$. In the general case the corresponding multipartite number is σ-partite.

For the tables of weights

$$2, 4, 6, 8, \ldots$$

we find the numbers of matrices

$$2, 9, 31, 109, \ldots,$$

while a reference to Combinatory Analysis shows that the bipartite numbers

$$(\overline{11}), (\overline{22}), (\overline{33}), (\overline{44}), \ldots$$

possess $\quad 2, \quad 9, \quad 31, \quad 109, \ldots$ partitions

respectively.

The generating function which enumerates the partitions of bipartite numbers is

$$1/(1-x_1)(1-x_2).(1-x_1^2)(1-x_1x_2)(1-x_2^2)$$
$$.(1-x_1^3)(1-x_1^2x_2)(1-x_1x_2^2)(1-x_2^3)\ldots.$$

We thus establish that there is a one-to-one correspondence between *groups* of symmetric functions of weight 2θ, and the partitions of the multipartite number $(\overline{\theta\theta})$.

We have thus before us the relation that exists between the double-suffix symmetric functions, and those which are bipartite, and we notice the nature of the increased complexity exhibited by the former.

In general there exists a one-to-one correspondence between *groups* of symmetric functions of weight $\sigma\theta$ and the partitions of the σ-partite number

$$(\overline{\theta\theta\theta\ldots}).$$

The generating function which enumerates the groups involves the symbols

$$x_1, x_2, \ldots x_\sigma$$

and is identical with that of the σ-partite theory.

The theory of the multiplicity of the group which corresponds to a given σ-partite partition must be reserved for the present.

The determinant of n-lines is a symmetric function of weight $2n$

and the functions which present themselves are all derived from those which appertain to the Greek-letter partition

$$(2^n).$$

Thus for $n = 3$, the development is

$$\Sigma\,{}^{\alpha}_{\alpha}\cdot{}^{\beta}_{\beta}\cdot{}^{\gamma}_{\gamma} - \Sigma\,{}^{\alpha}_{\alpha}\cdot{}^{\beta}_{\gamma}\cdot{}^{\gamma}_{\beta} + \Sigma\,{}^{\alpha}_{\beta}\cdot{}^{\beta}_{\gamma}\cdot{}^{\gamma}_{\alpha}$$

in three letters and it will be noticed that the functions

$$\Sigma\,{}^{\alpha}_{\alpha}\cdot{}^{\beta}_{\gamma}\cdot{}^{\beta}_{\gamma},\quad \Sigma\,{}^{\alpha}_{\beta}\cdot{}^{\beta}_{\gamma}\cdot{}^{\alpha}_{\gamma},$$

which also belong to the partition (2^3), do not appear in the determinant.

Appertaining to the Greek-letter partition (2^n), n being a repetitional exponent, is a group of multipartite partitions of the bipartite number $\overline{nn}$. Of this group there is one partition which involves units exclusively and is of form $(\overline{11}\ \overline{11}\ \overline{11}\dots)$. The corresponding matrix is

$$\begin{matrix}111\dots\\ 111\dots\end{matrix}$$

The symmetric functions, appertaining to this matrix, are, exclusively, those which present themselves in the n-line two-dimensional determinant.

When $\sigma > 2$ the matrix of σ rows which involves units exclusively and the corresponding multipartite partition of

$$\overline{nn\dots\sigma \text{ times}}$$

whose constituents are units exclusively yield the symmetric functions which present themselves is the n-stratum σ-dimensional determinants.

Such groups of symmetric functions of this nature as are found in the annexed Tables are enclosed in unbroken-line rectangles.

Other groups of symmetric functions which arise from matrices which involve, exclusively, some integer $b > 1$ are those which, in the next place, seem to be worthy of consideration. They are enclosed in broken-line rectangles.

The subject-matter of this communication appears to be an infinite series of Umbral Algebras. When $\sigma > 1$ the umbrae taken singly have no quantitative meaning but when associated in an entity so as to present a column of σ umbrae, such entity is a symbol of quantity*. Umbral Algebras involve much more than Determinant Theories. A continuation of the discussion, here commenced, must be reserved for a future occasion.

* See Sylvester, *Phil. Mag.* 1, 1851, p 295 *et seq.*

Degree 1, *Weight* 2

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions
(2)	1 1	α α
(1^2)	10 01	α β

Degree 2, *Weight* 4

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions
4	2 2	α α α α
31	20 11	α α α β
	11 20	α β α α
2^2	20 02	α α β β
	11 11	α β α β α β β α
21^2	200 011	α α β γ
	110 101	α β α β α γ β γ
	011 200	α γ β β
1^4	1100 0011	α γ β δ

Degree 3, *Weight* 6

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions
6	3 3	α α α α α α
51	30 21	α α α α α β
	21 30	α α β α α α
42	30 12	α α α α β β
	21 21	α α β α α β α α β α β α
	12 30	α β β α α α

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions
Degree 3, *Weight* 6 (*cont.*)		
3^2	30	$\alpha\alpha\alpha$
	03	$\beta\beta\beta$
	21	$\alpha\alpha\beta \quad \alpha\alpha\beta$
	12	$\alpha\beta\beta \quad \beta\beta\alpha$
41^2	300	$\alpha\alpha\alpha$
	111	$\alpha\beta\gamma$
	210	$\alpha\alpha\beta \quad \alpha\alpha\gamma$
	201	$\alpha\alpha\gamma \quad \alpha\beta\alpha$
	111	$\alpha\beta\gamma$
	300	$\alpha\alpha\alpha$
321	300	$\alpha\alpha\alpha$
	021	$\beta\beta\gamma$
	210	$\alpha\alpha\beta \quad \alpha\alpha\gamma \quad \alpha\alpha\beta$
	111	$\alpha\beta\gamma \quad \alpha\beta\gamma \quad \beta\gamma\alpha$
	201	$\alpha\alpha\gamma \quad \alpha\alpha\gamma$
	120	$\alpha\beta\beta \quad \beta\beta\alpha$
	120	$\alpha\beta\beta \quad \alpha\alpha\beta$
	201	$\alpha\alpha\gamma \quad \beta\beta\gamma$
	111	$\alpha\beta\gamma \quad \alpha\beta\gamma \quad \alpha\beta\gamma$
	210	$\alpha\alpha\beta \quad \alpha\alpha\gamma \quad \beta\alpha\beta$
	021	$\alpha\alpha\gamma$
	300	$\beta\beta\beta$
2^3	210	$\alpha\alpha\gamma \quad \alpha\alpha\beta$
	012	$\beta\beta\gamma \quad \beta\gamma\gamma$
	111	$\boxed{\alpha\beta\gamma \quad \alpha\beta\gamma \quad \alpha\beta\gamma}$
	111	$\boxed{\alpha\beta\gamma \quad \alpha\gamma\beta \quad \beta\gamma\alpha}$
31^3	3000	$\alpha\alpha\alpha$
	0111	$\beta\gamma\delta$
	2010	$\alpha\alpha\gamma \quad \alpha\alpha\delta$
	1101	$\alpha\beta\delta \quad \beta\gamma\alpha$
	1101	$\alpha\beta\gamma \quad \alpha\beta\delta$
	2010	$\alpha\alpha\delta \quad \beta\gamma\beta$
	0111	$\alpha\gamma\delta$
	3000	$\beta\beta\beta$
2^21^2	2010	$\alpha\alpha\gamma \quad \alpha\alpha\delta$
	0201	$\beta\beta\delta \quad \beta\gamma\beta$
	2100	$\alpha\beta\beta \quad \alpha\alpha\beta$
	0111	$\alpha\gamma\delta \quad \beta\gamma\delta$
	1110	$\alpha\beta\gamma \quad \alpha\beta\delta \quad \alpha\beta\gamma \quad \alpha\beta\delta$
	1101	$\alpha\beta\delta \quad \alpha\gamma\beta \quad \beta\alpha\delta \quad \beta\gamma\alpha$
	0111	$\alpha\beta\delta \quad \alpha\gamma\delta$
	2100	$\alpha\gamma\gamma \quad \beta\alpha\beta$

Degree 3, *Weight* 6 (*cont.*)

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions		
21^4	21000	$\alpha\alpha\delta$		
	00111	$\beta\gamma\epsilon$		
	11100	$\alpha\beta\delta$	$\alpha\gamma\delta$	
	10011	$\alpha\gamma\epsilon$	$\beta\alpha\epsilon$	
	00111	$\alpha\gamma\delta$		
	21000	$\beta\beta\epsilon$		
1^6	111000	$\alpha\gamma\epsilon$		
	000111	$\beta\delta\zeta$		

Degree 4, *Weight* 8

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions		
8	4	$\alpha\alpha\alpha\alpha$		
	4	$\alpha\alpha\alpha\alpha$		
71	40	$\alpha\alpha\alpha\alpha$		
	31	$\alpha\alpha\alpha\beta$		
	31	$\alpha\alpha\alpha\beta$		
	40	$\alpha\alpha\alpha\alpha$		
62	40	$\alpha\alpha\alpha\alpha$		
	22	$\alpha\alpha\beta\beta$		
	31	$\alpha\alpha\alpha\beta$	$\alpha\alpha\alpha\beta$	
	31	$\alpha\alpha\alpha\beta$	$\alpha\alpha\beta\alpha$	
	22	$\alpha\alpha\beta\beta$		
	40	$\alpha\alpha\alpha\alpha$		
53	40	$\alpha\alpha\alpha\alpha$		
	13	$\alpha\beta\beta\beta$		
	31	$\alpha\alpha\alpha\beta$	$\alpha\alpha\alpha\beta$	
	22	$\alpha\alpha\beta\beta$	$\alpha\beta\beta\alpha$	
	22	$\alpha\alpha\beta\beta$	$\alpha\alpha\beta\beta$	
	31	$\alpha\alpha\alpha\beta$	$\alpha\beta\alpha\alpha$	
	13	$\alpha\beta\beta\beta$		
	40	$\alpha\alpha\alpha\alpha$		
4^2	40	$\alpha\alpha\alpha\alpha$		
	04	$\beta\beta\beta\beta$		
	31	$\alpha\alpha\alpha\beta$	$\alpha\alpha\alpha\beta$	
	13	$\alpha\beta\beta\beta$	$\beta\beta\beta\alpha$	
	22	$\alpha\alpha\beta\beta$	$\alpha\alpha\beta\beta$	$\alpha\alpha\beta\beta$
	22	$\alpha\alpha\beta\beta$	$\alpha\beta\alpha\beta$	$\beta\beta\alpha\alpha$
61^2	400	$\alpha\alpha\alpha\alpha$		
	211	$\alpha\alpha\beta\gamma$		
	310	$\alpha\alpha\alpha\beta$	$\alpha\alpha\alpha\gamma$	
	301	$\alpha\alpha\alpha\gamma$	$\alpha\alpha\beta\alpha$	
	211	$\alpha\alpha\beta\gamma$		
	400	$\alpha\alpha\alpha\alpha$		

Degree 4, *Weight* 8 (*cont.*)

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions				
521	400	αααα				
	121	αββγ				
	310	αααβ	αααγ	αααβ		
	211	ααβγ	ααβγ	αβγα		
	301	αααγ	αααγ			
	220	ααββ	αββα			
	220	ααββ	ααγγ			
	301	αααγ	αβαα			
	211	ααβγ	ααβγ	ααβγ		
	310	αααβ	αααγ	αβαα		
	121	αββγ				
	400	αααα				
431	400	αααα				
	031	βββγ				
	301	αααγ	αααγ			
	130	αβββ	βββα			
	310	αααβ	αααβ	αααβ		
	121	αββγ	αβγβ	ββγα		
	220	ααββ	ααββ	ααγγ	ααββ	
	211	ααβγ	αβαγ	αβαγ	βγαα	
	211	ααβγ	ααβγ	ααβγ	ααβγ	
	220	ααββ	αβαβ	αββα	ββαα	
	121	αββγ	αββγ	ααβγ		
	310	αααβ	ααβα	ββαβ		
	130	αβββ	αααβ			
	301	αααγ	βββγ			
	031	αααγ				
	400	ββββ				
42^2	400	αααα				
	022	ββγγ				
	310	αααγ	αααβ	αααγ		
	112	αββγ	αβγγ	ββγα		
	220	ααββ	ααγγ	ααγγ		
	202	ααγγ	αβαβ	ββαα		
	211	ααβγ	ααβγ	ααβγ	ααβγ	ααβγ
	211	ααβγ	ααγβ	αβαγ	αβγα	βγαα
	112	αββγ	αββγ	ααβγ		
	310	αααγ	ααγα	ββγβ		
	022	ααγγ				
	400	ββββ				

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions						
Degree 4, *Weight* 8 (*cont.*)								
3^22	310	αβββ	αααβ					
	022	ααγγ	ββγγ					
	220	ααββ	ααγγ	ααββ				
	112	αβγγ	αββγ	βγαγ				
	301	αααγ	αααγ					
	031	βββγ	ββγβ					
	211	ααβγ	ααβγ	ααβγ	αββγ	αββγ	ααβγ	ααβγ
	121	αββγ	αβγβ	αβγγ	ααγβ	αγγβ	ββγα	βγαβ
	112	αββγ	αβγγ	αβγγ				
	220	ααγγ	ααββ	βααβ				
	022	ααγγ	ααγγ					
	310	αβββ	ββαβ					
51^3	4000	αααα						
	1111	αβγδ						
	3100	αααγ	αααδ					
	2011	ααβδ	αβγα					
	2011	ααβγ	ααγδ					
	3100	αααδ	αβαα					
	1111	αβγδ						
	4000	αααα						
421^2	4000	αααα						
	0211	ββγδ						
	3100	αααβ	αααδ	αααβ				
	1111	αβγδ	αβγδ	βγδα				
	3010	αααγ	αααδ	αααδ				
	1201	αββδ	αβγβ	ββγα				
	2200	ααββ	ααγγ	ααββ				
	2011	ααγδ	αβαδ	ββγδ				
	2110	ααβγ	ααβδ	ααβγ	ααβδ	ααγδ	ααγδ	ααβδ
	2101	ααβδ	ααγβ	αβαδ	αβγα	αβαδ	αβδα	βγαα
	2011	ααβδ	ααγδ	ααγδ				
	2200	ααγγ	αβαβ	ββαα				
	1201	αββγ	αββδ	ααβδ				
	3010	αααδ	ααγα	ββγβ				
	1111	αβγδ	αβγδ	αβγδ				
	3100	αααβ	ααγα	βααα				
	0211	ααγδ						
	4000	ββββ						

Degree 4, *Weight* 8 (*cont.*)

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions						
3^21^2	3010	αααγ	αααδ					
	0301	βββδ	ββγβ					
	3100	αβββ	αααβ					
	0211	ααγδ	ββγδ					
	2200	ααββ	ααγγ	ααββ				
	1111	αβγδ	αβγδ	βγαδ				
	2110	ααβγ	ααβδ	ααγδ	αββδ	ααβγ	ααβδ	ααβδ
	1201	αββδ	αβγβ	αβδδ	ααγβ	ββαδ	ββγα	βγαα
	1111	αβγδ	αβγδ	αβγδ				
	2200	ααββ	ααγγ	βααβ				
	0211	ααγδ	ααγδ					
	3100	αβββ	βββα					
32^21	3001	αααδ						
	0220	ββγγ						
	3100	αβββ	αααγ	αααβ				
	0121	αγγδ	ββγδ	βγδγ				
	2200	ααγγ	ααγγ	ααγγ	ααδδ			
	1021	αββδ	αβδδ	ββαδ	βγαβ			
	2110	ααβγ	ααβδ	ααγδ	ααγδ	αββγ	ααβγ	ααβδ
	1111	αβγδ	αβγδ	αβγδ	αβδγ	αγδβ	βγαβ	βγδα
	2101	ααβδ	ααγδ	ααγδ	αββδ	ααγδ	ααβδ	ααβδ
	1120	αβγγ	αββγ	αββδ	αγγβ	ββαγ	βγαγ	βγγα
	1120	αββγ	αββδ	αβγγ	αββγ	ααβγ	ααβγ	ααβδ
	2101	ααγδ	ααγδ	ααβδ	αγγδ	ββγδ	βγγδ	βγδβ
	1111	αβγδ	αβγδ	αβγδ	αβγδ	αβγδ	αβγδ	αβγδ
	2110	ααβγ	ααβδ	αβγα	αγβα	αγβγ	βαβγ	βγαβ
	1021	αββδ	αβγγ	ααβδ	ααβδ			
	2200	ααγγ	ααδδ	ββγγ	βγγβ			
	0121	αββδ	αγγδ	αγγδ				
	3100	αγγγ	βαββ	βββα				
	0220	ααγγ						
	3001	βββδ						
2^4	2200	ααγγ	ααδδ					
	0022	ββδδ	βγβγ					
	2110	αβγγ	αββγ	ααγδ	ααβγ	ααβδ		
	0112	αβδδ	αγδδ	ββδγ	βγδδ	βγδγ		
	1111	αβγδ	αβγδ	αβγδ	αβγδ	αβγδ		
	1111	αβγδ	αβδγ	αγδβ	βαδγ	βγδα		

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions						
Degree 4, Weight 8 (cont.)								
41^4	40000	αααα						
	01111	βγδε						
	31000	αααδ	αααε					
	10111	αβγε	βγδα					
	21100	ααβδ	ααγδ	ααδε				
	20011	ααγε	αβαε	βγαα				
	10111	αβγδ	αγδε					
	31000	αααε	βααα					
	01111	αγδε						
	40000	ββββ						
321^3	30100	αααδ	αααε					
	02011	ββγε	βγδβ					
	31000	αααβ	αααε					
	01111	βγδε	βγδε					
	22000	ααγγ	ααδδ					
	10111	αβδε	βγαε					
	21100	ααβδ	ααγε	αββε	ααβδ	ααβε	ααδε	
	11011	αβγε	αβδε	αγδβ	βγαε	βγδα	βγαδ	
	20110	ααγδ	ααγε	ααγδ	ααδε			
	12001	αββε	αβδδ	ββαε	βγαβ			
	12001	αββδ	αβδδ	ααβδ	ααβε			
	20110	ααγε	ααγε	ββγε	βγδβ			
	11011	αβγε	αβγδ	αβδε	αβγδ	αβδε	αβδε	
	21100	ααδβ	αβαε	αγββ	βααε	βγαα	βγαβ	
	10111	αβδε	αγδε					
	22000	ααγγ	βααβ					
	01111	αβδε	αγδε					
	31000	αγγγ	βαββ					
	02011	ααγδ	ααδε					
	30100	βββε	βγββ					
2^31^2	22000	ααγγ	ααδδ					
	00211	ββδε	βγβε					
	21100	αβγγ	αββγ	ααβγ	ααβδ	ααδε		
	01111	αβδε	αγδε	βγδε	βγδε	βγεδ		
	21010	αββδ	αββε	ααγε	ααβδ	ααβε	ααβε	ααδε
	01201	αγγε	αγδγ	ββδγ	βγγε	βγδγ	βγδδ	βγβδ
	11110	αβγδ	αβγε	αβγδ	αβγε	αβγε	αβγδ	αβγε
	11101	αβγε	αβδγ	αγβε	αγδβ	βαδγ	βγαε	βγδα
	01111	αβγε	αβδε	αγδε	αγδε	αγδε		
	21100	αβδδ	αγγβ	βαβγ	βαβδ	ββεδ		
	00211	ααγε	ααγε					
	22000	ββδδ	βγδβ					

Degree 4, Weight 8 (cont.)

Greek-letter Partition	Matrix and Multipartite Partition	Representative Terms of Symmetric Functions				
31^5	310000 001111	αααε βγδζ				
	211000 100111	ααγε αβδζ	ααδε βγαζ			
	100111 211000	αβγε ααδζ	αγδε βααζ			
	001111 310000	αγδε βββζ				
2^21^4	220000 001111	ααδδ βγεζ				
	211000 010111	αββε αγδζ	ααβε βγδζ	ααδζ βγεδ		
	201100 020011	ααγε ββδζ	ααδε βγβζ	ααδε βγζζ		
	111100 110011	αβγε αβδζ	αβδε αγβζ	αβγε βαδζ	αβδε βγαζ	αγεζ βδαγ
	010111 211000	αβδε αγγζ	αγδε ββαζ	αγδζ ββεδ		
	001111 220000	αγδζ ββεε				
21^6	2110000 0001111	ααδζ βγεη				
	1111000 1000111	αβδζ αγεη	αγδζ βαεη			
	0001111 2110000	αγδζ ββεη				
1^8	11110000 00001111	αγεη βδζθ				

In row four of the Greek-letter partition (321^3) (derived from $\alpha\alpha\alpha\beta\beta\gamma\delta\epsilon$) the matrix $\begin{matrix}21100\\11011\end{matrix}$ also connotes the multipartite partition $(\overline{21}\ \overline{11}\ \overline{10}\ \overline{01}\ \overline{01})$ of the multipartite number $\overline{44}$.

In the same row are the representative products of the associated symmetric functions.

In each of these products the four factors are in alphabetical order.

The representative products are in alphabetical order amongst themselves.

On simplexes doubly incident with a quadric. By R. VAIDYANATHASWAMY. (Communicated by Professor H. F. BAKER.)

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Definitions.

(1) The simplex in a space S_n of order n (or $n-1$ dimensions) is a set of n points $a_1 a_2 \dots a_n$ which do not all lie in a subspace of S_n.

The *elements* of the simplex $a_1 a_2 \dots a_n$ are the n points $a_1 a_2 \dots a_n$ which are elements of order one, the $\frac{n(n-1)}{2}$ lines $a_r a_s$ which are elements or faces of order two, etc., etc., and finally the n *primes*[*] $a_1 a_2 \dots a_{r-1} a_{r+1} \dots a_n$ which are elements or faces of order $n-1$.

A quadric Q in S_n is the kernel of a polarity $\{Q\}$ which associates with each region S_r its polar region S_{n-r}. Two regions S_r, S_r' of order r are said to be mutually *conjugate* in regard to the polarity $\{Q\}$, if either (and therefore each) meets the polar region of the other in respect of $\{Q\}$. The *self-conjugate regions*, namely, the regions which meet their own polar regions in regard to $\{Q\}$, are those which are *tangent* to the quadric Q in the usual sense.

The quadric Q may be considered as the totality of the self-conjugate *points* of $\{Q\}$, namely, the points which lie in their polar primes, and more generally as the totality of the self-conjugate S_r's of $\{Q\}$ $(r = 1, 2, \dots n-1)$.

A region S_r may be said to be *incident* with a quadric Q, if S_r is an element of Q, that is, if S_r is tangent to Q, or is conjugate to itself in the polarity $\{Q\}$.

A simplex will be said to have the incidence $\{r\}$ with Q, if every element (face) of order r of the simplex is incident with Q. If the simplex has at the same time the incidences $\{r\}$, $\{s\}$, $\{t\}$, ... with Q, ($r, s, t, \dots$ being unequal), it may be said to have *the multiple incidence* $\{r, s, t, \dots\}$ with it. For instance, a simplex which is both inscribed and circumscribed to a quadric Q in S_n is one which has the double incidence $\{1, n-1\}$ with it.

In what follows we concern ourselves, as a rule, only with non-singular quadrics.

Some types of doubly incident simplexes.

(2) The quadric in S_2—the space of one dimension—is a pair of points; the simplex in S_2 is also a pair of points. Also, in the

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[*] The word 'prime' (due to Prof. H. F. Baker) is a convenient one for denoting a flat space of order $n-1$ situated in a flat space of order n. Flat spaces of order $< n-1$ may be called generally *regions* or *spaces*.

binary domain S_2 the principle of duality loses its significance, the point being its own dual element. Hence the simplex a_1a_2 should be considered as not only inscribed to the quadric a_1a_2, but also as circumscribed to it. Thus the quadric in S_2 admits an in-circum-simplex. It should be said, however, that this simplex has only single (not double) incidence with the quadric.

(3) There do not exist in-circum-simplexes of quadrics in S_3; that is, there do not exist triangles which are both inscribed and circumscribed to a conic. To show this, consider generally a simplex having $\{1, 2\}$ incidence with a quadric Q in S_n. Choosing it as the simplex of reference, Q will have an equation of the form

$$\sum_{i,j} a_{ij}x_ix_j = 0; \quad i, j = 1, 2, \ldots n;$$

the conditions of $\{1, 2\}$ incidence give immediately

$$a_{ii} = 0; \quad \begin{vmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk} \end{vmatrix} = 0; \quad i, j, k = 1, 2, \ldots n.$$

These equations necessitate the identical vanishing of Q, demonstrating thereby the non-existence of a simplex with $\{1, 2\}$ incidence with Q.

We may prove a slightly more general result:

There do not exist simplexes having consecutive double incidence $\{r, r+1\}$ *with a quadric* Q*.

For, denote the containing space by S_n. Supposing that there exists a simplex having the double incidence $\{r, r+1\}$ with Q, let S_r be any one of its faces of order r, and S_{r+1} any face of order $r+1$ containing S_r. Then, by hypothesis, S_r, S_{r+1} both touch Q; let the points of contact be p, q respectively. S_{r+1} being tangent to the quadric at q is contained in the tangent prime at q; hence the tangent prime at q contains S_r and therefore the point p. p and q are thus conjugate points, so that the tangent prime at p (which contains S_r) passes through q and therefore contains S_{r+1}, if q does not lie in S_r. If every S_{r+1}-face through S_r has its point of contact q outside S_r, it will follow that the whole simplex is contained in the tangent prime at p. Hence there is at least one S_{r+1} through S_r, having its point of contact q on S_r, different from p (for, if q were identical with p, the S_{r+1} will still be contained in the tangent prime at p). This implies that S_r touches the quadric along the line pq, so that every face of order r of the simplex touches the quadric along a space of at least one dimension. Now any face of order $r-1$ will necessarily cut one of the lines of the type pq, and will therefore touch the quadric. Hence $\{r+1, r\}$ incidence implies the incidence $\{r-1\}$, and therefore every incidence lower than $\{r\}$.

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* The corresponding algebraic proposition may be found in Cullis, *Matrices and Determinoids*, Camb. Univ. Press, vol. 1.

Thus the impossibility of $\{r+1, r\}$ incidence reduces to the impossibility already proved of $\{1, 2\}$ incidence.

(4) A quadric Q in S_4 possesses in-circum-simplexes; the obvious ones are the ∞^2 tetrahedra $ABCD$, four of whose edges are generating lines of Q. These tetrahedra $ABCD$ are all those in reference to which the quadric Q reduces to the form $X_1X_2 + X_3X_4$.

It is, however, easy to see that Q possesses in-circum-tetrahedra which are not of this type. To obtain these, consider generally a simplex having $\{1, 3\}$ incidence with a quadric Q in S_m; choosing it as the simplex of reference, the equation to Q is of the form

$$\sum_{i,j} a_{ij}x_ix_j = 0; \quad i, j = 1, 2, \ldots m.$$

The conditions of $\{1, 3\}$ incidence are

$$a_{ii} = 0; \quad i = 1, 2, \ldots m,$$

$$\begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} = 2a_{ij}a_{jk}a_{ik} = 0; \quad i, j, k = 1, 2, \ldots m.$$

We have now to solve these equations in accordance with the condition that Q is non-singular; that is to say, has a non-vanishing determinant.

We shall suppose that none of the quantities a_{ij} vanishes by mere accident; that is, otherwise than in so far as it is forced to vanish by the above equations, or by the exercise of any legitimate choice in their solution. Let us now take a particular suffix i, and in reference to it let us divide all suffixes j ($j = 1, 2, \ldots m$) into two classes, according as a_{ij} does or does not vanish. The suffix i itself clearly falls into the first of these classes; we may accordingly speak of these classes as *the proper class of* i, and *the opposite class of* i, respectively. Supposing that j, k belong to the opposite class of i, it follows from

$$a_{ij} \neq 0, \quad a_{ik} \neq 0, \quad a_{ij}a_{jk}a_{ik} = 0,$$

that $a_{jk} = 0$. Thus any two members of the opposite class of i belong to the proper classes of each other. Again, if l and j belong respectively to the proper class and opposite class of i, then none of our equations necessitate that a_{lj} should vanish. Accordingly a_{lj} does not vanish, and therefore l and j belong to the opposite classes of each other. Combining this with the previous result, it follows, first, that the opposite class of i is the proper class of any one of its members, and secondly, that the opposite class of any member of the opposite class of i is the proper class of i. By a second application of these results it follows finally, that the proper class and opposite class of any suffix i are equally the proper class and opposite class of any member of the first, as well as the opposite class and proper class of any member of the second. The division into classes is therefore independent of the particular suffix i, in reference to which it was made; in other words, the m suffixes fall into two classes—containing, say, n and $m - n$ suffixes—such that a_{ij} does or does not vanish, according as i and j belong to the same, or to different classes.

From an inspection of the determinant of the quadric, it is now easy to see that it will vanish unless the two classes are equinumerous. We may also show this geometrically by remarking, that the two groups of reference points corresponding to the two classes of suffixes determine two generating regions, of orders n and $m - n$, of the quadric Q. If n is not equal to $m - n$, the greater of these numbers will be greater than $\frac{m}{2}$, and therefore Q will possess a generating region of order greater than $\frac{m}{2}$. But the maximum order of a generating region of a non-singular quadric in S_m is the greatest integer in $\frac{m}{2}$, and is therefore *not* greater than $\frac{m}{2}$. This shows that Q is singular unless $n = m - n$, or $m = 2n$. We have therefore the theorem:

Simplexes with the double incidence $\{1, 3\}$ *with a quadric* Q *exist only when the containing space* S_{2n} *is of even order; the most general simplex of this kind is constituted by two sets of* n *points chosen arbitrarily on two non-intersecting generating* S_n*'s of* Q.

In particular the most general in-circum-tetrahedra of a quadric Q in S_4 are constituted by two pairs of points chosen arbitrarily on two non-intersecting generators of Q. It may be noticed that in-circum-scription of a quadric to a tetrahedron in S_4 amounts not to $4 + 4 = 8$, but only to six conditions on the quadric, or on the tetrahedron.

(5) It is necessary to observe that the simplexes constructed, as above, to have the incidence $\{1, 3\}$ with a quadric Q in S_{2n} have, as a matter of fact, every odd incidence $\{1\}, \{3\}, \{5\}, \ldots \{2n - 1\}$ with it. To prove this, consider any face of odd order μ of a simplex having $\{1, 3\}$ incidence with Q. The $2n$ points of the simplex fall, as has been shown, into two groups of n, lying respectively on two non-intersecting generating S_n's of Q; the μ points of the face of order μ will thus consist (say) of λ and $\mu - \lambda$ points from the two groups of n points ($\lambda = 0, 1, 2, \ldots \mu$). Hence the section of Q by the face in question is a quadric Q' which contains two generating regions of orders λ and $\mu - \lambda$. Now, μ being odd, λ can never be equal to $\mu - \lambda$; hence, by repeating the argument used in (4), it follows that Q' is singular, so that any face of the simplex of any odd order μ touches Q. The result may be stated thus:

{1, 3} *incidence of a simplex with a quadric in S_{2n} implies complete odd incidence.*

A very special class of simplexes with {1, 3} (and therefore, complete odd) incidence should now be mentioned. These are the simplexes in reference to which the quadric Q in S_{2n} reduces to the form

$$X_1X_2 + X_3X_4 + \dots + X_{2n-1}X_{2n} = 0.$$

These simplexes have the property that their $2n$ points can be divided, in 2^{n-1} ways, into two sets of n points lying respectively on two generating S_n's of Q. We may see directly that a simplex of this kind has complete odd incidence with Q. Consider any face of order μ of the simplex; to form the equation of its intersection with Q, we put all the $2n-\mu$ coordinate-variables not pertaining to the face equal to zero in the equation of Q. But, from the form of the equation of Q, it is clear that, when we have done this, the surviving equation will always contain an *even* number of coordinate-variables. Hence, if μ is odd, one at least of the coordinate-variables pertaining to the face must disappear in the equation of the intersection, which shows that the intersection is a singular quadric. Thus every face of odd order touches Q.

(6) A quadric Ω in S_6 admits in-circum-simplexes; in fact, if we establish a correspondence between the points of Ω and the lines of S_4, then the property of the double-six implies, as has been shown by Dr Richmond*, the existence of mutually polar pairs of in-circum-simplexes of Ω. Further, the fact that the double-six admits an automorphic quadric polarity shows that any in-circum-simplex determines a pair of polar planes of Ω, the involutoric collineation defined by which transforms the simplex into its polar in-circum-simplex. A similar property also holds in regard to in-circum-tetrahedra of a quadric Q in S_4; there is however this difference, that there is now an infinite number of polar line-pairs of Q, the involutoric collineations defined by which carry a given in-circum-tetrahedron of Q into its polar in-circum-tetrahedron.

Among the in-circum-simplexes of Ω are included the special simplexes in reference to which Ω takes the form

$$X_1X_2 + X_3X_4 + X_5X_6;$$

these latter in fact correspond to the degenerate line-sixers consisting of the six edges of a tetrahedron.

It has been shown by Dr Richmond (*l.c.*) that in-circumscription of a quadric to a simplex in S_6 amounts not to $6 + 6 = 12$, but only to eleven conditions.

(7) A quadric L in S_5 possesses in-circum-simplexes; we prove this by availing ourselves of the correspondence between the points

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* *Proc. Camb. Phil. Soc.*, vol. 14, p. 475 (1908).

of L and the lines of a non-singular linear complex L in S_4. An in-circum-simplex $a_1 a_2 a_3 a_4 a_5$ of L will then correspond to five lines $a_1 a_2 a_3 a_4 a_5$ of L, not all belonging to a congruence, with the property that any four of the lines have a double transversal. Calling these double transversals $b_1 b_2 b_3 b_4 b_5$ (b_r being the double transversal of the four lines a other than a_r), it is clear that the simplex $b_1 b_2 b_3 b_4 b_5$ is the polar simplex of $a_1 a_2 a_3 a_4 a_5$ in regard to L, and is therefore equally an in-circum-simplex of L. Thus the two sets of lines

$$a_1 a_2 a_3 a_4 a_5,$$
$$b_1 b_2 b_3 b_4 b_5$$

have the same character, and are symmetrically related, each set determining the other. The two sets, which may be said to form *a double-pentad of lines*, have the property that any a_r is the only transversal of the four lines b other than b_r, and a like property for the b's.

If we start with four non-intersecting lines $a_1 a_2 a_3 a_4$ of L with a double transversal b_5, it will not in general be possible to find a line a_5 such that each of the sets

$$(a_2 a_3 a_4 a_5), \quad (a_1 a_3 a_4 a_5), \quad (a_1 a_2 a_4 a_5), \quad (a_1 a_2 a_3 a_5)$$

has a double transversal. For, a line a_5 of this character is a common tangent of the four quadrics each containing three of the lines $a_1 a_2 a_3 a_4$. These four quadrics do not in general have a common tangent; but, if they have one common tangent, they have an infinite number which belong to a quadric regulus, the complementary regulus of which is contained in the congruence determined by $a_1 a_2 a_3 a_4$. Thus, when it is possible to find lines a_5, it is possible to find only one which belongs to any assigned complex L containing $a_1 a_2 a_3 a_4$. Further, it is known* that the four quadrics have a common tangent only when the four lines $a_1 a_2 a_3 a_4$ touch one, and therefore an infinite number of cubic curves. It can be shown that, for the particular case in which $a_1 a_2 a_3 a_4$ possess a double transversal b_5, the condition that they touch a cubic curve is that they cut b_5 in an equi-anharmonic range†.

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* Baker, *Principles of Geometry*, vol. 3, pp. 144, 145.

† If by 'line' we mean a line of the complex L, and if we agree to consider for a moment all lines not belonging to L as non-existent, we can throw the theorem of the double-pentad in the following form, which is a fairly accurate caricature of the theory of the double-six:

Three lines have in general two common transversals; four lines have in general none. Consider four lines $a_1 a_2 a_3 a_4$ which have a transversal b_5; every three of the lines have then a second transversal. Calling these second transversals $b_1 b_2 b_3 b_4$, the theorem of the double-pentad states that the four lines $b_1 b_2 b_3 b_4$ have themselves a common transversal a_5 (provided $a_1 a_2 a_3 a_4$ cut out an equi-anharmonic range on b_5). The two sets of five lines thus obtained are then symmetrically related, the lines $a_r b_s$ meeting each other if $r \neq s$, and each set being such, that any four of its five lines have a common transversal which belongs to the other set.

These considerations practically prove that, if $a_1a_2a_3a_4$ be an equi-anharmonic quadruple of tangents of a twisted cubic C which belongs to a complex L, there exists a unique fifth line a_5 of L, which forms with $a_1a_2a_3a_4$ a pentad of the desired kind. We may also prove this directly by a binary-geometric method depending on two lemmas.

Lemma I. If a tetrad of tangents to a cubic curve is a self-apolar (or equi-anharmonic) tetrad, then it is well known that the transversal-pair of the tetrad coalesces into a line of the tangent linear complex L of the curve. Our first lemma is, that if two self-apolar tetrads of tangents of the cubic curve have the respective transversals l_1, l_2, then the intersection of l_1, l_2 is the condition for the mutual apolarity of the two tetrads. This may be deduced as a special case from the theorem on page 77 of Meyer, *Apolarität und rationale Kurven* (1883); it may be also obtained as a direct consequence of Theorem (C), "The Null Pencil of Binary Quartics," *Proc. London Math. Soc.*, Ser. 2, vol. 23, p. 319.

Lemma II. The condition of self-apolarity of a tetrad is of the second degree. Hence, if $A_1A_2A_3$ is a given triad, there are two points δ which make $A_1A_2A_3\delta$ a self-apolar tetrad. These two points must be a covariant pair of $A_1A_2A_3$, and hence its Hessian pair A_4B_4 (say). Further, the symmetry of the self-apolar condition shows that a self-apolar tetrad $A_1A_2A_3A_4$ must have the property that each of its points must be a Hessian point of the other three. Let the Hessian points of

$$(A_2A_3A_4),\quad (A_1A_3A_4),\quad (A_1A_2A_4),\quad (A_1A_2A_3)$$

other than A_1, A_2, A_3, A_4 be B_1, B_2, B_3, B_4 respectively. Then $B_1B_2B_3B_4$ is a covariant tetrad of $A_1A_2A_3A_4$, and is therefore a member of its syzygetic pencil. Also from the known property of the Hessian pair of a triad, the homography of period three, whose fixed points are A_4B_4, carries the triad $A_1A_2A_3$ into itself. Hence, since $B_1B_2B_3B_4$ is a covariant tetrad of $A_1A_2A_3A_4$, it must carry the triad $B_1B_2B_3$ into itself as well. Thus A_4B_4 are also the Hessian points of $B_1B_2B_3$. This shows that $B_1B_2B_3B_4$ is also a self-apolar tetrad, and it clearly follows that the tetrads $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ are related in mutual symmetry. They are in fact the two self-apolar tetrads of a syzygetic pencil, and may be called *the covariant self-apolar tetrads* of each other.

Our second lemma is the following property of these tetrads:

The self-apolar tetrad $B_1B_2B_3B_4$ is apolar to each of the four self-apolar tetrads of the type $A_1A_2A_3B_4$.

To prove this, we have only to show that the triad $A_1A_2A_3$ is apolar to the first polar triad of B_4 in regard to $B_1B_2B_3B_4$. Since B_4 is a point of the tetrad $B_1B_2B_3B_4$, its first polar triad in regard to it will consist of B_4 itself and the first polar pair of B_4 in regard to

$B_1B_2B_3$. But, since it has been shown that the Hessian pair of $B_1B_2B_3$ is A_4B_4, this first polar pair is simply (A_4A_4). Thus the first polar triad of B_4 in regard to $B_1B_2B_3B_4$ is $(B_4A_4A_4)$; and this is obviously apolar to $A_1A_2A_3$ since A_4B_4 is the Hessian pair of $A_1A_2A_3$. The lemma is therefore proved.

Let now $a_1a_2a_3a_4$ be four lines of a complex L, possessing a double transversal b_5, and capable of touching one, and therefore ∞^1 cubic curves, among which there would be only one—say, C—belonging to the complex L. The lines $a_1a_2a_3a_4$ form then a self-apolar tetrad on C, and their double transversal b_5 is necessarily a line of L. Let $c_1c_2c_3c_4$ be the covariant self-apolar tetrad of tangents of C. The lines of L other than b_5 which meet threes of the four lines $a_1a_2a_3a_4$, are clearly the double transversals b_1, b_2, b_3, b_4 of the four self-apolar tangent-tetrads of C, of the type $a_2a_3a_4c_1$. These four self-apolar tetrads are, by Lemma II, apolar to the self-apolar tetrad $c_1c_2c_3c_4$; hence by our first lemma, $b_1b_2b_3b_4$ meet the double transversal a_5 of $c_1c_2c_3c_4$. Since all the lines concerned belong to L, a_5 must be the double transversal of $b_1b_2b_3b_4$, which proves our theorem*.

The self-polar simplex.

(8) The special simplexes in respect of which a quadric Q in S_{2n} reduces to the form $X_1X_2 + X_3X_4 + \dots + X_{2n-1}X_{2n}$ have been frequently mentioned; their true character will appear on an analysis of the concept of 'self-polar simplex.'

A simplex $\delta^{(m)} = \delta_1\delta_2 \dots \delta_m$ in S_m is usually considered to be a self-polar simplex of a quadric Q in S_m, if the polar prime of any point δ_r in regard to Q is the opposite prime $\delta_1\delta_2 \dots \delta_{r-1}\delta_{r+1} \dots \delta_m$. We shall however give a more general definition of the self-polar simplex.

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* Two lines like a_5, b_5, which are the transversals of covariant self-apolar tetrads of tangents of a cubic curve, have some interesting properties. They are conjugate to each other in regard to any quadric circumscribed or inscribed to the cubic curve. From the binary point of view, these lines determine on the cubic curve two null pencils of triads (meaning by a null pencil, one which has any two of its members apolar), which may be called *associated null pencils*. The characteristic property of two associated null pencils Γ_1, Γ_2 is that, if abc, $ab'c'$ be members of Γ_1, Γ_2 respectively, the pairs bc, $b'c'$ separate each other harmonically. The converse theorem, that two pencils of cubics with this property are necessarily associated null pencils, is true with certain trivial exceptions. An interesting application of the idea of associated null pencils to the rational space-quartic curve C_4 may be mentioned. The curve C_4 possesses an infinity of tri-secants which lie on the unique quadric containing it. The pencil of collinear triads on C_4 is a null pencil, if its super-osculation points are coplanar. In this case we can accordingly seek for the associated null pencil, using the characteristic property stated. We easily find that the associated null pencil must consist of triads of points on the curve, the tangents at which are concurrent; we can show that the locus of the meets of the concurrent tangents is a conic through the super-osculation points. The existence of triads of concurrent tangents of a rational quartic with co-planar super-osculation points has been known for a long time (see R. A. Roberts, "The Unicursal Twisted Quartic," *Proc. London Math. Soc.*, vol. 14, 1st series).

We shall say that $\delta^{(m)}$ is *a self-polar simplex of Q, if it is its own polar simplex in regard to Q.* It follows from this definition that the polar prime of any point δ_r is some prime of the simplex—say, the prime opposite to δ_s; thus, for every point δ_r there is a point δ_s, such that δ_s is the only point of the simplex not conjugate to δ_r. Clearly this relation between δ_r, δ_s is symmetrical; we shall refer to them as *associated points of the simplex.* So far it has been left open, whether δ_r and δ_s are identical or distinct. If every point δ is associated with itself, then the polar prime of any δ is its opposite prime, and we have the usual type of self-polar simplex. We shall say in this case, that $\delta^{(m)}$ is *a properly self-polar simplex.*

If, on the contrary, no point of the simplex is associated with itself, then we shall say that $\delta^{(m)}$ is *an improperly self-polar simplex.* In this case the m points of the simplex fall into associated pairs, so that m must be even. Also since no point of the simplex is associated with itself, every point δ must be conjugate to itself and therefore lies on the quadric. Thus a synonym of 'improperly self-polar simplex' is 'inscribed self-polar simplex.'

In the intermediate case, in which $\delta^{(m)}$ contains points which are, as well as points which are not, their own associates, we may say that $\delta^{(m)}$ is *a mixed self-polar simplex.* A mixed self-polar simplex can clearly be resolved into its improper and proper components, $\delta^{(2\nu)}$ and $\delta^{(m-2\nu)}$ respectively, where $\delta^{(2\nu)}$ consists of ν pairs of distinct associated points, and $\delta^{(m-2\nu)}$ of the remaining $(m - 2\nu)$ self-associated points. The spaces $S_{2\nu}$ and $S_{m-2\nu}$ determined by these components are mutually polar in regard to Q; $\delta^{(2\nu)}$ and $\delta^{(m-2\nu)}$ are respectively improper and proper self-polar simplexes of the intersection of Q with $S_{2\nu}$ and $S_{m-2\nu}$.

From the definition of properly self-polar simplex it follows that *a properly self-polar simplex of Q can not be incident with Q.* Since an improperly self-polar simplex of Q in S_{2n} has pairs of associated vertices X_1X_2, X_3X_4, ... $X_{2n-1}X_{2n}$, it follows that, by taking it as the simplex of reference, Q reduces to a form in which

$$\frac{dQ}{dX_1} = \text{multiple of } X_2,\ \frac{dQ}{dX_2} = \text{multiple of } X_1;\ \text{etc.}$$

Thus Q reduces to

$$X_1X_2 + \ldots + X_{2n-1}X_{2n} = 0.$$

Thus the improperly self-polar simplexes are none other than the special simplexes in reference to which Q takes this form. Hence by (5), the improperly self-polar simplexes of Q are not only inscribed to Q, but have complete odd incidence with it.

The n lines X_1X_2, ... $X_{2n-1}X_{2n}$, which join associated pairs of an improperly self-polar simplex of Q, are clearly such that the polar region of any one of them in regard to Q is the join of all the rest. Hence the following construction for an improperly self-polar simplex:

Take any properly self-polar simplex of Q and join its $2n$ points in pairs, obtaining n lines. The $2n$ points where these n lines cut Q constitute an improperly self-polar simplex of Q.

The number of properly self-polar simplexes of Q in S_m is $\infty^{\frac{m(m-1)}{2}}$; the number of improperly self-polar simplexes in S_{m-2m} is $\infty^{\frac{m(m-2)}{2}}$.

A class of doubly incident simplexes in S_{2n}.

(9) We shall now show that there exist families of doubly incident simplexes of a quadric Q in S_{2n}, associated with a pair of polar regions U_n, V_n of Q. The double incidence which occurs is of the form $\{\mu, 2n-\mu\}$, where μ is an odd number.

Let Q denote also the polarity of the quadric Q. Let I denote the involutoric collineation determined by the polar regions U_n, V_n, of order n, of Q. Then, if p be any point, puv the unique transversal through p meeting U_n, V_n in u, v respectively, and p' the harmonic conjugate of p in regard to uv, the collineation I carries p to p' and *vice versa*. Since U_n, V_n are polar regions of Q, it follows that *I transforms the quadric Q into itself*.

Theorem I. *The transformations I, Q are permutable, and their product $IQ = QI$ is a quadric polarity Q'.*

Proof. Consider QIQ; this is a product of a polarity, a collineation, and a polarity, and is hence a collineation. Also, since Q is a polarity, $Q^{-1} = Q$, and so $QIQ = Q^{-1}IQ$ must be an involutoric collineation, whose latent regions are the transforms by Q of the latent regions U_n, V_n of I. But, since U_n, V_n are polar regions of Q, they are only interchanged by Q. Hence QIQ is an involutoric collineation having the same latent regions as I, so that we conclude:

$$QIQ = I,$$

and therefore $$QI = IQ = Q' \text{ (say)}.$$

It is clear that Q' is a polarity, since

$$Q'^{-1} = (QI)^{-1} = I^{-1}Q^{-1} = IQ = Q'.$$

Q' cannot be a null system, since obviously $Q'(x) = IQ(x)$ cannot be incident with x for every point x. Thus Q' is a quadric polarity.

Theorem II. *The quadric polarities Q, Q' are symmetrically related, and are permutable, and $QQ' = Q'Q = I$. Thus the three involutoric operations Q, Q', I, form with identity the fundamental type of geometrical group, of order 4.*

The quadrics Q, Q' have double ring-contact along U_n, V_n, and *have the property that each is its own polar reciprocal in regard to the other.*

The first part is obvious.

To prove that Q is its own polar reciprocal in regard to Q', let x be any point of Q. Then $I(x)$ is also a point of Q, that is to say, $QI(x)$ is incident with $I(x)$. Substituting Q' for QI and QQ' for I, it follows that $QQ'(x)$ is incident with $Q'(x)$. Hence $Q'(x)$ is a prime of Q, which proves what is required.

To show that Q, Q' have ring-contact along U_n, V_n, let x be any point on either of the latent regions U_n, V_n of I, so that $I(x) = x$. Then

$$Q'(x) = QI(x) = Q(x).$$

This shows that Q, Q' have identical intersections with U_n, V_n, and hence have contact along these intersections.

(10) By choosing the points of reference $x_1 x_2 \ldots x_n$, $y_1 y_2 \ldots y_n$ suitably in U_n, V_n respectively, we can reduce the equations of U_n, V_n, Q to the forms

$$U_n:\ y_1 = y_2 = \ldots = y_n = 0,$$
$$V_n:\ x_1 = x_2 = \ldots = x_n = 0,$$
$$Q:\ x_1^2 + x_2^2 + \ldots + x_n^2 + y_1^2 + y_2^2 + \ldots + y_n^2 = 0.$$

Then I is the collineation

$$x_r' = x_r;\quad y_r' = -y_r\quad (r = 1, 2, \ldots n);$$

and the quadric Q' takes the form

$$x_1^2 + x_2^2 + \ldots + x_n^2 - y_1^2 - y_2^2 - \ldots - y_n^2 = 0.$$

The characteristic equation of $\lambda Q + Q'$ is

$$(\lambda + 1)^n (\lambda - 1)^n = (\lambda^2 - 1)^n = 0.$$

Odd powers of λ do not appear in this equation. Therefore, from the known geometrical significance of vanishing coefficients in the characteristic equation, it follows that:

There exist families of simplexes $[\delta_{(\mu)}]$ *which are (properly) self-polar in regard to* Q' *and have the incidence* $\{\mu\}$ *with* Q $(\mu = 1, 3, 5, \ldots 2n - 1)$.

We shall now show that these families are not all distinct, but coalesce in pairs into families of doubly incident simplexes. In fact, let $\delta_{(\mu)}$ be any simplex of the family $[\delta_{(\mu)}]$. Then, by hypothesis, $\delta_{(\mu)}$ is a self-polar simplex of Q', and its faces of order μ are tangent to Q. But it has been shown (Theorem II) that Q is its own polar reciprocal in regard to Q'. Hence by reciprocating with respect to Q', it follows at once that the faces of order $2n - \mu$ of $\delta_{(\mu)}$ must also be tangent to Q. Thus $\delta_{(\mu)}$ is also a $\delta_{(2n-\mu)}$; that is, if μ is not equal to n, the simplexes $\delta_{(\mu)}$ have the *double* incidence $\{\mu, 2n - \mu\}$. We have therefore:

Of the n families of simplexes $[\delta_{(\mu)}]$ ($\mu = 1, 3, \ldots 2n-1$), *only the first* $\frac{n}{2}$ *if n is even, and the first* $\frac{n+1}{2}$ *if n is odd, are distinct. In the former case all the* $\frac{n}{2}$ *families are families of doubly incident simplexes; in the latter, all the distinct families excepting* $[\delta_{(n)}]$ *are doubly incident.*

To indicate the double incidence, we may write the family $[\delta_{(\mu)}]$ in the form $[\delta_{(\mu.2n-\mu)}]$.

The simplexes of any doubly incident family can be associated in pairs having mutual incidence relations analogous to the Möbius simplexes.

For, since I transforms each of the quadrics Q, Q' into itself, it follows that it must transform each of the families $[\delta_{(\mu,2n-\mu)}]$ into itself. Let I transform a particular simplex $\delta_{(\mu,2n-\mu)}$ into another simplex $\delta'_{(\mu,2n-\mu)}$ of the same family. This may be written

$$I\delta_{(\mu,2n-\mu)} = \delta'_{(\mu,2n-\mu)}.$$

But $I = QQ'$, and from the definition of the simplexes,

$$Q'\delta_{(\mu,2n-\mu)} = \delta_{(\mu,2n-\mu)}.$$

Hence
$$Q\delta_{(\mu,2n-\mu)} = \delta'_{(\mu,2n-\mu)}.$$

Thus δ, δ' not only correspond in I, but are also mutually polar simplexes of Q. Therefore, since any face of order μ of either δ or δ' is tangent to Q, it must meet the face of order $2n-\mu$ of the other, which is its polar region. Thus the simplexes δ, δ' are so related that any face of order μ of either meets the face of order $2n-\mu$ determined by the non-corresponding vertices of the other. In particular for $\mu = 1$ the two simplexes will each be in-circumscribed to the other. For $\mu = 1$, $n = 3$ the two simplexes thus associated are those which correspond to the two sixers of a double-six in space (cf. (6)).

We may now identify the sub-family $[\epsilon]$ of simplexes, which is common to all the families $[\delta_{(\mu,2n-\mu)}]$. The simplexes ϵ must be those self-polar simplexes of Q' which have complete odd incidence with Q. By (5), the most general simplex which has complete odd incidence with Q, is the general simplex which has $\{1, 3\}$ incidence with Q; that is to say, it is the simplex constituted by two sets of n points chosen at will on two non-intersecting generating regions G_n, G_n' of Q. In order that the simplex so chosen may be self-polar in regard to Q', it is necessary firstly, that G_n, G_n' be mutually polar regions of Q', and secondly, that the two sets of n points should form properly self-polar simplexes of the intersections of G_n, G_n' with Q'. The former condition implies that G_n' is the transform of G_n by I; for

$$Q'G_n = IQG_n = IG_n.$$

We may therefore obtain all the simplexes ϵ, by choosing at will a generating region G_n of Q (not intersecting U_n, V_n), and taking the second generating region $G_n' = IG_n$, and choosing two sets of n points on G_n, G_n' forming properly self-polar simplexes of their intersections with Q'. We notice also that the family $[\epsilon]$ is carried into itself by I, and therefore that the simplexes ϵ can be associated in pairs, such that two associated simplexes correspond to each other in I, and are also polar simplexes of each other in regard to Q.

The simplexes ϵ are not in general their own associated simplexes; let $[\phi]$ be the sub-family of $[\epsilon]$, consisting of simplexes associated with themselves. Any simplex ϕ must then have complete odd incidence with Q, and must in addition be its own polar simplex in regard to Q. Thus $[\phi]$ must consist of all the improperly self-polar simplexes of Q, which are properly self-polar in regard to Q'. It is also obvious that 'the associated points' of any simplex ϕ (see (8)) must correspond in I. Hence the following construction for the simplexes ϕ:

Choose two sets of n points $u_1 u_2 \ldots u_n$, $v_1 v_2 \ldots v_n$ on U_n, V_n respectively, so as to form properly self-polar simplexes in regard to the intersections of Q with U_n, V_n. The $2n$ intersections of Q with the n lines $u_1 v_1$, $u_2 v_2$, $\ldots$ $u_n v_n$ give the general simplex ϕ, improperly self-polar in regard to Q, and properly self-polar in regard to Q'.

In conclusion, it is necessary to remark that the simplexes $\delta_{(\mu, 2n-\mu)}$ are not the most general simplexes possessing the double incidence indicated; their speciality may in fact be said to consist in their association with a pair of polar spaces U_n, V_n of Q. This fact appears from an enumerative comparison. Consider, for instance, the number of in-circum-simplexes of a quadric in S_{m-2n}. The in-circum-scription cannot clearly amount to more than $2m$ conditions* on the simplex; hence there must be at least $\infty^{m(m-1)-2m} = \infty^{m(m-3)}$. in-circum-simplexes. Now the number of simplexes $\delta_{(1, 2n-1)}$ is the number $\infty^{f(m)}$ of simplexes inscribed to Q so as to be self-polar in regard to an in-polar quadric Q' in S_{m-2n}. Now

$$\begin{aligned} f(m) &= m - 2 + f(m-1) \\ &= (m-2) + (m-3) + \ldots + 2 + 1 \\ &= \frac{(m-1)(m-2)}{2}. \end{aligned}$$

Also the total number of polar spaces U_n, V_n is $\infty^{n^2} = \infty^{\frac{m^2}{4}}$. Thus the number of simplexes of the type $\delta_{(1, 2n-1)}$ is

$$\infty^{\frac{3m^2 - 6m + 4}{4}} = \infty^{m(m-3) - \frac{m^2 - 6m - 4}{4}}$$

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* It amounts, as already remarked, to $2m - 1$ conditions for $m = 6$, to $2m$ conditions for $m = 5$, and to $2m - 2$ conditions for $m = 4$. It is perhaps not unlikely that it would amount to fully $2m$ conditions for higher values of m.

Thus the in-circum-simplexes $\delta_{(1, 2n-1)}$ are at least $\frac{m^2-6m-4}{4}$ times specialised ($m = 2n$). We notice that

$$\frac{m^2-6m-4}{4} > 0 \text{ for } m > 6.$$

Thus the in-circum-simplexes obtained by our method may be expected to be general of their kind for S_4, S_6, but not for S_8, S_{10}, ...

On the phrase "equally probable." By Mr W. BURNSIDE, Honorary Fellow of Pembroke College.

[*Received* 30 January, *read* 2 February 1925.]

The following is a simple example of what Poincaré calls *probabilité du continu*.

Two points are marked at random on a unit line and it is assumed that all positions of either point are equally probable. What is the average or probable positive distance between the two points?

The solution given is that $dxdx'$ is the probability that the two points lie respectively in the segments dx and dx', and that the positive distance between the points is $x' - x$ or $x - x'$ according as x' is greater or less than x; so that the required average positive distance

$$= \int_0^1 dx \int_0^x dx' (x - x') + \int_0^1 dx \int_x^1 dx' (x' - x)$$
$$= \tfrac{1}{3}.$$

A person, unacquainted with the subject of probability, would be sure to ask, in connection with this, What is implied in stating that "all positions of either point are equally probable"? It is possible that, if he got an answer to this question, he would not regard it as satisfactory. He might then ask, Is there any property of points or sets of points on the unit line which conveys the same information as the result of the above calculation? What follows is a suggested answer to this second question.

When n points are marked on a unit line the average of the positive distances between the $\frac{1}{2}n(n-1)$ pairs of points is readily seen to lie between 0 and $\frac{1}{2}$. If the unit line is divided into n equal parts, and one point is marked in each part (n points in all), the same average can be shown by simple arithmetic to lie between $\frac{1}{3} + \frac{a}{n}$ and $\frac{1}{3} - \frac{b}{n}$, where a and b are two comparatively small positive numbers. A similar result is true, if n is large enough, when a fixed number of the n parts have no points in them (n points in all). If the identity of the points is neglected, there will be just one way of distributing the points among the parts corresponding to each solution (in zeros and positive integers) of the equation

$$n_1 + n_2 + \dots + n_n = n,$$

where n_i is the number of points in the ith part reckoning from one end. For a particular solution of this equation the average of the positive distances between the $\frac{1}{2}n(n-1)$ pairs of points differs from

$$\frac{\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} n_i n_j (j-i)}{\frac{1}{2}n(n-1)} = l_{n_1 n_2 \dots n_n} \text{ say}$$

by something equal to or less than $\frac{A}{n}$, where A is a fixed finite number.

Also, when the identity of each point is taken account of, the number of ways in which the n points can be distributed among the n parts, in accordance with the particular solution, is

$$\frac{n!}{n_1!\, n_2! \dots n_n!}.$$

The total number of ways in which the n points can be distributed among the n parts is n^n.

Hence the frequency of the distribution which corresponds to a particular solution of the equation is

$$\frac{n!}{n_1!\, n_2! \dots n_n!}\,\frac{1}{n^n}.$$

It follows that the average of the averages of the positive distances between the $\frac{1}{2}n(n-1)$ pairs of points, taken for all possible distributions of the n points among the n parts, differs from

$$\Sigma \frac{n!}{n_1!\, n_2! \dots n_n!}\,\frac{1}{n^n}\, l_{n_1 n_2 \dots n_n},$$

when the sum is taken for all distinct solutions of the equation, by a quantity less than $\frac{B}{n}$, where B is a fixed finite number.

Now the frequency of a distribution for which $l_{n_1 n_2 \dots n_n}$ differs materially from $\frac{1}{3}$ is clearly very small compared with the frequency of a distribution for which $l_{n_1 n_2 \dots n_n}$ differs very little from $\frac{1}{3}$: and it may, in fact, be shown that the above sum differs from $\frac{1}{3}$ by a quantity less than $\frac{C}{n}$, where C is a fixed finite number. This is an arithmetical fact, and as such would be readily understood by a person unacquainted with the idea of probability.

It is suggested that this arithmetical statement is equivalent to the statement that:

When two points are marked on a unit line and it is assumed that all positions of the points are equally probable, the average positive distance between the two points is $\frac{1}{3}$.

If an authority on probability, who makes this latter statement, refuses to accept the suggestion, he may be fairly asked what meaning he actually attaches to his statement.

If the suggestion is accepted, questions similar to the one considered can hardly be regarded as belonging to *probabilité du continu*, since their equivalents are concerned with a finite number, or at most an enumerable infinity of points in the continuum.

A note on wireless signal strength measurements made during the Solar Eclipse of 24 *January,* 1925. By Mr E. V. Appleton and M. A. F. Barnett.

[*Read* 2 March, 1925.]

In the course of a series of signal intensity measurements made at Cambridge on the electric waves received from broadcasting stations a marked difference between day and night conditions has been observed. The day-time signals are sensibly constant but marked fluctuations of intensity become apparent about sunset and continue throughout the night. These variations are detectable at Cambridge on the signals from London, where they represent a change in strength of about 5 per cent.

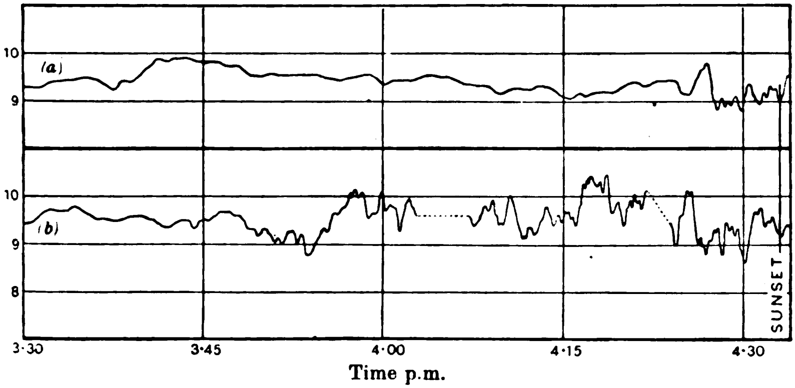
These results may be explained in a general way if it is assumed that there is a variable ray returned by ionic deviation from the upper atmosphere which interferes with the direct ray which travels along the ground. During the day-time the indirect ray returned from the atmosphere is practically negligible, due to its being deviated in the lower layers of the atmosphere where the absorption is considerable. As the sun is setting the ionization in the lower layers of the atmosphere begins to disappear, due to recombination, and the result is an increase in the height (and therefore decrease in pressure) of the stratum which deviates the indirect ray. Deviation without undue absorption is thus possible, and the increase in the intensity of the indirect ray is accounted for.

Since the daily intensity curves show a general similarity with respect to the time of sunset as origin it was thought of interest to investigate during the recent solar eclipse the effect of the partial withdrawal of solar radiation on the strength of the ray returned from the upper atmosphere. The intensity curve for the period of the eclipse together with a curve for a typical sunset are shown together in the accompanying figure.

The abscissae of the upper curve (*a*), which represents the typical sunset effect, have been arranged so that sunset occurs at the same point as on the eclipse curve (*b*), so that the actual times only refer to the eclipse curve. It will be seen that the effect of the eclipse was that of an abnormally early sunset. It should be noted that the intensity changes are not relatively as large as appear at first sight from the diagram as the ordinates have not been extended to the origin. The dotted lines on the curve indicate breaks in the observations. Mr F. G. G. Davey of Caius College, who also made observations during the eclipse, got similar results.

We now proceed to consider the causes of these signal variations and also the fading effects which are observed at greater distances. If we assume that the stratum responsible for the deviation of the indirect ray is at a height greater than say 50 kilometres, the above interpretation of the short-distance variations indicates that the ray is bent back through relatively large angles. Such deviation without undue absorption appears possible if we assume, with Larmor, that the mean free path of the effective ions is large, so that the ion makes many oscillations between the time of two collisions. In this way the low ionic absorption of waves of high frequency is accounted for.

The ionic deviation theories of Eccles* and Larmor†, in terms of which the action of the upper atmosphere must be explained, are incomplete, as mentioned in a previous paper‡, since the effect of the earth's magnetic field on the phase-velocity has not been taken into account. The need for a revision of these theories is specially marked if, as suggested by the work of Loeb, the negative carriers in nitrogen at low pressures are of electronic mass.



We shall now consider some consequences of the propagation of waves through the earth's magnetic field in terms of the revised magneto-ionic theory. It is a simple matter to derive expressions for the phase-velocity of waves travelling in an ionized medium along or across a magnetic field since the classical equations of physical optics apply. We shall, for simplicity, give here only the formulae relating to cases in which collisional "friction" is absent.

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* Eccles, *Proc. Roy. Soc.*, A, vol. 87, p. 79, 1912.

† Larmor, *Nature*, Nov. 1, 1924.

‡ Appleton, *Physical Society Discussion Report*, Nov. 28th, 1924; *Proc. Physical Soc.*, vol. 37, Part II, 1925.

If we consider transmission along the earth's magnetic field we find that the expression for the phase-velocity v is altered* from

$$v = c\left(1 - \frac{4\pi Ne^2}{\omega^2 m}\right)^{-\frac{1}{2}} \qquad \text{......(1)}$$

to

$$v = c\left(1 - \frac{4\pi Ne^2}{\omega^2 m \mp He\omega}\right)^{-\frac{1}{2}}, \qquad \text{......(2)}$$

where e and m are the charge and mass respectively of the electron, N the number of electrons per c.c., ω the angular frequency of the waves, H the strength of the magnetic field, and c the velocity of electromagnetic radiation in free space. Thus any linearly polarized wave is split up into two circularly polarized waves (corresponding to the two signs of $He\omega$) travelling with different phase-velocity (and thus having different atmospheric trajectories) so that all the phenomena associated with rotatory polarization may be expected to occur since the aerial system of a wireless receiver may be regarded as playing the part of the analyser in the corresponding optical problem. The two circularly polarized rays may be differentially absorbed, in which case their combination yields an elliptically polarized wave. Thus we must regard the variations of intensity of the London signals as due to the interference phenomena produced by the direct ray and the two circularly polarized waves produced by magneto-ionic deviation.

As has been previously pointed out anomalous effects may be expected when the terms $\omega^2 m$ and $He\omega$ are of the same order of magnitude where the difference between the phase-velocities of the two rays is specially marked. Now if we assume that the waves travel along a magnetic field of average value of say 0·3 gauss in some part of their path, anomalous effects are to be expected for a wave-length of about 350 metres. It is remarkable that the fading of wireless signals seems to be more marked within a wave-length band of 200–500 metres than for much shorter and much longer wave-lengths and it is suggested that these effects are due to the large alterations in the phase-velocity and absorption which may be brought about by small alterations in N or H for wave-lengths round about this critical value.

For reception at great distances from the transmitter the direct ray is highly attenuated, the signal being almost wholly due to the indirect rays. Multiple reflection may also play a part. Fading in such cases is regarded as due to the varying amplitude, phase, and polarization of these rays acting on a polarized receiver.

The above considerations however apply for wave-lengths below the critical value, but for wave-lengths much greater an examination of (2) shows that in certain cases only one of the two circularly polarized rays will have a phase-velocity greater than c, in which the returned ray will be circularly polarized. For very long waves, however, the transitional layer may be comparable in thickness with a wave-length, in which case the ordinary process of deviation might be more appropriately termed reflection. In such cases the Kerr effect would replace the Faraday effect discussed above.

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* Appleton, *loc. cit.*

For propagation in a direction at right angles to the magnetic field we may consider the electric vector of the wave as resolved parallel to and at right angles to the field. The phase-velocity v_0 for vibrations parallel to the field (ordinary ray) is unaltered and is given by

$$v_0 = c\left(1 - \frac{4\pi N e^2}{\omega^2 m}\right)^{-\frac{1}{2}}, \qquad \ldots\ldots(3)$$

while the phase-velocity v_e of the other (extraordinary) ray is given by

$$v_e = c\left\{1 - \frac{4\pi N e^2}{\omega^2 m - e^2 H^2\left(\dfrac{\omega^2}{\omega^2 m - 4\pi N e^2}\right)}\right\}^{-\frac{1}{2}}. \qquad \ldots\ldots(4)$$

Double refraction effects will therefore be present, the case being analogous to the familiar optical example of transmission of parallel light through a doubly refracting crystal and parallel nicols, the vibration directions of the nicols not coinciding with the vibration directions of the crystal.

But perhaps the most interesting possibility raised by the magneto-ionic theory is that the reciprocity relation between two wireless stations may not always be fulfilled. The mechanical forces on the moving electrons arising from the magnetic field are independent of the direction of propagation of the waves so that the magneto-ionic effects may be relatively different for the two directions of propagation.

The Action of the Electrical Counter. By K. G. Emeléus, B.A., St John's College. (Communicated by Dr J. Chadwick.)

[*Received* 17 March. *Read* 4 May 1925.]

When the potential applied to a discharge tube is less than that required for sparking, entry of a single ionising particle will cause, under certain conditions, the momentary passage of a detectable current. This principle is employed in the electrical counter. One electrode is usually the wall of the vessel, the other a thin wire[*], a small ball[†], or a fine point[‡]. The counter appears to work best in the last case when the inner electrode is negatively charged, in the other cases with it positive. The pressure of the contained gas may range from atmospheric to a few millimetres of mercury. It has been much used for detection and enumeration of α and β particles, as well as in investigations of X-rays and γ-rays.

The point counter has also been studied as a unit of an electrical circuit containing capacity and resistance, and an analogy established between its discharges, and the "flashing" of a neon lamp[§]. That is to say, the ionisation due directly to the particle recorded is increased by collision till it reaches a certain threshold value, after which a discharge sets in characteristic no longer of the ionising particle, but of the tube itself. The potential across the tube then falls till the current ceases to pass at some lower critical voltage. The main object of the present communication is to give a picture of the ionic mechanism involved, and in particular to explain why the current should not continue to flow once it has started. The drop in the difference of potential between the electrodes is sufficient to extinguish the discharge at low pressures[||], but not at pressures approaching that of the atmosphere[¶].

To explain this it has been suggested that the inner electrode is covered with a thin film of high resistance, which, when a discharge starts, becomes polarised by ions collecting on its outer surface[**]. The fall of potential across the film would then reduce the field in the gas so much that ionisation by collision would stop. Such an effect could be produced by grease[††], but the usual preliminary heating of the inner electrode might be expected to eliminate this. The observation that an iron point works best when covered with a smooth layer of oxide points to a film of some compound of the underlying metal, but platinum, which is relatively inert chemically, gives as good a counting point as steel. In fact, the nature of the metal used appears to be of little importance, provided its surface does not become rough on use. The action appears to be mainly determined by the size and disposition of the electrodes, and the nature and pressure of the gas. For these reasons, an attempt has been made to give a picture of the ionic process which shall involve only the phenomena of ionisation by collision, and of the motion of ions in gases.

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[*] Rutherford and Geiger, *Proc. Roy. Soc.* A, vol. 81, p. 141 (1908).

[†] Rutherford and Geiger, *Phil. Mag.* 24, p. 618 (1912).

[‡] Geiger, *Phys. Zeit.* 14, p. 1129 (1913).

[§] Appleton, Emeléus and Barnett, *Proc. Camb. Phil. Soc.* 22, p. 434 (1924).

[||] Appleton, Emeléus and Barnett, *loc. cit.*

[¶] Geiger, *Zeit. für Phys.* 27, p. 7 (1924).

[**] Geiger, *Zeit. für Phys.* 27, p. 7 (1924); Zeleny, *Phys. Rev.* 19, p. 566 (1922).

[††] K. Compton and Ross, *Phys. Rev.* 6, p. 207 (1915).

It is not necessary for the ionising particle to strike the inner electrode, or even to pass close to it[*], so long as some of the ions it produces are drawn into a sufficiently intense part of the field. Nevertheless, the major part of the magnification of the initial ionisation takes place in a very small region, which makes the problem difficult to study directly. Existing data for the counter itself must therefore be supplemented by other relations for the ionising powers and mobilities of ions, and analogies sought in discharges which are sustained, instead of irregularly intermittent as in the case under consideration.

It has been shown that negative ions occur mainly as free electrons in point discharges[†], and in general in intense fields. Also, under these conditions, the number of complex ions of low mobility is small[‡]. It will be assumed that the current is one of molecular ions and electrons, and that the mobility of the negative carriers is much greater than that of the positive, at least in the strongest parts of the field. Since negative ions produce a large ionisation of gas molecules compared with that due to positive ions, the further assumption will be made that the latter produce secondary ions only when they impinge on an electrode[§].

The counter with a positive wire or ball will be considered first. It is doubtful if in this instance the change in the potential between the electrodes consequent on a discharge ever exceeds a few volts, as it appears necessary to use a sensitive electrometer as the recording instrument. The corresponding experimental results for a sustained discharge are that at low voltages a positive point gives a greater current than a negative[||], the disparity being greater the blunter the point[¶], and that the current increases gradually as the applied voltage is increased.

These facts can be explained in terms of the different ionising powers of positive and negative ions. Magnification of the initial ionisation will first start at a surface (A, Fig. 1) at a distance from the surface of the inner electrode, where the field becomes sufficiently

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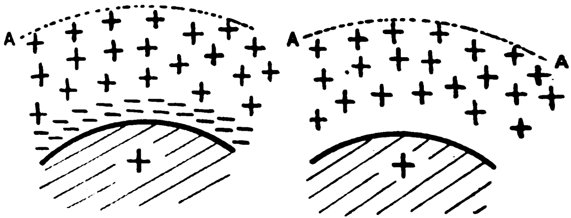
[*] Cf. Kutzner, *Zeit. für Phys.* 23, p. 117 (1924).
[†] Franck, *Ann. der Phys.* 21, p. 984 (1906).
[‡] Hess, *Wien. Ber.* Bd. 129, Heft 6 (1920).
[§] Cf. Holst and Oosterhuis, *Phil. Mag.* 46, p. 1117 (1923).
[||] Cf. Zeleny, *Phys. Rev.* 24, p. 255 (1924).
[¶] Cf. Townsend, *Electricity in Gases*, Chap. x.



(*a*) (*b*)

Fig. 1

strong for the incoming negative ions to produce fresh ions by collision. The negative ions move inwards, the total number of ions increasing rapidly, whilst the positives formed move outwards without ionising. So long as this is true of any positives liberated from the point itself, the current will cease with removal of the primary ionising agent, for the negative ions, which alone are supposed capable of producing ionisation by collision in the gas, are all quickly attracted to and neutralised by the point. Evidently the quantity of electricity discharged is proportional to the initial ionisation.

The dimensions of the inner electrode in such a counter must be a compromise between

(*a*) a small electrode, to produce intense fields, and

(*b*) a large electrode, to make use of all the initial ionisation.

It is of interest to follow the variations of space-charge during this process. The negative ions, because of their great mobility, will reach the positive electrode in a short time, and before the positive ions have moved far from where they were produced. Just before the negative ions disappear, space-charges will exist as in Fig. 1 (*a*): immediately after the negative ions have disappeared, the state of affairs is shown by Fig. 1 (*b*). The repulsive force now acting on the gas containing the positive space charge will tend to make it travel away, probably as a puff, unionised gas streaming in laterally to take its place. Even if this fresh gas were to contain negative ions, it would be unlikely to produce a second discharge of the same magnitude as the first, because they would come in to a diminished field between the positive electrode and the positive space charge moving away. The chance of a second discharge will however increase with increase of the potential of the point, and photographs by Toepler's method show that a series of such puffs, travelling in the predicted direction, does actually precede a brush ischarge from a positive point[*]. Observation of a similar discharge n the Wilson cloud apparatus also often shows an axial stream of onised air in the same direction[†].

For a counter with the point negative the case is more com-licated, and essentially different in that

(*a*) a very fine point is used;

(*b*) the extent of the discharge is almost independent of the nitial ionisation, if this exceeds some very small quantity[‡].

The allied phenomena for a sustained discharge have recently een studied very carefully[§]. The characteristics for the point egative and positive are represented by the curves *N* and *P* re-pectively of Fig. 2, the gas being ionised by a strong source of α-rays. The form of *P* follows readily from the previous discussion, the surface *A* (Fig. 1) receding from the point as the voltage is increased, and possibly at the higher voltages ionisation of gas molecules by positive ions becoming of importance. The curve *N* shows, however, an almost constant current, due directly to the initial ionisation, until a steep rise sets in at some definite voltage. It is not unlikely that this large current is intermittent, with each unit discharge similar in wave-form to that excited by a single ionising particle in a counter[||]. A tentative explanation is outlined below.

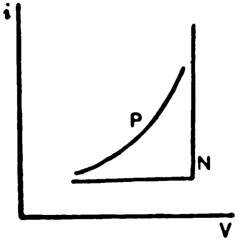


Fig. 2

The main fall of potential between the electrodes occurs in the immediate neighbourhood of the point, and with the latter negative any large magnification of the initial effect must arise in the first instance from ionisation by positive ions. It is probable, as already mentioned, that little ionisation is produced by these in a gas compared with that produced at a solid surface. Therefore little increase in the number of ions can occur unless the incoming positive ions acquire sufficient energy in their last free path in the gas to enable them to produce electronic δ-radiation from the point when they collide with it. A large discharge will then take place, for although the positive ions were unable to ionise the gas-molecules, the electrons passing away from the point do so readily, as well as the secondary electrons which they themselves produce. The electrons move away relatively quickly, and leave a positive space-charge, whose immediate effect is to diminish the field farther away from the point, and so hinder further ionisation by the electrons. Such a space-charge seems required by the existence of a Crookes' dark space in helium at atmospheric pressure*. The force resulting on the gas in which it is present will cause an electrical wind towards the point, accompanied by an increase of pressure.

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[*] Trey, *Phys. Zeit.* 22, p. 406 (1921).

[†] Douglas, *Trans. Roy. Soc. Canada*, 3, 18, iii, p. 133 (1924).

[‡] Geiger, *Verh. d. D. Phys. Ges.* 15, p. 534 (1913).

[§] Zeleny, *loc. cit.*

[||] Appleton, Emeléus and Barnett, *loc. cit.*

The wind will spread the gas containing positive ions over the tip of the point (Fig. 3). If the lateral field is sufficiently intense, these positive ions will in turn liberate electrons from an increased area, and so give a bigger discharge. There is, indeed, considerable experimental evidence that the lateral field plays an important role. The writer has found that a good counting point made by grinding down a wire has usually the shape shown in Fig. 3. Roughly speaking, its tip is a cylinder with a hemispherical end†. Further, on substituting an ebonite cylinder for the brass of the outer electrode, leaving only a flat metal end opposite the point, the initial voltage at which counting was possible rose from 1300 to 2300: axial concentration of the field at the point had made the counter less sensitive. On the other hand, a lateral increase of the field, by having the central pointed wire surrounded by a coaxial brass cylinder with an ebonite front, has the opposite effect, and makes the counting far more regular‡. In fact, it is not necessary to have a cap to the cylinder at all§.

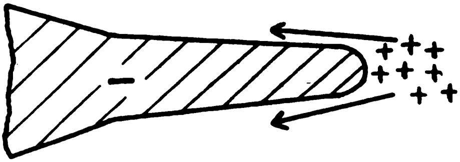

Fig. 3

The result of the increase in gas-pressure close to the point will be to diminish the mean free path of the positive ions. These will now no longer gain sufficient momentum in the field to enable them to liberate δ-rays from the metal, and the discharge will cease, in absence of a further supply of negative ions, which alone are supposed capable of ionising the molecules of the gas.

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* Zeleny, *loc. cit.*

† This is in agreement with a private communication from Professor Kovarik.

‡ Shrader, *Phys. Rev.* 6, p. 292 (1915).

§ Kovarik, *Phys. Rev.* 23, p. 559 (1924).

On this view the total number of ions formed should be practically independent of the initial ionisation, and determined mainly by the effective area of the point. Nevertheless, an α particle might well produce a slightly greater effect than a β particle, as has sometimes been observed*, because with greater initial ionisation there is less chance that any potentially effective part of the point shall not receive some positive ions from the gas.

Another factor, when there is a large series resistance, will be the change in potential between the electrodes due to their neutralisation of the ions. If this effect predominates, a definite fall in the potential across the tube should accompany each discharge. If extinction is brought about by the increase of pressure, the change of potential should be inversely proportional to the capacity of the counter. Intermediate stages are to be expected. Direct measurements by the string electrometer and the cathode ray oscillograph show that on this interpretation of the results the pressure effect becomes of more importance as the initial pressure is increased†.

The well-known phenomenon of the electric wind directed away from a highly charged negative point would appear at first sight to contradict what has been suggested above. It must, however, be remembered that outside the positive space-charge will be a negative region, and that in that region some of the electrons will probably have been converted into molecular ions of lower mobility. The increased pressure at the point, and the outer electrical wind away from it, are related to the mechanical forces on cathode and anode in a low-pressure discharge, due to bombardment by positive and negative ions respectively.

So far it has not been possible to demonstrate directly the wind towards the point, and the corresponding increase of pressure. Additional indirect evidence may however be brought forward in support of the idea. For example, with a discharge from a negative point at the appropriate voltage and gas-pressure the shape of the glow is shown in Fig. 4 (*a*), strongly suggesting the existence of convection currents in the direction of the arrows. Similarly, isolated discharges from a negative point in the Wilson expansion apparatus often give clouds of the shape of Fig. 4 (*b*)‡. Again, the glow on a blunt negative electrode is trumpet-shaped, and its fine end, on the metal, moves about erratically. This could be explained by positive ions being spread over the surface by the hypothetical wind, and starting discharges at local irregularities. Similar considerations would account for the instability of the corona on a thin negative wire, which takes the form of mobile beads*. The positive space-charge round the negative wire is unstable, like a cylindrical film of water on a fine thread.

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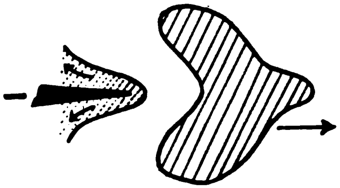
* Cf. Greinacher, *Zeit. für Phys.* 23, p. 361 (1924).

† Appleton, Emeléus and Barnett, *loc. cit.*; Geiger, *Zeit. für Phys.* 27, p. 7 (1924).

‡ Douglas, *loc. cit.*



(*a*) (*b*)

Fig 4

As previously mentioned, a good counting point needs careful preliminary treatment. It must be smooth, and although sharp, the curvature of the tip must not be too great. With too fine a point, or a speck of dust on an otherwise suitable electrode, or with a crack in the film of oxide on a steel needle, spontaneous discharges start in the intense local fields, for applied potentials less than that at which a regular counting action can be obtained. In the writer's opinion, the great value of the preliminary heating is that it assists in the removal of such irregularities. It may also, however, affect adsorbed gas, which seems to play an important part in emission of δ-rays on direct bombardment with α particles†.

The existence of differences of pressure and convection currents in low pressure discharges has been recognised by various authors‡. It remains to show that their magnitude is sufficient in the present instance to account for the actions attributed to them. For a counter whose point can be taken as equivalent to a sphere of radius 0·02 mm.§, the maximum field is $7{\cdot}5 \times 10^5$ volts/cm. for an applied potential of 1500 volts. Taking the mean free path of an ion to be $8{\cdot}5 \times 10^{-6}$ cm., its value for an air molecule at N.T.P., the fall of potential in this distance is 6·4 volts. The quantity of electricity on the point by virtue of its capacity is 0·01 E.S.U. The quantity passing at each discharge is of the order of 0·5 E.S.U. Hence it is probable, on the ideas developed above, that during the greater part of the time of discharge most of the lines of force from the tip of the point end on the positive space-charge, and that the excess pressure in the gas is of the same order as the force per unit area of the point due to its charge, or about 0·2 atmosphere. These calculations are necessarily approximate, but they serve to illustrate the large increase of pressure to be expected locally, probably sufficient to account for extinction of the discharge by the diminution in the mean free path of the positive ions, and for a motion of the gas of the type contemplated.

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* Kunz, *Phys. Rev.* 19, p. 165 (1922).

† Bumstead, *Phil. Mag.* 22, p. 907 (1911).

‡ Cf. Langmuir, *Journ. Franklin Inst.* 196, p. 751 (1923); J. J. Thomson, *Phil. Mag.* 48, p. 1 (1924).

§ Cf. Shrader, *loc. cit.*

SUMMARY.

Some previous work on the theory of the electrical counter is extended by a detailed consideration of the ionic process involved. It is shown that with a positive point the total number of ions produced should be proportional to the initial ionisation. Evidence is adduced to show that with a negative point extinction of the discharge is probably brought about by a local increase of pressure, accompanied by an electrical wind along the sides of the point. The latter would bring about the discharge of a definite quantity of electricity, independent of the initial ionisation, provided the initial pressure was not too low. It is shown that these considerations account for a number of other phenomena connected with discharges from small electrodes, and that the quantities involved are of the right order of magnitude.

The experimental work in connection with this paper has been carried out in the Cavendish Laboratory. The author desires to thank Professor Sir E. Rutherford for his continual interest in its progress, and further to thank Professor Appleton, and Dr Chadwick for much helpful discussion of several of the points involved.

An extension of Wallace's, Miquel's and Clifford's theorems on circles. By Mr F. P. White, St John's College.

[*Received* 19 March, *read* 4 May 1925]

The theorem of Wallace* that if four arbitrary lines be taken in a plane the circumcircles of the four triangles formed by them in threes meet in a point—in which for circles we may equally well take conics drawn through two arbitrary points I, J of the plane—may be proved very easily from three dimensions. Through the four given lines draw four arbitrary planes, forming a tetrahedron X, Y, Z, T. A unique cubic curve can be drawn through I, J and the four vertices. Projecting this curve from the point T upon the original plane, we get a conic which passes through I, J and the vertices of the triangle formed by the lines in which the plane meets the planes TYZ, TZX, TXY, that is, by three of the given lines. This conic, which is one of the conics of the theorem, also passes through the third point K, other than I, J, in which the original plane meets the cubic curve. Projecting in turn from the other three vertices of the tetrahedron, we get the other three conics of the theorem, which also pass through K.

Similarly, in space of three dimensions, the five cubic curves, which pass through two points I, J and through the vertices of the tetrahedra formed by fours of five planes, can be obtained as the projections from five points of a single rational quartic curve in space of four dimensions, and thus have two points in common besides I and J, where the three-dimensional space of the original figure meets the quartic curve. This theorem was remarked by W. F. Meyer and proved by him and by K. Rohn†, but by methods, involving only three dimensions, quite different from that suggested here.

The generalisation to space of n dimensions is easily obtained, and not only so, but by successive projections we may obtain the whole chain of theorems which Clifford has associated with the name of Miquel‡, as well as extensions thereof.

1. We shall denote a (linear) space of n dimensions by $[n]$ and a variety of r dimensions by V_r.

A curve of order n in $[n]$, and not in a space of lower dimensions, is rational and is uniquely determined by $n + 3$ points of general position. It will have chords, trisecant planes, quadrisecant solids and so on; any l points of it determine, in fact, a l-secant $[l-1]$. If $2l \leqslant n$, the l-secant $[l-1]$ lie on a variety V_{2l-1}, of which the order is $r = \binom{n-l+1}{l}$*. This variety is met by a general $[n-2l+1]$ in r points, and by a general $[n-2l+2]$ in a curve of order r; we shall, in the sequel, require to know the intersections with a $[n-2l+1]$ and a $[n-2l+2]$ *containing a chord* of the curve. If we project from a point I of the curve upon a $[n-1]$ we get a curve of order $n-1$ in this space, and the points in which a $[n-2l+1]$ through I meets the V_{2l-1} again give rise to the points in which a general $[n-2l]$ meets the V_{2l-1} of the new curve, their number is thus $\binom{n-l}{l}$. Similarly, projecting from a chord IJ of the curve upon a $[n-2]$, the number of points in which a $[n-2l+1]$ through IJ meets the V_{2l-1} again is seen to be $\binom{n-l-1}{l}$.

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* Given by W. Wallace ("Scoticus") in 1804 in Leybourn's *Mathematical Repositor* .New Series, p. 22; see Mackay, *Proc. Edin. Math. Soc.* 9 (1891), p. 87.

† *Leipzig Berichte*, 65 (1914), p. 329. The method of proof given here was suggested by Prof. H. F. Baker, whose attention I had drawn to Meyer's theorem.

‡ *Mathematical Papers*, pp. 38–54: "Synthetic Proof of Miquel's Theorem." The date of the paper is 1870.

Now take a $[n-2l+2]$ through IJ, which meets the V_{2l-1} again in a curve of order s, say, passing t times through each of the points I, J. A $[n-2l+1]$ in this space, which passes through I and not through J, meets this curve again in $s-t$ points; if it passes through both I and J the number is $s-2t$. Thus

$$s - t = \binom{n-l}{l}, \quad s - 2t = \binom{n-l-1}{l},$$

whence
$$s = \frac{n\,(n-l-1)!}{l!\,(n-2l)!} \text{ and } t = \binom{n-l-1}{l-1}.$$

2. Suppose that we are given in $[n]$ two points I, J and $(n+2)$ *primes*, or $[n-1]$. In a $[n+1]$ containing the $[n]$ draw through each of the $[n-1]$ a $[n]$; let these meet $(n+1)$ at a time in the points $A_1, A_2, \dots A_{n+2}$, which will form the vertices of a *simplex* in $[n+1]$. Through these $(n+2)$ points and through I, J there passes a single rational curve C_{n+1} of order $(n+1)$, which is met by the original $[n]$ in the two points I, J and in $(n-1)$ further points K_1. Now project the curve C_{n+1} in turn from each of the points A upon the $[n]$. We obtain, in $[n]$, $(n+2)$ rational curves C'_n all passing through I, J and the set of points K_1. The curve obtained by projection from A_1 will also pass through the vertices of the simplex in $[n]$ obtained by the intersection of this $[n]$ with the faces of the simplex in $[n+1]$, that is through the vertices of the simplex formed by $(n+1)$ of the original $[n-1]$; this curve C'_n is uniquely determined by the $(n+1)$ vertices and the two points I, J. Hence, omitting in turn each of the $[n-1]$, we obtain

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* See Segre in *Encyk. Math. Wiss.*, III. c. 7, p. 894, footnote 368.

$(n+2)$ curves through I, J and the vertices of the simplex formed by the remaining $[n-1]$, and *they have in common* $(n-1)$ *further points.*

3. Now take in $[n]$ two points I, J and $(n+3)$ primes. In a $[n+2]$ containing the $[n]$ draw through each of the primes a $[n+1]$; we obtain a simplex in $[n+2]$ and a curve C_{n+2} passing through I, J and the vertices $A_1, A_2, \ldots A_{n+3}$ of this simplex.

Select one of the vertices, say A_1, and project the curve upon a $[n+1]$ containing the $[n]$. We get a C_{n+1} through I, J and through the vertices of a simplex $A_2', A_3', \ldots A'_{n+3}$. If this curve be now projected in turn from $A_2', A_3', \ldots A'_{n+3}$ upon the $[n]$, we obtain, from the theorem of the preceding paragraph, $(n+2)$ curves C_n through I, J and the vertices of $(n+2)$ simplexes, which have $(n-1)$ further common points. Now successive projections from A_1 and A_2' are equivalent to a projection from the line A_1A_2. Thus projecting in turn from the edges $A_1A_2, A_1A_3, \ldots A_1A_{n+3}$ upon the $[n]$ we get $(n+2)$ curves C_n with $(n-1)$ further common points K_1. These arise from the chords of the curve C_{n+2} through A_1 which meet the $[n]$; if we join A_1 to $[n]$ by a $[n+1]$, this will meet the C_{n+2} in A_1, I, J and $(n-1)$ further points and these last project into the points required.

If we now substitute for A_1 any one of the other vertices of the simplex, we equally get $(n-1)$ points K_1; and clearly all these $(n+3)$ sets K_1 lie upon the curve in which the $[n]$ meets the V_3 of chords of the C_{n+2}. This curve, G_2, is of order $\frac{1}{2}(n-1)(n+2)$ and passes $(n-1)$ times through each of the points I, J.

4. We now take $(n+4)$ primes in $[n]$, build up, in the same way, a simplex in $[n+3]$ and draw the corresponding curve C_{n+3} through I, J and the vertices of the simplex. Project from one vertex A_1 upon a $[n+2]$ containing the $[n]$ and we get the figure of § 3, from which by double projections we get, in $[n]$, $(n+3)$ sets K_1 of $(n-1)$ points lying upon the curve G_2 in which the V_3 of chords of the C_{n+2} in $[n+2]$ meets the $[n]$. A chord of the C_{n+2} is the projection from A_1 of a chord of the C_{n+3}, and thus the points (finite in number) in which the variety V_3 of chords of the C_{n+3} meets the $[n]$ will lie upon this G_2, and they will equally lie upon the curves G_2 obtained by projecting from $A_2, \ldots A_{n+4}$. Hence the $(n+4)$ curves G_2 in $[n]$ obtained by omitting each of the $(n+4)$ primes in turn have in common, besides I, J, the set of points K_2, in number $\frac{1}{2}n(n-1)$, in which the variety of chords of the C_{n+3} meets the $[n]$.

5. Once again, taking $(n+5)$ primes in $[n]$, and drawing the simplex and curve C_{n+4} in $[n+4]$, we get $(n+5)$ sets of points K_2, one such set consisting of the points in which the V_3 of chords of the C_{n+3}, obtained by projection from one vertex A_1, meets the

$[n]$. These points arise from the trisecant planes of the C_{n+4} which pass through A_1 and meet $[n]$; this set and similarly all the $(n+5)$ sets K_2 lie on the curve G_3 in which the V_5 of trisecant planes of the C_{n+4} meets the $[n]$. This curve G_3 is of order $\frac{1}{6}n(n-1)(n+4)$ and passes $\frac{1}{6}n(n-1)(n-2)$ times through each of the points I, J.

Taking $(n+6)$ primes in $[n]$ we obtain $(n+6)$ curves G_3 which have in common, besides I, J, a set K_3 of $\frac{1}{6}(n+1)\,n\,(n-1)$ points in which the V_5 of trisecant planes of the C_{n+5} meets the $[n]$.

6. The argument can clearly be continued indefinitely. The results may be stated as follows:

In $[n]$ two points I, J and

(1) $(n+1)$ primes determine a curve G_1 of order n passing once through I, J.

(2) $(n+2)$ primes determine $(n+2)$ curves G_1 with $(n-1)$ further common points K_1.

(3) $(n+3)$ primes determine $(n+3)$ sets K_1 lying on a curve G_2 of order $\frac{1}{2}(n-1)(n+2)$ passing $(n-1)$ times through I, J.

(4) $(n+4)$ primes determine $(n+4)$ curves G_2 with $\frac{1}{2}n(n-1)$ further common points K_2.

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$(2l-1)$ $(n+2l-1)$ primes determine $(n+2l-1)$ sets K_{l-1} lying on a curve G_l of order $(n+2l-2)\dfrac{(n+l-3)!}{l!\,(n-2)!}$ passing $\dbinom{n+l-3}{l-1}$ times through I, J.

$(2l)$ $(n+2l)$ primes determine $(n+2l)$ curves G_l with $\dbinom{n+l-2}{l}$ further common points K_l*.

For $n=2$, the number of points K_l is $\dbinom{l}{l}=1$, and the order of G_l is $2l\,(l-1)!/l!=2$, i.e. the curve is in every case a *conic*, passing through I and J, and the $(2l+2)$ conics have *one* further common point. This is Clifford's chain of theorems.

For $n=3$, we begin with a space cubic through I, J and the vertices of a tetrahedron; K_l consists of $(l+1)$ points and G_l is a curve of order $(2l+1)$ passing l times through I and J—it is therefore a rational curve. We have thus a chain of theorems in three-dimensions, of which Meyer's is the first; and equally in space of any greater number of dimensions.

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* Compare the statement (for $n=2$) in Clifford, *Common Sense of the Exact Sciences* (1885), p. 80.

Triple Binary Forms; the complete system for a single (1, 1, 1) *form with its geometrical interpretation.* By Mr W. SADDLER, St John's College. (Communicated by Mr H. W. TURNBULL.)

[*Received* 28 January, *read* 2 February, 1925.]

In the *Transactions of the American Math. Society*, vol. I, 1900, and vol. IV, 1904, Mr E. Kasner treats exhaustively the (2, 2) double binary form and discusses the theory connected with double binary forms and multiple binary forms in general terms. He shows the relations between the systems of multiple binary forms with digredient variables and the forms with cogredient variables. Hitherto nothing seems to have been written on systems of *triple* binary forms (with regard to higher forms see a paper on the (1, 1, 1, 1) form by C. Segre[*]); so here I propose to discuss the complete system of a (1, 1, 1) binary form which consists of six forms connected by one syzygy. When two of the variables are the same we naturally get the (2, 1) form and when the three are the same we get the (3) or cubic binary form.

§ 1. Let

$$f = \sum_{ijk} \binom{m}{i} \binom{\mu}{j} \binom{\omega}{k} a_{ijk} x_1^{m-i} x_2^{i} y_1^{\mu-j} y_2^{j} z_1^{\omega-k} z_2^{k}$$

be a triple binary form of orders (m, μ, ω) in the variables (x, y, z). Symbolically we write it as $a_x^m b_y^\mu c_z^\omega$ where

$$a_{ijk} = a_1^{m-i} a_2^{i} b_1^{\mu-j} b_2^{j} c_1^{\omega-k} c_2^{k}.$$

We assume the variables (x, y, z) are independent and undergo independent projective linear transformations; and hence what distinguishes a triple binary form from a double or single (i.e. ordinary) binary form is the independence of symbols a, b, c originating from x, y, z respectively. Thus in order to secure the complete independence of the linear transformations for x, y, z, we must only convolve a with a', a''; b with b' ...; c with c' ...; where the symbols with dashes are cogredient to those without.

If
$$f = a_x^m b_y^n c_z^p = a'^m_x b'^n_y c'^p_z = \dots$$
$$g = \alpha_x^m \beta_y^n \gamma_z^p = \alpha'^m_x \dots$$

the (h, k, l) *transvectant* of (f, g) is defined as

$$(f, g)_{h,k,l} = (a\alpha)^{m-h} (b\beta)^{n-k} (c\gamma)^{p-l} \dots\dots\dagger$$
$$(f, g)_{h,k,l} = (-1)^{h+k+l} (g, f)_{h,k,l}.$$

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[*] *Annali di Mat.* (3), 29, 105–140 (1920).

[†] The necessary x, y, z factors are to be added.

The usual identities

$$(ab)\,c_x + (bc)\,a_x + (ca)\,b_x = 0,$$
$$(ab)\,(cd) + (bc)\,(ad) + (ca)\,(bd) = 0,$$

are used in reducing the transvectants. Also the fundamental property of transvectants for ordinary binary forms, viz. that the whole transvectant differs from any term by a sum of terms which may be expressed as lower transvectants of forms obtained by convolution from the original forms, is true for the extended transvectants used here.

§ 2. Let $\quad f = a_x b_y c_z = a_x' b_y' c_z' = a_x'' b_y'' c_z$

be the single (1, 1, 1) form.

To find the invariant system, we proceed step by step by transvection to forms of ascending order of degree, utilising the extension of Gordan's method for binary forms*. The system obtained will be shown by induction to be complete.

For degree *one* there is only one form f. Hence all concomitants of degree *two* are included in $(f, f)_{i,j,k}$. $i, j, k = 0, 1$, only.

I. $\quad (f,f)_{1,0,0} \equiv 0;\ (f,f)_{0,1,0} \equiv 0;\ (f,f)_{0,0,1} \equiv 0.$

II. $\quad (f,f)_{1,1,0} = (aa')\,(bb')\,c_z c_z' = \psi_z^2 = P_1,$

$\quad (f,f)_{1,0,1} = (aa')\,(cc')\,b_y b_y' = \phi_y^2 = P_2,$

$\quad (f,f)_{0,1,1} = (bb')\,(cc')\,a_x a_x' = \theta_x^2 = P_3.$

III. $\quad (f,f)_{1,1,1} \equiv 0.$

Degree 3.

IV. $\quad (\theta_x^2, f)_{1,0,0} = (\theta a)\,\theta_x b_y c_z = (aa'')\,(bb')\,(cc')\,a_x' b_y'' c_z'' = Q_3,$

$\quad (\phi_y^2, f)_{0,1,0} = (\phi b)\,a_x \phi_y c_z = (aa')\,(bb'')\,(cc') \ldots† = Q_2,$

$\quad (\psi_z^2, f)_{0,0,1} = (\psi c)\,a_x b_y \psi_z = (aa')\,(bb')\,(cc'') \ldots = Q_1.$

Here we may write $Q_1 = Q_2 = Q_3 = A_x B_y C_z = Q$, say.

For

$$\begin{aligned}(aa')\,(bb'')\,(cc') \ldots &= \tfrac{1}{2}\,(bb'')\,\{(aa')\,(cc') \ldots - (a''a')\,(c''c') \ldots\}\\ &= \tfrac{1}{2}\,(bb'')\,\{(a''a')\,(cc'') \ldots + (aa'')\,(cc') \ldots\}\\ &= \tfrac{1}{2}\,(aa')\,(bb'')\,(cc'') \ldots + \tfrac{1}{2}\,(aa')\,(bb')\,(cc'') \ldots\end{aligned}$$

by fundamental identities and by interchanging equivalent symbols.

Thus $\qquad Q_2 = \tfrac{1}{2}Q_3 + \tfrac{1}{2}Q_1;$

similarly $\qquad Q_3 = \tfrac{1}{2}Q_1 + \tfrac{1}{2}Q_2;$

and since $Q_1, Q_2, Q_3 \not\equiv 0$ then $Q_1 = Q_2 = Q_3 = Q$ say.

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* Grace and Young, *Algebra of Invariants*, pp. 85–100; Peano, *Battaglini*, vol. 20 (1882).

† In ϕ_2, ϕ_1 and in what follows, indicates the implied necessary x, y, z factors.

v. All irreducibles of degree 4 are included in $(f, Q)_{i,j,k}$ but the alternative degree 4 transvectants involving $(P, P)^2$ etc., are added for completeness.

$$(P_1, P_1)_{0,0,2} = (\psi\psi')^2 = X \quad = (aa')(bb')(\psi c)(\psi c')$$

$$= (aa')(bb')(a''a''')(b''b''')(cc'')(c'c''')$$

$$(P_2, P_2)_{0,2,0} = (\phi_y{}^2, \phi_y{}^2)_{0,2,0} = (\phi\phi')^2 = Y$$

$$= (aa')(cc')(\phi b)(\phi b')$$

$$= (aa')(cc')(a''a''')(c''c''')(bb'')(b'b''')$$

$$(P_3, P_3)_{0,0,2} = (\theta\theta')^2 = Z.$$

Then it may be shown that $X = 2Z - Y$ and similarly for the others. Thus $X = Y = Z$, since $X, Y, Z \not\equiv 0$.

$$(f, Q)_{0,0,1} = -((aa')(bb')(cc'')\ldots, a_x{}'''b_y{}'''c_z{}''')_{0,0,1}$$

$$= -(aa')(bb')(cc'')(c'c''')\ldots = -(\psi_z{}^2, f^2)_{0,0,2}$$

$$= -R \text{ (say)} = \tfrac{1}{2}P_2P_3$$

for the following reason.

$$(aa')(bb')(cc'')(c'c''')\ldots = -(aa''')(bb''')(cc'')(c'c''')\ldots$$

$$= -(a''a''')(bb''')(cc'')(c'c''')\ldots - (aa'')(bb''')(cc'')(c'c''')\ldots$$

$$= -(a''a''')(bb''')(cc'')(c'c''')\ldots - \tfrac{1}{2}P_2.P_3$$

$$= -\tfrac{1}{2}(a''a''')(cc'')(c'c''')\{(bb''')\ldots - (b'b'')\ldots\} - \tfrac{1}{2}P_2.P_3.$$

Thus

$$\tfrac{3}{2}R = -\tfrac{1}{4}(a''a''')(c''c''')(cc')(bb')\ldots - \tfrac{1}{2}P_2.P_3$$

whence

$$R = -\tfrac{1}{2}P_2.P_3.$$

$$(f, Q)_{0,1,0} = -(\phi_y{}^2, f^2)_{0,2,0} = \tfrac{1}{2}P_3.P_1,$$

$$(f, Q)_{1,0,0} = -(\theta_x{}^2, f^2)_{2,0,0} = \tfrac{1}{2}P_1.P_2.$$

Compare the results for the binary cubic $(ab)^2(ac)(bd) = -\frac{1}{2}H^2$ which may be deduced by putting $a = b = c$.

$$(f, Q)_{1,1,0} \equiv 0 \equiv (f, Q)_{0,1,1} \equiv (f, Q)_{1,0,1}$$

$$(f, Q)_{1,1,1} = (a_xb_yc_z, A_xB_yC_z)_{1,1,1}$$

$$= -((aa')(bb')(cc'')\ldots, a_x{}'''b_y{}'''c_z{}''')_{1,1,1}$$

$$= -(aa')(bb')(cc'')(a''a''')(b''b''')(c'c''')$$

$$= -X.$$

Degree 5.

VI. $$\begin{aligned}(P_1 P_3, f)_{1,0,1} &= (\theta a)(\psi c)\,\theta_x b_y \psi_z = (\theta_x^2, Q)_{1,0,0}\\ &= -\,((\psi c)\,a_x b_y c_z,\ \theta_x^2)_{1,0,0}\\ &= -\,((\theta a)\,\theta_x b_y c_z,\ \theta_x'^2)_{1,0,0}\\ &= -\tfrac{1}{2}(\theta\theta')^2 f = -\tfrac{1}{2}X.f,\end{aligned}$$

similarly $$\begin{aligned}(P_1 P_2, f)_{0,1,1} &= (P_2 P_3, f)_{1,1,0}\\ &= (\phi_y^2, Q)_{0,1,0} = (\psi_z^2, Q)_{0,0,1} = -\tfrac{1}{2}X.f.\\ (P_1 P_2, f^2)_{0,1,1} &= -\tfrac{1}{2}X.f^2 - \tfrac{1}{4}P_1 P_2 P_3,\end{aligned}$$

as it is a reducible part $$\begin{aligned}&+ (\psi c)(\phi b')\ \ldots\\ &= \ldots + (\psi c')(\phi b')\ \ldots - (cc')(\phi b')\ \ldots\\ &= \ldots + (\psi c')(\phi b')\ \ldots - \tfrac{1}{2}P_1 P_2 P_3.\end{aligned}$$

$$\begin{aligned}(Q_1, Q_1)_{0,1,1} &= \tfrac{1}{2}X.P_3,\\ (Q_1, Q_1)_{1,1,0} &= \tfrac{1}{2}X.P_1,\\ (Q_1, Q_1)_{1,0,1} &= \tfrac{1}{2}X.P_2,\\ (P_1 P_3, fQ_3)_{2,0,2} &\equiv 0.\end{aligned}$$

VI. *There is one syzygy*, $-\tfrac{1}{2}P_1 P_2 P_3 = Q_1^2 + \tfrac{1}{2}f^2 X$.

§ 3. The irreducible system consists of the forms f, P_1, P_2, P_3, Q, X, which we shall show are complete.

Let $$F = f^{\alpha} P_1^{\beta} P_2^{\gamma} P_3^{\delta} Q^{\lambda} X^{\rho}.$$

If the system up to degree m is complete then we shall show that that of degree $m + 1$ is also complete; now it has been proved true for $m = 1, 2, 3, \ldots 6$ and hence it will be true in general.

Suppose $$\alpha + 2(\beta + \gamma + \delta) + 3\lambda + 4\rho = m.$$

The transvectant of a degree $m + 1$ will be given by $\{F, f\}_{\lambda,\mu,\nu}$. Now if $\rho > 0$ then this transvectant contains the invariant X and hence is reducible; thus ρ is zero.

If α or λ are > 0 then $(f, f)_{i,j,k}$ or $(\phi, f)_{i,j,k}$ reduces it. Thus α and $\lambda = 0$.

If β, γ, δ are all > 0 this reduces to the last case by using the syzygy.

If β and γ are > 0 then $(P_1 P_2, f)_{\lambda,\mu,\nu}$ reduces it; similarly for α and β both > 0.

The only case to consider is when β (say) is > 0 and the transvectant (P_1, f) reduces it.

Hence if it is complete for degree m, it is complete for $m + 1$. This proves the theorem.

Combinants. Let

$$\begin{aligned} f_\kappa &= \kappa f + \kappa' Q = \kappa a_x b_y c_z + \kappa' A_x B_y C_z, \\ P_{1_\kappa} &= (\kappa^2 + \tfrac{1}{2} X . \kappa'^2)\, P_1, \\ P_{2_\kappa} &= (\kappa^2 + \tfrac{1}{2} X . \kappa'^2)\, P_2, \\ P_{3_\kappa} &= (\kappa^2 + \tfrac{1}{2} X . \kappa'^2)\, P_3, \\ Q_\kappa &= ((\kappa^2 + \tfrac{1}{2} X . \kappa'^2)\, P_1, \kappa f + \kappa' Q)_{0.0.1} \\ &= (\kappa^3 + \tfrac{1}{2} X . \kappa\kappa'^2)\, Q_1 - \tfrac{1}{2}(\kappa^2\kappa' + X\kappa'^3)\, X . f, \\ X_\kappa &= (\kappa^2 + \tfrac{1}{2} X . \kappa'^2)^2 . X. \end{aligned}$$

The combinants of the form f are thus f, P_1, P_2, P_3, X.

Connection between binary and double binary forms. It is to be noticed that when $x = y$, and hence when a and b are equivalent, $P_2 = P_3$, but the other covariants are quite distinct; this case leads to the (2, 1) form with 5 concomitants, connected by one syzygy, as given by Peano (*loc. cit.*).

When $z = x = y$ and a, b, c are equivalent symbols $P_1 = P_2 = P_3$ and leads to the binary cubic with 4 concomitants and one syzygy.

§ 4. *Geometrical Interpretation.* If we consider the form as $a_x b_y c_z = \ldots = f$ and look on x, y as circular coordinates

$$x = X + iY = Z,$$
$$y = X - iY = Z',$$
$$z = \lambda,$$

then the equation $f = 0$ will represent a system of coaxal circles whose parameter is λ, in rectangular Cartesian coordinates X, Y. Now if $C_x D_y = C_x' D_y' = 0$ and $E_x F_y = 0$ represent a pair of circles, $(CE)(DF) = 0$ determines the condition that the circles are orthogonal; while $(CC')(DD') = 0$ determines the condition that $C_x D_y = 0$ reduces to a point circle and

$$(CC)'(DD')(EE')(FF') - \{(CE)(DF)\}^2 = 0$$

that the circles touch.

$$P_1 = \psi_\lambda{}^2 = (aa')(bb')\, c_\lambda c_\lambda' = 0$$

will thus determine the two values of λ, for which the circles reduce to point circles; otherwise it determines the limiting points of the coaxal system.

Now the actual Z, Z' coordinates for these two points are got by eliminating λ between $(aa')(bb')\, c_\lambda c_\lambda' = 0$ and $a_Z b_{Z'} c_\lambda = 0$, and are given by $(P_1, f^2)_{0.0.2} = P_2 . P_3 = 0$.

Thus $P_2 = 0 = (aa')(cc')\, b_{Z'} b'_{Z'}$ will give the two Z' lines corresponding to the two limiting points.

$P_3 = 0 = (bb')(cc')\,a_Z a_Z'$ will give the two Z lines corresponding to the two limiting points.

$A_Z B_{Z'} C_\lambda = Q = 0$ will represent a second system of coaxal circles and since $(A_Z B_{Z'} C_\lambda,\ a_Z b_{Z'} c_\lambda)_{1,1,0} \equiv 0$ the two systems of circles are mutually orthogonal and hence $A_Z B_{Z'} C_\lambda = 0$ determines the second system of coaxal circles orthogonal to the original set.

Since $$(Q, Q)_{1,1,0} = \tfrac{1}{2} X \,.\, \psi_\lambda{}^2.$$

The two values of λ which determine the limiting points of the first system will also determine the limiting points of the second system.

$X = (\psi\psi')^2 = (\phi\phi')^2 = (\theta\theta')^2 = 0$. X is thus the discriminant of the quadratic forms and it vanishes if the limiting points coincide and hence the two Z, and Z' lines also coincide. But since

$$\tfrac{1}{2} P_1 P_2 P_3 = -Q^2 - \tfrac{1}{2} f^2 X$$

when X vanishes $Q = A_Z B_{Z'} C_\lambda$ becomes the product of three independent linear forms and thus degenerates.

The Z line corresponding to $f = a_Z b_{Z'} c_\lambda = 0$ will coincide with the Z line corresponding to $Q = A_Z B_{Z'} C_\lambda = 0$ if

$$(f, Q)_{1,0,0} = \tfrac{1}{2} P_1 P_2 = 0,$$

i.e. if the Z', λ satisfy the equations for the limiting points and thus the second coaxal system passes through the limiting points of the first.

$\psi_\lambda \psi_\mu = 0$, the polar of "$\mu$" with regard to $\psi_\lambda{}^2$, gives the condition that the two circles whose parameters are λ and μ may be orthogonal: $a_Z b_{Z'} c_\lambda = 0$, $a_Z b_{Z'} c_\mu = 0$ are orthogonal if

$$(aa')(bb')\,c_\lambda c_\mu' = 0 = \psi_\lambda \psi_\mu.$$

A Geometrical treatment of the Correspondence between Lines in Threefold Space and Points of a Quadric in Fivefold space.

By Mr H. W. TURNBULL, Trinity College.

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§ 1. The six Plücker coordinates of a straight line in three dimensional space satisfy an identical quadratic relation

$$p_{12}p_{34} + p_{23}p_{14} + p_{31}p_{24} = 0,$$

which immediately shows that a one-one correspondence may be set up between lines in three dimensional space, Λ, and points on a quadric manifold Σ_4^2 of four dimensions in five dimensional space, S_5. For these six numbers p_{ij} may be considered to be six homogeneous coordinates of such a point.

Although this correspondence is well known, there does not appear to be any published work which proves the fact by the methods of pure geometry. It therefore seems worth while to record such a proof, and for completeness to add several well-known preliminaries which are inserted to avoid assuming theorems which are usually proved by analytical methods.

§ 2. The simplest plan appears to be this. Let Euclidean space S_5 be postulated up to and including five dimensions. Let all finite real points be called *proper*; and all other real points of S_5 be called *improper* (or points at infinity). Let all the fundamental properties of the incidence of linear elements—points (S_0), lines (S_1), planes (S_2), with other flat regions (S_3), (S_4)—be assumed, together with the property of *distance* between two given proper points. Let lines (planes, etc.) be called *improper* if they can be completely determined by two (three, etc.) improper points.

These incidence properties are the basis of projective geometry: not so the distance postulate. This however may be replaced by a straightforward circumlocution involving cross ratios so as to become a projective relation. But if in the first instance simplicity of transformation, from lines in Λ to points on Σ_4^2, is sought, the more homely postulate is to be preferred.

Next let the facts of the three types of *involution* be postulated —the elliptic involution with all real pairs of elements separated by a central element; the parabolic, with one element of each of its pairs coinciding with the central element; and the hyperbolic, with all real pairs of elements not separated by the central element.

Lastly let it be possible to describe a *circle*, *sphere*, *hypersphere* in the flat region S_2, S_3, S_4 or S_5 with a given centre and a given radius (distance from centre to the locus considered). Here hypersphere is used of a three-fold locus in S_4 or a four-fold locus in S_5. This postulate implies that a given centre and a given point of the locus completely define the locus. Let *circular manifold* denote sphere or hypersphere. Also let *positive* and *zero* qualify the locus when the centre *is not* or *is* on the locus.

§ 3. With these assumptions as a basis it is now possible to define the *power* of a point with regard to a circular manifold, by simply taking any plane section through the centre and the point, for the cases S_3, S_4, S_5, and using the plane properties of a point and a circle. All such sections give congruent figures.

Further, a positive circle determines a hyperbolic involution upon any diameter, wherein inverse points are point pairs of the involution. By reflecting one of the points of each pair in the centre we obtain an elliptic involution which may be used conversely to define a *negative circle*[*] with real centre. The power of the centre C with regard to a negative circle is equal to the square of the distance of either point P, Q from C, where P, Q form the unique symmetrical point pair of the involution circle.

Let *real* circle denote indifferently positive, zero or negative circle.

§ 4. This leads directly to a corresponding definition of *real circular loci*, including negative sphere or negative hypersphere. It also leads to the definition of complex points (or imaginary points), after the manner of definition which we owe to von Staudt[†]. Thus by regarding the common points of a circle and co-planar line as being defined by the coaxal system of circles conjugate to the system of which the given circle and line are members, we obtain by pure geometry a proof that

Every proper line has two points in common with a coplanar circle.

For example, a line actually intersecting the circle determines a conjugate coaxal system of which these points of intersection are the limiting points. A tangent line determines coincident points, while each of all other proper lines (i.e. excluding the line at infinity) determines a conjugate coaxal system with no real limiting points, but giving a real elliptic involution on the axis of this coaxal system. This involution defines two complex limiting points and therefore two complex points of intersection of the given line and circle.

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[*] It seems unnecessary here to develop the pure geometrical theory of negative circles in full. It corresponds to the analytical fact that a locus

$$x^2+y^2+2gx+2fy+c=0$$

exists where g, f, c are real and $g^2+f^2<c$.

[†] Cf. Baker, *Principles of Geometry*, I (Cambridge, 1922), 165–172, 182.

Also we may apply these principles to flat spaces up to five dimensions and prove, within the meaning of our definitions, that

Every proper line has two points in common with a real circular manifold:

and that

Every flat manifold S_2, S_3, S_4 intersects a real circular manifold in a circular manifold or a circle.

§ 5. The main proposition may now be enunciated and proved:

The straight lines of three dimensional space may in general be put into one-one correspondence with the points of a quadric manifold in five dimensional space.

The proof will deal entirely with real elements and real transformations. The quadric selected will be a hypersphere Σ_4^2.

PROOF. Let O be a point on a positive hypersphere, and let S_4 be a linear space of four dimensions neither containing O nor being parallel with the tangent four-fold S_4' whose point of contact is O. Then by hypothesis Σ_4^2 and S_4 are distinct loci with a real circular manifold Σ_3^2 for their complete intersection.

Next let all points P of Σ_4^2 be projected from O to S_4, by joining OP and finding the common point Q of this line and S_4. This puts points of Σ_4^2 into one-one correspondence with points of S_4, with the exception of the linear three-fold space S_3, in S_4, common to the given S_4 and the tangent space S_4' to Σ_4^2 at O. All points of this S_3 correspond with O.

Now let Λ, in this S_4, be any other linear three-fold space parallel with this S_3, and let one proper point O' of S_3, and one positive hypersphere Σ_3^2 touching S_3 at O', be chosen, the whole figure consequently lying in S_4. Since this hypersphere Σ_3^2 is positive it has an actual circumference, and consequently all points of S_4 are either inside, upon, or outside this circumference. Also each point has a unique polar three-fold locus which intersects the hypersphere Σ_3^2 in a real sphere. It follows that

A positive sphere answers to an external point,
A zero ,, ,, ,, a circumference point,
and A negative ,, ,, ,, an internal point.

Now let a stereographic projection of these spheres be made from the origin O' to the space Λ. Since this projection transforms a sphere situated on the hypersphere Σ_3^2 into a sphere of Λ—in particular preserving the character of positive, zero, or negative, of spheres—it sets up a one-one correspondence between all spheres of Λ and all polar S_3's of points of S_4 with respect to the hypersphere Σ_3^2, and therefore ultimately between spheres of Λ and points on the original hypersphere Σ_4^2 in S_5. The exceptional case is the original point O, which corresponds with the totality of planes in

Λ, through the stereographic projection of spheres on Σ_3^2 which are polars of points in S_3.

§ 6. The figure is now confined to three dimensional space Λ, and a correspondence between spheres and lines is provided by the help of coaxal systems of circles.

In fact, let π, π' be two intersecting planes containing the proper line $\pi\pi'$: further let A be a given point of π', not on the line $\pi\pi'$.

Then a sphere is completely determined by its real circular sections in π and π' respectively. But, conversely, answering to one circle σ in π is a coaxal system of spheres, which determine in π' a coaxal system of circles with $\pi\pi'$ for radical axis. Since polars of A with regard to these circles pass through a fixed point B, not on $\pi\pi'$, and not coinciding with A, the polars intersect $\pi\pi'$ in points P which are in one-one correspondence with circles of the system and therefore with the spheres containing σ. Also if, in the plane π, V is a fixed point external to the line $\pi\pi'$, and δ is the diameter of the circle σ parallel with VP, there is a one-one correspondence, so constructed, between spheres containing σ and diameters δ of σ, excluding the diameter parallel with $\pi\pi'$. But δ and σ determine a coaxal system λ, of circles in the plane π, whose limiting points are the common (real or complex) points of δ and the circumference of σ. There is therefore a one-one correspondence between real spheres (excluding planes) of Λ and systems of coaxal circles of a plane π (excluding systems with axes parallel to a given line $\pi\pi'$).

Finally if Σ_2^2 is a given positive sphere in Λ, these coaxal systems of π may be projected stereographically on to Σ_2^2, so determining a one-one correspondence between a line p of Λ and a coaxal system λ of the plane π: for the coaxal system upon the sphere Σ_2^2 forms a pencil of planes through a line.

If O'' is the origin of this last projection, O'' lies on Σ_2^2 and one tangent line p' through O'' is parallel with $\pi\pi'$. The projection associates all those diameters δ, which happen to be parallel with $\pi\pi'$, with lines p which meet p'. *Excluding these lines, which form a special linear complex K, all lines Λ are now in one-one correspondence with all spheres of Λ, and therefore with all but one point of Σ_4^2.* This proves the theorem.

§ 7. The last few steps have been cut short to avoid unduly prolonging the proof. Detailed investigation would show that all real improper lines of Λ are included in the correspondence.

The case of lines p which actually intersect the sphere Σ_2^2 could of course be treated by noticing that the two points of intersection are stereographic projections of the common points of the line δ

and the circle σ. The method used above however also covers cases where the lines p are entirely external to Σ_4^2.

It is unfortunate that a linear complex (∞^3) of lines remains unpaired. But it is not easy to avoid some such singularity in the correspondence, although the analytical statement of § 1 shows that the singularity is an actual blemish.

The transformation of lines into spheres, and vice versa, has been well known since the days of Sophus Lie. His method however was analytical and essentially involved complex numbers. In the present instance the transformation is real. It is also, except for specifically excluded elements, continuous.

§ 8. We may however complete the correspondence by starting with a ruled hyperbolic fourfold R_4^2 in S_5 instead of a hypersphere. For in this case, instead of one point O uncorrelated on the quadric, there are all ∞^3 points common to the quadric and its tangent linear S_4 at O. These can be correlated with the ∞^3 lines of the special linear complex, while other points of R_4^2 are treated as before. This would account for *all real points of the quadric* and *all real lines of the threefold space* Λ, although the complete transformation is composed of two quite distinct parts, the original general and now this special part.

This special correlation may be effected as follows:

All ∞^1 points Q on a line p' may be correlated with all ∞^1 generators of a regulus* $\mathscr{R}$, and therefore with the ∞^1 planes ξ joining these several generators to a fixed point O external to the threefold S_3 which contains $\mathscr{R}$. Such planes have no points in common at all except O, and they lie in a linear fourfold space S_4. Again all ∞^2 lines p in Λ which contain the point Q, form a bundle which may be correlated with all points of a plane ξ. Thus the lines of a special linear complex in Λ may be correlated with points of this ∞^1 of planes ξ in S_4, all containing O: in particular the line p' corresponds with O.

But such a configuration of planes ξ is provided by the intersection of the fourfold quadric R_4^2 and its tangent S_4 at a point O. The desired correlation is therefore established by this last fact, which rests on the two following theorems:

(1) *If two non-intersecting planes, ρ and σ, are taken in S_5, and a correlation is established between points R of ρ and lines s of σ, then the ∞^2 planes Rs, joining corresponding elements R, s, generate a quadric fourfold R_4^2; each plane lying entirely on the quadric.*

(2) *If O is any point of the plane σ, the ∞^1 lines s through O determine on the quadric ∞^1 planes Rs, which form the complete intersection of the tangent linear S_4 at O and the quadric.*

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[*] One system of generators of an ordinary ruled quadric surface.

The first of these is known. There is no difficulty in proving than an arbitrary line through O cuts the locus at most once elsewhere, or else lies on the locus. The second follows directly from the first.

§ 9. Lastly it seems to the writer worth while calling attention to the importance of a geometrical figure apart from its boundary or circumference. This return to the Euclidean distinction between circle and circumference appears to be not without value. It corresponds with the distinction between an algebraic *form* and the *equation*, obtained by equating the form to zero. It immediately yields a geometrical interpretation of a form whose coefficients are real even when the equation provides no real points on the circumference. There is no impropriety in calling a circle real even when it has no circumference! It also leaves the designation *imaginary* or *complex* for the geometrical figure answering to the algebraic form or equation, the ratios of whose coefficients are not all entirely real.

Theory of Statistical Estimation. By Mr R. A. FISHER, Gonville and Caius College.

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PREFATORY NOTE.

It has been pointed out to me that some of the statistical ideas employed in the following investigation have never received a strictly logical definition and analysis. The idea of a frequency curve, for example, evidently implies an infinite hypothetical population distributed in a definite manner; but equally evidently the idea of an infinite hypothetical population requires a more precise logical specification than is contained in that phrase. The same may be said of the intimately connected idea of random sampling. These ideas have grown up in the minds of practical statisticians and lie at the basis especially of recent work; there can be no question of their pragmatic value. It was no part of my original intention to deal with the logical bases of these ideas, but some comments which Dr Burnside has kindly made have convinced me that it may be desirable to set out for criticism the manner in which I believe the logical foundations of these ideas may be established.

The idea of an infinite hypothetical population is, I believe, implicit in all statements involving mathematical probability. If, in a Mendelian experiment, we say that the probability is one half that a mouse born of a certain mating shall be white, we must conceive of our mouse as one of an infinite population of mice which might have been produced by that mating. The population must be infinite for in sampling from a finite population the fact of one mouse being white would affect the probability of others being white, and this is not the hypothesis which we wish to consider; moreover, the probability may not always be a rational number. Being infinite the population is clearly hypothetical, for not only must the actual number produced by any parents be finite, but we might wish to consider the possibility that the probability should depend on the age of the parents, or their nutritional conditions. We can, however, imagine an unlimited number of mice produced upon the conditions of our experiment, that is, by similar parents, of the same age, in the same environment. The proportion of white mice in this imaginary population appears to be the actual meaning to be assigned to our statement of probability. Briefly, the hypothetical population is the conceptual resultant of the conditions which we are studying. The probability, like other statistical parameters, is a numerical characteristic of that population.

We only need the conception of an infinite hypothetical population, in connection with random sampling. The ultimate logical elucidation of the one idea implies that of the other. Also, the word infinite is to be taken in its proper mathematical sense as denoting the limiting conditions approached by increasing a finite number indefinitely. I imagine that an exact meaning can be given to all the ideas required by some process such as the following.

Imagine a population of N individuals belonging to s classes, the number in class k being $p_k N$. This population can be arranged in order in $N!$ ways. Let it be so arranged and let us call the first n individuals in each arrangement a sample of n. Neglecting the order within the sample, these samples can be classified into the several possible types of sample according to the number of individuals of each class which appear. Let this be done, and denote the proportion of samples which belong to type j by q_j, the number of types being t. Consider the following proposition.

Given any series of proper fractions $P_1, P_2, \ldots, P_s$, such that $S(P_k) = 1$, and any series of positive numbers $\eta_1, \eta_2, \ldots, \eta_k$, however small, it is possible to find a series of proper fractions $Q_1, Q_2, \ldots, Q_t$, and a series of positive numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_s$, and an integer N_0, such that, if

$$N > N_0$$

and $|p_k - P_k| < \epsilon_k$ for all values of k,

then will $|q_j - Q_j| < \eta_j$ for all values of j.

I imagine it possible to provide a rigorous proof of this proposition, but I do not propose to do so. If it be true, we may evidently speak without ambiguity or lack of precision of an infinite population characterised by the proper fractions, P, in relation to the random sampling distribution of samples of a finite size n.

It will be noticed that I provide no definition of *a* random sample, and it is not necessary to do so. What we have to deal with in all cases is a random sampling distribution of samples, and it is only as a typical member of such a distribution that a random sample is ever considered.

When in 1921 the author put forward in the *Phil. Trans.* a paper (1)* on mathematical statistics he was principally concerned, in respect of problems of estimation, with the practical importance of making estimates of high efficiency, i.e. of using statistics which embody a large proportion of the relevant information available in the data, and which ignore, or reject along with the irrelevant information, only a small proportion of that which is relevant. Many of the properties of efficient statistics, such as that even moderate inefficiency of estimation will vitiate tests of goodness of fit, were at that time unknown, and the further discrimination among statistics within the efficient group, a discrimination which is essential to the advance of the theory of small samples, was left in much obscurity. Further work along the lines of the 1921 paper has, however, cleared up the main outstanding difficulties, and seems to make possible a theory of statistical estimation with some approach to logical completeness.

1. *The problem of estimation.*

Any body of numerical observations, or qualitative data thrown into a numerical form as frequencies, may be interpreted as a random sample of some infinite hypothetical population of possible values. Problems of estimation arise when we know, or are willing to assume, the form of the frequency distribution of the population, as a mathematical function involving one or more unknown parameters, and wish to estimate the values of these parameters by means of the observational record available. A statistic may be defined as a function of the observations designed as an estimate of any such parameter. The primary qualifications of satisfactory statistics may most readily be seen by their behaviour when derived from large samples.

A statistic, to be of any service, must tend to some fixed value as the number in the sample is increased; more precisely if T be any statistic calculated from a sample of n observations, there must be a limiting value T_∞ such that if ϵ be any positive number however small, the frequency (or probability) with which $|T - T_\infty|$ exceeds ϵ, tends to zero as n tends to infinity. One example will suffice to illustrate the class of statistics which fail to fulfil this condition.

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* See numbered list of references on p. 725.

If the frequency with which the variate x falls into the range dx, be given by

$$df = \frac{1}{\pi} \cdot \frac{dx}{1 + (x - m)^2},$$

where m is the unknown parameter representing the centre of the symmetrical frequency curve of x, then it is not difficult to show that the arithmetic mean of any number of independent values of x, will be distributed in exactly the same distribution as a single value of x. If the observational material consisted of 1000 values of x, we should be able from it to estimate the value of m with some precision; but if we were to replace the actual observations by their mean, our action would be equivalent to discarding 999 of the observations, and retaining one of them chosen at random. Clearly the mean is a useless statistic for our purpose in that it does not tend to a fixed value as the size of the sample is increased.

2. *Consistent statistics.*

When T tends to a limiting value T_∞, the latter will be some determinate function of the unknown parameters. If, therefore, T is to be used for purposes of estimation, it must be equated to one particular parameter, or function of the parameters, and if it is equated to some other function its use will be inconsistent, though perhaps approximately accurate. A statistic is said to be a consistent estimate of any parameter, if when calculated from an indefinitely large sample it tends to be accurately equal to that parameter. The criterion of consistency has been widely used in the development of statistical methods, and too often it has been the only criterion employed. For example the "method of moments" consists merely in evaluating a number of arbitrarily chosen statistics, and equating as many of them as may be necessary to the corresponding series of parametric functions. Estimates of the parameters may be obtained from these equations, but they are often estimates of little value. In the example given above we have shown how little value has the mean, the first moment, in locating a particular curve, one of the Pearsonian types, in fitting which the method of moments has been so extensively used.

In a special group of cases the criterion of consistency is adequate alone to give a complete solution. If the number of frequency classes is only one greater than the number of adjustable parameters, then for each parameter there is only one consistent statistic, and this of course is the one which must be used. Generally, however, there are a great number of possible statistics available, all of them consistent, but by no means all of equal value.

3. *Efficient statistics.*

In a large and important class of consistent statistics the random sampling distribution tends to the normal (Gaussian) form as the size of the sample is increased, and in such a way that the variance (the square of the standard deviation) falls off inversely to the size of the sample. In such cases the characteristics of any particular statistic, for large samples, are completely specified by (i) its bias, and (ii) its variance. The question of bias is only of preliminary interest; if b is the mean value of $T - T_\infty$, then for consistent statistics b must tend to zero with increasing samples. If we wish to make tests of significance for the deviation of T from some hypothetical value, then b must fall off more rapidly than $n^{-\frac{1}{2}}$; if, finally, we wish to use mean values of T from a number of finite samples, then b must be actually zero, or at least a small quantity of an order determined by the number of such samples to be used. In any case a knowledge of the exact form of the distribution of T will enable us to eliminate any disadvantages from which a statistic might seem to suffer by reason of bias.

Such knowledge is, however, of no avail to repair the defects of a statistic in respect of variance. The criterion of efficiency requires that the fixed value to which the variance of a statistic (of the class of which we are speaking) multiplied by n, tends, shall be as small as possible. An efficient statistic is one for which this criterion is satisfied. If we know the variance of any efficient statistic and that of any other statistic under discussion, then the *efficiency* of the latter may be calculated from the ratio of the two values. The efficiency of a statistic represents the fraction, of the relevant information available, actually utilised, in large samples, by the statistic in question.

For example, in estimating the value of the standard deviation of a normal distribution from a sample of n values, two methods have frequently been employed. If

$$ns_1 = \sqrt{\frac{\pi}{2}}\, S\,(|\,x - \bar{x}\,|),$$

where S stands for summation over the sample, s_1 is an estimate of the true value σ, based on the method of the mean error. It has been shown (2) that the mean value of s_1 in random samples is

$$\sigma\sqrt{\frac{n-1}{n}}$$

while the variance of s_1 is

$$\frac{(n-1)\sigma^2}{n^2}\left(\frac{\pi}{2}+\sqrt{n(n-2)}-n+\sin^{-1}\frac{1}{n-1}\right).$$

If, also, s_2 is given by the equation of the mean square error

$$ns_2^2 = S(x-\bar{x})^2$$

then the mean value of s_2 is

$$\sigma\sqrt{\frac{2}{n}}\cdot\frac{\frac{n-2}{2}!}{\frac{n-3}{2}!},$$

(where $x!$ is used as equivalent to $\Gamma(x+1)$, whether x is an integer or not); while the variance of s_2 is

$$\frac{\sigma^2}{n}\left[n-1-2\left(\frac{\frac{n-2}{2}!}{\frac{n-3}{2}!}\right)^2\right].$$

The latter happens to be an efficient statistic, and for large samples the variance reduces to $\frac{\sigma^2}{2n}$ while for large samples the variance of s_1 reduces to

$$\frac{\sigma^2}{2n}(\pi-2).$$

Evidently s_1 is not an efficient statistic, but has an efficiency of nearly 88 per cent. From a body of 800 observations it will derive an estimate of about the same value, as that obtained by s_2 from 700 observations. That is to say the behaviour of the two statistics for large samples indicates that about one-eighth of the information available is rejected if s_1 is used, while if s_2 is employed the whole is retained. An exact knowledge of the distribution of s_1 would not enable us to recover the lost information, for if the sample were increased without limit, and consequently the distribution brought infinitely near to the normal form, nevertheless the fraction of information lost tends to a fixed value.

4. *Properties of efficient statistics.*

Some simple properties of efficient statistics may be derived directly from their definition, e.g. their correlational properties (3). The correlation between any two statistics, both efficient estimates of the same parameter, tends to + 1 as the sample is increased.

For if A and B be two such statistics, let the variance of each be $\frac{\sigma^2}{n}$ and the correlation between them be r; also let

$$C = \tfrac{1}{2}(A + B),$$

then C will be a statistic providing a consistent estimate of the same parameter, but the variance of C is

$$\frac{\sigma^2}{n}\left(\frac{1+r}{2}\right),$$

and this by hypothesis must not be less than $\frac{\sigma^2}{r}$, therefore r cannot be less than $+1$; but r cannot be greater than $+1$; therefore $r = +1$.

For large samples therefore all efficient statistics are equivalent, and if in practical work we were only concerned with infinitely large samples, the theory of estimation would not require development beyond the stipulation that statistics should be efficient.

If A is an efficient statistic, and B is an inefficient estimate of the same parameter with efficiency equal to E, then in large samples the correlation between A and B tends to a limit $r = +\sqrt{E}$. For if from them we compound a new statistic C, such that

$$(1 + E - 2r\sqrt{E})\,C = (1 - r\sqrt{E})\,A + (E - r\sqrt{E})\,B,$$

then C will be an estimate of the same parameter with variance

$$\frac{\sigma^2}{n}\cdot\frac{1 - r^2}{(1 - r^2) + (r - \sqrt{E})^2},$$

and if r does not tend to the limiting value $+\sqrt{E}$, this will be less than the variance of A, which is impossible; hence $r = +\sqrt{E}$. It should be noted that in making a new statistic with variance as low as that of A, when $r = +\sqrt{E}$, the above equation for C degenerates into $C = A$. In other words if we have an efficient statistic and an inefficient estimate of the same parameter, the best use we can make of these two values, at any rate with large samples, will be to ignore the latter entirely. Any compound of the two will be less efficient than A.

For example, if a quantity x be normally distributed with variance σ^2, then it is well known that the mean of a sample of n is also distributed normally with variance σ^2/n. The mean in this case is an efficient statistic; the median is a second statistic which may be used to locate the curve. If

$$\phi(a) = \sqrt{\frac{2}{\pi}}\int_0^a e^{-\frac{1}{2}t^2}\,dt$$

then if a is the central value of a sample of n values (n being odd) it appears that the probability that a shall fall into the range da is

$$\frac{n!}{\left(\frac{n-1}{2}!\right)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}} da \, . \, 2^{-(n-1)} \left\{1 - \phi^2\left(\frac{a}{\sigma}\right)\right\}^{\frac{n-1}{2}}$$

When n is large, $\phi\left(\frac{a}{\sigma}\right)$ in this expression must be small, and may be replaced by

$$\frac{a}{\sigma}\sqrt{\frac{2}{\pi}},$$

and the factor involving ϕ by

$$e^{-\frac{(n-1)a^2}{\pi\sigma^2}},$$

so that the variance of a for large samples multiplied by n tends to the limit

$$\frac{\pi}{2}\sigma^2.$$

The efficiency of the median in locating the normal curve is therefore

$$E = \frac{2}{\pi} = 63{\cdot}66 \text{ per cent.};$$

from this value may be deduced the correlation, in large samples, between the mean and the median derived from the same sample

$$r = \sqrt{E} = {\cdot}7979.$$

The median thus utilises about 64 per cent. of the information provided by the sample, its correlation with the mean of the same sample is about ·8, but any value obtained by combining the values of the median and the mean will result in an estimate inferior to that given by the mean.

A further consequence of this relation between the efficiency of a statistic and its correlation with any efficient statistic, is that if A be any efficient statistic, and B any inefficient statistic, then the difference $B - A$, will tend to zero correlation with A. We may thus divide the error of B into two parts, which in large samples at least will be independent; the first part is equal to the error in A, and is the error of random sampling properly so called; the second $B - A$ is not properly speaking an error of random sampling, but an error of estimation. It is the property of efficient statistics that, when applied to large samples, they shall have no errors of estimation of order comparable with the errors of random sampling.

In all tests of significance an observed deviation is compared with the random sampling variation to be anticipated; in tests of goodness of fit in particular the "expectation" from which the deviations are measured is usually the product of a process of estimation, the basis of which is the actual sample of observations with which the expectation is to be compared. If, therefore, the process of estimation employed involves errors of the same order as the errors of random sampling the test of goodness of fit will be vitiated; the apparent discrepancy between observation and hypothesis will in fact involve errors of estimation of the same order as the errors of random sampling to which it is to be compared. The effects of such errors upon tests of goodness of fit have been shown in more detail in (3).

5. *Derivation of efficient statistics.*

To discover the efficiency of any statistic it is necessary that we should have found at least one statistic efficient for the estimation of the same parameter, and should know the variance in large samples of the latter. We shall see that the method of maximum likelihood will always provide a statistic which, if normally distributed in large samples with variance falling off inversely to the sample number, will be an efficient statistic. The variance in large samples of such solutions may be obtained directly from the equations by which they were obtained (1).

For example, if we have a number of observations drawn from a population, of which the distribution is given by

$$df = \frac{1}{\pi} \cdot \frac{dx}{1 + (x - m)^2},$$

and wish from the observations to obtain an estimate of the value of m, we may write down in terms of m the actual probability of such a sample as ours occurring. This probability will be

$$\pi^{-n}\, dx_1 dx_2 \ldots dx_n \{1 + (x_1 - m)^2\}^{-1} \{1 + (x_2 - m)^2\}^{-1} \ldots \{1 + (x_n - m)^2\}^{-1}.$$

The likelihood of any value of m, in relation to such a sample, is defined as a quantity, of which the maximum value is unity, and which shall be proportional to the above probability. It is therefore independent of the elements $dx_1 \ldots dx_n$ which enter into the probability, but which do not involve m. *Likelihood* in this sense is not a synonym for probability, and is a quantity which does not obey the laws of probability; it is a property of the values of the parameters, which can be determined from the observations without antecedent knowledge. An exact knowledge of the likelihood of different values of m tells us nothing whatever about the probability that m will fall in any given range.

If we write for simplicity,

$$S(\log df) = L,$$

then

$$\frac{\partial L}{\partial m} = 0$$

is the equation of maximum likelihood, the solution of which gives an estimate of m, which we shall write $\hat{m}$. In the example before us this reduces to

$$S\left\{\frac{2(x-m)}{1+(x-m)^2}\right\} = 0.$$

To find the value of the variance of $\hat{m}$ derived from a large sample, it is only necessary to differentiate a second time;

$$\frac{\partial^2 L}{\partial m^2} = S\left\{\frac{2(x-m)^2-2}{\{1+(x-m)^2\}^2}\right\},$$

and for large samples the value of the right-hand side divided by n, tends to the limiting value $-\frac{1}{2}$. If $V(\hat{m})$ is the variance of $\hat{m}$, we therefore equate

$$-\frac{n}{2} = \frac{-1}{V(\hat{m})}$$

or

$$V(\hat{m}) = \frac{2}{n}.$$

Knowing this value it is easy to determine the efficiency of any other proposed statistic; in particular, since the equations of maximum likelihood do not always lend themselves to direct solution, it is of importance that, starting with an inefficient estimate, we can, by a single process of approximation, obtain an efficient estimate.

For example, if m_1, the median of the above distribution, be chosen as starting point, it is easy to show that the variance of m_1 in large samples is

$$\frac{\pi^2}{4n}$$

so that its efficiency is $8/\pi^2$. The median will differ from the maximum likelihood solution by errors of estimation, of which the variance will be

$$\frac{\pi^2-8}{4n}.$$

It is sufficient for our purpose that the error of estimation is of the order $n^{-\frac{1}{2}}$. If now we evaluate

$$S\left\{\frac{2(x-m_1)}{1+(x-m_1)^2}\right\}$$

from the observations, and calculate a new estimate m_2 from the equation

$$m_2 = m_1 + \frac{2}{n} S \left\{ \frac{2(x - m_1)}{1 + (x - m_1)^2} \right\},$$

it is easy to see that the error of estimation of m_2 will be of the order n^{-1}, and therefore that m_2 will be an efficient statistic.

6. *Intrinsic accuracy of error curves.*

The variance of efficient statistics from a distribution of any form affords us a measure of an important property of the distribution itself. The fact that from a large sample of n it is possible to obtain an estimate of the value of a parameter with variance $2/n$, shows that regarded as an error curve the above distribution is intrinsically of the same accuracy as, for example, a normal error curve with variance 2. We may thus obtain a measure of the intrinsic accuracy of an error curve, and so compare together curves of entirely different form. If the variance of an efficient estimate derived from a large sample of n is A/n, then the intrinsic accuracy of the distribution is defined as $1/A$.

If a frequency curve is defined by

$$df = y\,dx$$

where y is a function of a parameter θ, then the intrinsic accuracy of the curve, as a means of estimating θ, is

$$-\int y \frac{\partial}{\partial \theta^2} (\log y) \,.\, dx$$

over the whole range of possible values. Since

$$y \frac{\partial^2}{\partial \theta^2} (\log y) = \frac{\partial^2 y}{\partial \theta^2} - \frac{1}{y} \left(\frac{\partial y}{\partial \theta} \right)^2,$$

while the integral of the first term over all possible values must vanish, the intrinsic accuracy may equally be written

$$\int \frac{1}{y} \left(\frac{\partial y}{\partial \theta} \right)^2 dx,$$

over all possible values; in this form it is clearly seen to be necessarily positive.

What we have spoken of as the intrinsic accuracy of an error curve may equally be conceived as the amount of information in a single observation belonging to such a distribution. If for instance two independent observations were available from the same or different distributions, the distribution of the *pair* of values would be

$$df = yy'\,dx\,dx',$$

and the intrinsic accuracy of such a pair would be

$$-\iint yy'\left(\frac{\partial^2}{\partial\theta^2}\log y+\frac{\partial^2}{\partial\theta^2}\log y'\right)dx\,dx',$$

which, with the identities,

$$\int y dx = 1,\quad \int y' dx' = 1,$$

reduces to $$-\int y\,\frac{\partial^2}{\partial\theta^2}\log y\,.\,dx-\int y'\,\frac{\partial^2}{\partial\theta^2}\log y'\,.\,dx';$$

the amount of information provided by a combination of two or more independent observations is thus merely the sum of the amounts of information in each piece separately.

It is a common case for a sample of n observations to be distributed into a finite number of classes, the numbers "expected" in each class being functions of one or more unknown parameters, if p is the probability of an observation falling into any one class, the amount of information in the sample is

$$S\left\{\frac{1}{m}\left(\frac{\partial m}{\partial\theta}\right)^2\right\}$$

where $m = np$, is the expectation in any one class. The variance of an efficient statistic derived from a large sample may, of course, be calculated from this expression.

7. *Efficiency of the maximum likelihood solution.*

We shall now prove that when an efficient statistic as defined above exists one may be found by the method of maximum likelihood.

If f stand for the probability that any particular type of observation should occur, and ϕ for the probability that any particular type of sample should occur, then

$$\log\phi = C + S(\log f)$$

when C is a constant which does not involve the parameters, the summation extending over all observations.

As regards the variation of ϕ with varying θ, it is to be noted that

$$\frac{1}{n}\frac{\partial^2}{\partial\theta^2}\log\phi=\frac{1}{n}S\left(\frac{\partial^2}{\partial\theta^2}\log f\right)$$

will tend to a fixed limit for large samples. Set $(-A)$ for this limiting value. Then since

$$\frac{\partial}{\partial\theta}\log\phi = 0,$$

when $\theta = \hat{\theta}$, and $\hat{\theta}$ is the solution of the equations of maximum ikelihood, it follows that

$$\frac{\partial}{\partial\theta}\log\phi = -nA(\theta - \hat{\theta})$$

f $\theta - \hat{\theta}$ is a small quantity of order $n^{-\frac{1}{2}}$.

Now if T is any statistic used as an estimate of θ, the probability, Φ, of T having any assigned value, will be the sum of the probabilities of those samples which yield the said value T, that is

$$\Phi = S(\phi)$$

when S stands for summation over all the samples which yield the same value for the statistic T. Also, if T is in large samples normally distributed with variance σ^2,

$$\Phi = \frac{1}{\sqrt{2\pi}} e^{-\frac{(T-\theta)^2}{2\sigma^2}} dT,$$

whence
$$\frac{\partial^2}{\partial\theta^2}\log\Phi = -\frac{1}{\sigma^2}.$$

The problem of making σ^2 as small as possible, is the problem of so grouping the several sorts of samples under the same values of the estimate T, that the second differential coefficient of $\log\Phi$ shall be a negative quantity as large as possible. Now

$$\frac{1}{\sigma^2} = -\frac{\partial^2}{\partial\theta^2}\log\Phi = \frac{S^2(\phi')}{S^2(\phi)} - \frac{S(\phi'')}{S(\phi)}$$

and $S(\phi') \div S(\phi)$ is the mean value, within the group, of $-nA(\theta - \hat{\theta})$, while $S(\phi'') \div S(\phi)$ is the mean value of

$$-nA + n^2A^2(\theta - \hat{\theta})^2,$$

consequently
$$\frac{1}{n\sigma^2} = A - nA^2V'(\hat{\theta})$$

when $V'(\hat{\theta})$ is the variance of $\hat{\theta}$ within the group. If $T = \hat{\theta}$, then $\hat{\theta}$ will be constant within the group, so that the variance of $\hat{\theta}$ in random samples will be $1/nA$. For any statistic T which has the same value in sets of samples for which the variance of $\hat{\theta}$ is of order n^{-1} the value of $1/n\sigma^2$ will be reduced, for the variance within the group is necessarily a positive quantity, and consequently the variance of any such statistic will be greater than that of $\hat{\theta}$.

8. *Efficiency of weighting.*

The effects of the familiar process of weighting observations may be well shown in terms of efficiency. If w is the weight of a normally distributed observation x, so that its variance is $1/w$, and if a number of such observations be combined with false weights w', then the variance of the weighted mean will be

$$\frac{1}{S^2(w')}.S\left(\frac{w'^2}{w}\right),$$

and, when $w' = w$, this reduces to $1/S(w)$, the minimum value.

The efficiency is therefore

$$\frac{S^2(w')}{S(w).S\left(\frac{w'^2}{w}\right)}.$$

The loss of weight is

$$S(w) - \frac{S^2(w')}{S\left(\frac{w'^2}{w}\right)}.$$

If now the inaccuracy in weighting is due to a slight variation in w, and we have chosen weights w' equal to the mean values of w, then

$$w = w' + \epsilon,$$

$$S\left(\frac{w'^2}{w}\right) = S(w') - S(\epsilon) + S\left(\frac{\epsilon^2}{w'}\right) - \ldots,$$

$$S^2(w') \div S\left(\frac{w'^2}{w}\right) = S(w') + S(\epsilon) - S\left(\frac{\epsilon^2}{w'}\right) + \ldots,$$

whence the mean loss of weight is, approximately

$$S\left(\frac{1}{w'}V(w)\right),$$

when $V(w)$ is the variance of w.

9. *Small samples; Sufficient statistics.*

It is now possible to approach the more general problem of the estimation of statistics from finite samples, when the distributions of the statistics considered will not generally be of the normal form, nor will the errors of random sampling be small quantities. The different possible efficient statistics will no longer be equivalent. and it will be necessary to discriminate among them. In previous work on this subject(1) two circumstances seemed to point to fruitful lines of development. In the first place attention was called to a class of statistics possessing very remarkable properties, which contain in themselves the whole of the relevant information available in the data. These statistics were therefore distinguished by the term sufficient. In the second place it was suggested that the idea of intrinsic accuracy might be applied to the random sampling distributions of statistics when these were not normal, so as to afford a method of comparing their relative values.

As an example of a sufficient statistic consider the mean of the Poisson Series. A variate, confined to whole numbers, is distributed in a Poisson Series if the probability of its taking any particular value, x, is

$$e^{-m}\cdot\frac{m^x}{x!}.$$

The parameter m may be estimated from the mean of the observed sample. This is evidently the solution of the equation of maximum likelihood. If $\bar{x}$ is the mean of a sample of n, the distribution of $n\bar{x}$ may readily be proved to be given by the Poisson Series

$$e^{-nm}\frac{(nm)^{n\bar{x}}}{(n\bar{x})!}.$$

Now the probability of drawing in order any particular sample $x_1, x_2, \ldots, x_n$ is

$$e^{-nm}\frac{m^{n\bar{x}}}{x_1!\,x_2!\ldots x_n!},$$

and this may be divided into two factors,

$$e^{-nm}\frac{(nm)^{n\bar{x}}}{(n\bar{x})!}\cdot\frac{(n\bar{x})!}{n^{n\bar{x}}\,x_1!\,x_2!\ldots x_n!},$$

of which the first represents the probability that the actual total $n\bar{x}$ should have been scored, and the second the probability, given this total, that the partition of it among the n observations should be that actually observed. In the latter factor, m, the parameter sought, does not appear. Now when the mean is known any further information which the sample has to give must depend on the observed partition; but the probability of any particular partition is wholly independent of the value of m. Consequently no statistic calculated from the sample can give any information whatever respecting the value of m, beyond that supplied by the value of the mean.

In general, if θ is any parameter, T_1 a statistic sufficient in estimating that parameter, and T_2 any other statistic, the sampling distribution of simultaneous values of T_1 and T_2 must be such that for any given value of T_1, the distribution of T_2 does not involve θ. This will evidently be the case, if $f(\theta, T_1, T_2)\,dT_1\,dT_2$ be the probability that T_1 and T_2 should fall in the ranges dT_1, dT_2, and if

$$f(\theta, T_1, T_2) = \phi(\theta, T_1)\cdot\phi'(T_1, T_2).$$

If this condition is fulfilled for all possible statistics T_2, then will T_1 be a sufficient statistic.

When a sufficient statistic exists it is equivalent, for all subsequent purposes of estimation, to the original data from which it was derived. For example the mean of a normal sample is a sufficient statistic, and the mean possesses the property that it can be combined with the mean of a second sample from the same population to find the mean of the combined sample. If $f(x_1, \dots, x_n)$ be for any distribution a sufficient estimate of some parameter, then

$$f(x_1, \dots, x_n) = \phi\{f(x_1, \dots, x_p), f(x_{p+1} \dots, x_n)\};$$

this circumstance much limits the functions which can possibly be sufficient statistics.

For example, the function

$$\frac{1}{k}\log S(e^{kx}) - \frac{1}{k}\log n,$$

is a function which might, for the right distribution, be a sufficient statistic. As k is made to increase without limit the above function tends to be simply the greatest value observed in the sample; just as the mean of a number of means is the mean of the aggregate, so the greatest of a series of greatest observations, will be the greatest of the aggregate.

When sufficient statistics exist it has been shown that they will be solutions of the equations of maximum likelihood.

10. *Intrinsic accuracy of error curves of statistics.*

The fact that sufficient statistics do not always exist renders it necessary to explore the possibilities of comparing statistics by means of the intrinsic accuracy of their random sampling distributions.

We may, in fact, give an extended meaning to the word efficiency by the definition

The efficiency of a statistic is the ratio of the intrinsic accuracy of its random sampling distribution to the amount of information in the data from which it has been derived.

This definition is in accordance with the definition previously given of efficiency for the case of large samples with normally distributed statistics. For if $\frac{\sigma^2}{n}$ is the variance in large samples of an efficient statistic, the intrinsic accuracy of the original distribution will be $1/\sigma^2$, and the whole information in the data will be n/σ^2. Moreover if in large samples any statistic has variance σ^2/En, its intrinsic accuracy will be En/σ^2, and its efficiency, by either definition, will be E.

The extended definition has the advantage of applying to finite samples and to other cases where the distribution is not normal.

As an example of the calculation of the efficiency of statistics derived from finite samples, consider the median of an odd number, $2s + 1$, of observations the error curve of which is given by

$$df = \frac{1}{\pi} \cdot \frac{dx}{1 + (x - m)^2}.$$

It is easy to see that the frequency distribution of the median will be given by

$$df = \frac{(2s + 1)!}{(s!)^2 \pi^{2s+1}} \left(\frac{\pi^2}{4} - \theta^2\right)^s \frac{dx}{1 + (x - m)^2},$$

where $\tan \theta = x - m$, and θ lies between $\pm \frac{1}{2}\pi$.

To find the intrinsic accuracy of the distribution, we differentiate the logarithm of the above expression with respect to the unknown parameter, m; then since

$$\frac{d\theta}{dm} = -\cos^2\theta$$

we have

$$\frac{2s\theta \cos^2 \theta}{\frac{\pi^2}{4} - \theta^2} + \sin 2\theta.$$

The intrinsic accuracy will be the average value of the square of this quantity, or

$$\frac{(2s + 1)!}{(s!)^2 \pi^{2s+1}} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left\{2s\theta \cos^2 \theta + \left(\frac{\pi^2}{4} - \theta^2\right) \sin 2\theta\right\}^2 \left(\frac{\pi^2}{4} - \theta^2\right)^{s-2} d\theta.$$

The definite integral may be expressed in terms of the Bessel functions of π and 2π, in the form

$$\frac{1}{2} + \frac{3s(2s + 1)}{2(s - 1)\pi^2} + \frac{(s + \frac{1}{2})!}{2s - 1} \cdot \left(\frac{2}{\pi}\right)^{s+\frac{1}{2}} \left\{\frac{2s}{s - 1} J_{s-\frac{3}{2}}(\pi) - 2J_{s+\frac{1}{2}}(\pi)\right\}$$
$$\frac{(s + \frac{1}{2})!}{2s - 1} \cdot \left(\frac{1}{\pi}\right)^{s+\frac{1}{2}} \left\{\frac{2s}{s - 1} J_{s-\frac{3}{2}}(2\pi) - \frac{2s + 3}{2} J_{s+\frac{1}{2}}(2\pi)\right\}.$$

The Bessel functions are easily evaluated, for $J_{\frac{1}{2}} = 0$, for both values of the argument; while the values of $J_{\frac{3}{2}}$ are $\sqrt{2}/\pi$ and $-1/\pi$, the others being thence obtainable by the recurrence formula

$$J_{n+1} = \frac{2n}{z} J_n - J_{n-1}.$$

Thus for $s = 2$, we find the intrinsic accuracy of the median of five observations to be

$$\frac{1}{2} + \frac{15}{\pi^2} - \frac{855}{32\pi^4}.$$

The following table shows the numerical values.

Number in sample	s	Accuracy	Increase for two observations	Efficiency
1	0	·50000	·59064	100·00 %
3	1	1·09064	·65488	72·71 %
5	2	1·74552	·69490	69·82 %
7	3	2·44042	·72121	69·73 %
9	4	3·16164	·73945	70·26 %
11	5	3·90109		70·93

The efficiency appears to have a minimum between 5 and 7 observations, and is not approaching its limiting value, 81·06 per cent., very rapidly. This is perhaps to be anticipated since for large samples it falls short of its limiting efficiency by $1{\cdot}19/s$, and the discrepancy in the above table is considerably less than this.

Statistics which are efficient for large samples may, of course, have comparatively low efficiencies for finite samples, and in certain cases the efficiency may tend to its limiting value so slowly that even samples of over 100 are not very efficiently treated. The median, for example, is an efficient statistic for locating the centre of the double exponential curve,

$$df = \tfrac{1}{2} e^{-|x-m|}\, dx,$$

when the sample is increased without limit, but owing to the discontinuity at the apex, its efficiency approaches its limiting value somewhat slowly. The intrinsic accuracy of the original distribution is unity, and that of the median of $(2s + 1)$ observations may be shown to be

$$\frac{(s+1)(2s+1)}{s-1}\left\{1 - \frac{(2s)!}{2^{2s-1}(s!)^2}\right\}.$$

The numerical values are:

Number in sample	s	Accuracy	Efficiency %	Loss of information
1	0	1	100	0
3	1	2·3178	77·26	·6822
5	2	3·7500	75·00	1·2500
7	3	5·2500	75·00	1·7500
9		6·7969	75·52	2·2031
19		14·940	78·63	4·060
33	6	26·932	81·61	6·068
51	25	42·844	84·01	8·156
73	36	62·709	85·90	10·291
99	49	86·544	87·42	12·456
129	64	114·36	88·65	14·64
163	81	146·16	89·67	16·84
201	100	181·95	90·52	19·05

The case is unusual in that the loss of information does not tend to a fixed limit, but increases ultimately as $4\sqrt{s/\pi}-4$; the cause of this exceptional behaviour lies in the fact that at the apex of the curve the second differential coefficient with respect to the unknown, m, is infinite. The example stresses the importance of investigating the actual behaviour of statistics from finite samples, instead of relying wholly upon their calculated behaviour in infinitely large samples.

We can now prove in general that the efficiency can never exceed unity, and derive the condition that there shall be no loss of information.

If the probability that any statistic, T, should take a particular value is Φ, then the intrinsic accuracy of the distribution of T is

$$S\left\{\frac{\Phi'^2}{\Phi}\right\},$$

the summation being taken over all possible values. If now every possible sample, having frequency ϕ gave a different value of T, then the intrinsic accuracy would be

$$S\left\{\frac{\phi'^2}{\phi}\right\}$$

and would be independent of T. If, however, a number of different samples give the same value of T, then the effect of this amalgamation will be to decrease the intrinsic accuracy by the amount

$$S\left(\frac{\phi'^2}{\phi}\right)-\frac{\Phi'^2}{\Phi},$$

$$S\left\{\phi\left(\frac{\phi'}{\phi}-\frac{\Phi'}{\Phi}\right)^2\right\}.$$

This quantity is never negative, so that the intrinsic accuracy of T can never be greater than when every possible sample yields a different value of T. This is obvious because, in such a case, the actual sample can be reconstructed without ambiguity from the value of T, and so the value of T, which is merely a kind of short-hand statement of the original sample, must contain the whole of the information provided by the sample.

The condition that there shall be no loss of information when different samples give the same value of T is that the sets of possible samples for which T is constant shall be those for which

$$\frac{\phi'}{\phi}=\frac{\partial L}{\partial\theta}$$

is constant. If these sets are the same for all values of θ, then the equation of maximum likelihood

$$\frac{\partial L}{\partial \theta} = 0$$

will provide a sufficient statistic.

For if this is the case $\partial L/\partial \theta$ depends, apart from θ, only on the set to which the sample belongs; in other words it is a function of θ and $\hat{\theta}$ only. Thus if f is the frequency with which any sample, or group of samples having the same $\hat{\theta}$, occurs, then

$$\frac{1}{f} \cdot \frac{\partial f}{\partial \theta} = \phi(\theta, \hat{\theta}),$$

now let the frequency of samples such that $\hat{\theta}$ lies in the range $d\hat{\theta}$ and a second statistic, T, lies in the range dT, be $f(\theta, \hat{\theta}, T)\, d\hat{\theta}\, dT$, then since the above equation will be true for all values of θ, we shall integrate it with respect to θ and obtain

$$\log f = \int \phi(\theta, \hat{\theta})\, d\theta + C$$

where C does not involve θ, and is a function therefore of $\hat{\theta}$ and T only. Hence f is of the form

$$\phi(\theta, \hat{\theta}) . \phi'(\hat{\theta}, T)$$

whatever statistic may be taken as T, and so $\hat{\theta}$ must be a sufficient statistic.

11. *Minimal loss of accuracy.*

When the sets of samples which for one value of θ have the same value of $\partial L/\partial \theta$, have no longer the same value for other values of θ, there exists no sufficient statistic, and some loss of information will necessarily ensue upon the substitution of a single estimate for the original data upon which it was based.

The extent of this loss, in large samples, for which presumably it will be greatest, may now be calculated. If the sample consist of observed numbers $x_1, \ldots x_s$ in categories in which the expectations are $m_1, \ldots, m_s$, then

$$L = S(x \log m),$$

$$\partial L/\partial \theta = S\left(x \frac{m'}{m}\right),$$

$$\partial^2 L/\partial \theta^2 = S\left\{x\left(\frac{m''}{m} - \frac{m'^2}{m^2}\right)\right\},$$

if now $\partial L/\partial\hat{\theta} = 0$, then to a first approximation

$$\frac{\partial L}{\partial \theta} = (\theta - \hat{\theta}) \frac{\partial^2 L}{\partial \theta^2},$$

and the variance of $\partial L/\partial\theta$ in a set of samples for which $\hat{\theta}$ is constant, will be given by the variance of $\partial^2 L/\partial\theta^2$ within the set multiplied by $(\theta - \hat{\theta})^2$, or the total loss of information will be given by the general variance within such sets multiplied by $V(\hat{\theta})$.

Now the random sampling distribution of the values of x will be the multinomial distribution, and the simplest method of regarding this distribution is to consider each value of x independently distributed in a Poisson Series about a mean value m; the whole being subject to the restriction that

$$S(x) = S(m) = n.$$

In such a system any quantity $S(kx)$ is easily seen to have a mean value $S(km)$; its variance, if there were no restriction, would be $S(k^2m)$, and in introducing any linear restriction we have only to remove that portion of the variance produced by varying in the prohibited direction. Two restrictions are here necessary, for the first

$$S(x) = n$$

we have to deduct $\qquad S^2(km) \div n.$

The second restriction arises from the fact that we require the variance within the groups for which $\partial L/\partial\hat{\theta}$ is constant, since

$$\frac{\partial L}{\partial \theta} = S\left(x . \frac{m'}{m}\right),$$

the deduction will be $\quad S^2(km') \div S\left(\dfrac{m'^2}{m}\right).$

Writing

$$k = \frac{m''}{m} - \frac{m'^2}{m^2},$$

and remembering that $V(\hat{\theta}) = 1/S\left(\dfrac{m'^2}{m}\right)$, we have for the loss of information in large samples,

$$\frac{S\left\{\dfrac{1}{m}\left(m'' - \dfrac{m'^2}{m}\right)^2\right\}}{S\left(\dfrac{m'^2}{m}\right)} - \frac{1}{n} S\left(\frac{m'^2}{m}\right) - \frac{S^2\left\{\dfrac{m'}{m}\left(m'' - \dfrac{m'^2}{m}\right)\right\}}{S^2\left(\dfrac{m'^2}{m}\right)}.$$

The method of calculation by differences has the advantage that if, by the estimation of other parameters, further restrictions upon variation are introduced, we may choose such parameters that their maximum likelihood estimates will be in large samples uncorrelated with each other and with $\hat{\theta}$, and the whole effect of the restrictions will be further to reduce the loss of accuracy by terms of the form

$$\frac{S^2\left\{\frac{1}{m}\frac{\partial m}{\partial\phi}\left(m''-\frac{m'^2}{m}\right)\right\}}{S\left\{\frac{1}{m}\left(\frac{\partial m}{\partial\phi}\right)^2\right\}.S\left(\frac{m'^2}{m}\right)},$$

without further examination of the restrictions.

12. *Example of loss of intrinsic accuracy.*

In the curve $$df=\frac{1}{\pi}.\frac{dx}{1+(x-\theta)^2}$$

we may write $$m=\frac{n}{\pi}.\frac{dx}{1+(x-\theta)^2};$$

then for the determination of θ,

$$S\left(\frac{m'^2}{m}\right)=\frac{n}{\pi}\int_{-\infty}^{\infty}\frac{4t^2dt}{(1+t^2)^3}=\frac{n}{2},$$

$$S\left\{\frac{1}{m}\left(m''-\frac{m'^2}{m}\right)^2\right\}=\frac{n}{\pi}\int_{-\infty}^{\infty}\frac{4(t^2-1)^2}{(1+t^2)^5}dt=\frac{7n}{8},$$

$$S\left\{\frac{m'}{m}\left(m''-\frac{m'^2}{m}\right)\right\}=0.$$

The loss of information is therefore

$$\tfrac{7}{4}-\tfrac{1}{2}=\tfrac{5}{4},$$

and since the intrinsic accuracy of the original distribution is $\frac{1}{2}$, the loss on statistical reduction is equivalent in large samples to $2\frac{1}{2}$ observations. For small samples the loss will presumably be less since it vanishes for samples of one.

In the location of the centre of this curve, therefore, we see that the mean is a statistic which throws away the greater part of the information available; the median is an inefficient statistic which makes use of a fraction, approaching in large samples the limit $8/\pi^2$ of the information. The solution of the equation of maximum likelihood, like other efficient statistics makes use of all but an amount which tends to a finite limit as the sample is increased. The amount lost may differ in different efficient statistics; it will be least for the solution of the equation of maximum likelihood.

13. *Loss of accuracy with other statistics.*

With efficient statistics other than the maximum likelihood solution, the loss of accuracy will be somewhat greater, though still tending to a finite limit for large samples. The variance of $\partial L/\partial\theta$ for sets of samples yielding the same statistic will be due to two independent causes. The first depends upon the sampling error, $T - \theta$, and upon the fact that $\partial^2 L/\partial\theta^2$ is not the same for all samples yielding the same statistic. Since all efficient statistics tend to equivalence with increasing samples, this portion will be the same for all efficient statistics. The second portion is approximately independent of the sampling error, and depends upon the deviation of T from the maximum likelihood solution $\hat{\theta}$.

For example, if $\chi^2 = S\left\{\dfrac{(x-m)^2}{m}\right\}$,

the equation for minimum χ^2,

$$S\left(\frac{x^2 - m^2}{m^2}\frac{\partial m}{\partial\theta}\right) = 0,$$

gives an efficient estimate of θ; for the maximum likelihood equation is

$$S\left(\frac{x - m}{m}\frac{\partial m}{\partial\theta}\right) = 0,$$

and the ratio $\dfrac{x+m}{m}$ tends to the constant value, 2, for large samples. Now, since

$$(x^2 - m^2) = 2m(x - m) + (x - m)^2,$$

the deviation in $\partial L/\partial\theta$ will be

$$\tfrac{1}{2}S\left\{\frac{(x-m)^2}{m^2}\frac{\partial m}{\partial\theta}\right\},$$

and it is the variance of this quantity for samples of equal sampling error which gives the loss of information. By the same device as before we evaluate the variance of $S\{k(x-m)^2\}$ in the form

$$2S(k^2m^2) - \frac{2}{n}S^2(km^2) - 2\frac{S^2(km'^2)}{S^2\left(\dfrac{m'^2}{m}\right)},$$

or, substituting $$k = \frac{1}{2}\frac{m'}{m^2},$$

we have $$\tfrac{1}{2}S\left(\frac{m'^2}{m^2}\right) - \frac{1}{2}\frac{S^2\left(\dfrac{m'^3}{m^2}\right)}{S^2\left(\dfrac{m'^2}{m}\right)}.$$

This quantity remains finite as the size of the sample is increased without limit, but increases without limit as the number of classes is increased. Consequently, as one might expect, the method of minimising χ^2 breaks down for fine grouping.

For example, suppose we have 5 classes only, in a population distributed according to the binomial distribution

$$(p+q)^4.$$

Let p be calculated from numbers observed in the 5 possible classes in a large sample. Then

$$m_1 = np^4, \quad m_2 = 4np^3q, \text{ etc.},$$

and the intrinsic accuracy is

$$S\left(\frac{m'^2}{m}\right) = \frac{4n}{pq}.$$

The loss of information due to using the minimum χ^2 solution is calculated from

$$\tfrac{1}{2}S\left(\frac{m'^2}{m^2}\right) = \frac{5}{p^2q^2}(3p^2 - 2pq + 3q^2)$$

and

$$S\left(\frac{m'^3}{m^2}\right) = -\frac{4(p-q)}{p^2q^2}(p^4 - 2p^3q + 18p^2q^2 - 2pq^3 + q^4)\,n,$$

and is in fact

$$\frac{5}{p^2q^2}(3p^2 - 2pq + 3q^2) - \frac{(p-q)^2}{2p^2q^2}(p^4 - 2p^3q + 18p^2q^2 - 2pq^3 + q^4)^2.$$

This is least when $p = q$, and is then equal to

$$\frac{20}{pq},$$

or equivalent to the loss of 5 observations.

In approaching the maximum likelihood solution by successive approximations we have seen that starting with an inefficient statistic, a single process of approximation will in ordinary cases give an efficient statistic differing from the maximum likelihood solution, by a quantity which with increasing samples decreases as n^{-1}. The loss of information of such efficient statistics is therefore finite for large samples, for the additional variance of $\partial L/\partial\theta$ will be

$$L''^2 V(T - \hat{\theta}),$$

and L'' increases proportionately to the sample. If the process of approximation be repeated a statistic is obtained differing from the maximum likelihood solution only to the order of $n^{-\frac{3}{2}}$, and for such a statistic the loss of accuracy, beyond that suffered by the maximum likelihood solution will tend to zero for large samples.

The practical procedure of fitting will thus not ordinarily require more than a second process of approximation.

14. *Theoretical existence of fully accurate statistics.*

In the manner in which we have developed the theory it would appear that the loss of information inherent in the process of replacing a quantity of observational material by a single value, arose from the circumstance that the groups of samples, which ought to give us the same estimate, change with the value of the unknown parameter, and so that no way of grouping can be the best for all values of the parameter. The method of maximum likelihood takes that way of grouping which is most accurate for the particular value which is equal to the estimate arrived at. The loss of information with which we are concerned is the difference in accuracy between the solution of the equations of maximum likelihood, and another statistic which might conceivably be arrived at by chance, but which cannot be specified without knowledge of the true value of the parameter.

If, for example, the quantity θ in the following expression *happened* to be equal to the value of m in the population specified by

$$df = \frac{1}{\pi} \frac{dx}{1 + (x - m)^2},$$

and we used the equation of estimation

$$S \left\{ \frac{2(x - \theta)}{1 + (x - \theta)^2} \right\} = nf(T),$$

where it will be noticed that the left-hand side is $\partial L/\partial\theta$, then $\partial L/\partial\theta$ is constant for samples giving the same T; the form of f can be ascertained from the condition of consistency, for the equation will only be consistent if

$$f(m) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2(x - \theta)}{1 + (x - \theta)^2} \cdot \frac{dx}{1 + (x - m)^2} = \frac{2(m - \theta)}{(m - \theta)^2 + 4}.$$

The equation for T will therefore be

$$S \left\{ \frac{2(x - \theta)}{1 + (x - \theta)^2} \right\} = \frac{2n(T - \theta)}{(T - \theta)^2 + 4};$$

this equation is nearly equivalent to the equation we have given for improving an approximate value, in this case θ. If, however, θ were in fact equal to the true value of the parameter, the statistic T would be distributed in random samples with an intrinsic accuracy equal to the maximum possible. We cannot, however, utilise this fact, for if we could rely upon our putting the right value of θ into this equation, we could choose θ as our estimate and so avoid errors of random sampling altogether.

15. *Use of ancillary statistics.*

Since the original data cannot be replaced by a single statistic, without loss of accuracy, it is of interest to see what can be done by calculating, in addition to our estimate, an ancillary statistic which shall be available in combination with our estimate in future calculations.

If our two statistics specify the values of $\partial L/\partial\theta$ and $\partial^2 L/\partial\theta^2$ for some central value of θ, such as $\hat{\theta}$, then the variance of $\partial L/\partial\theta$ over the sets of samples for which both statistics are constant, will be that of

$$\tfrac{1}{2}(\theta - \hat{\theta})^2 \frac{\partial^3 L}{\partial\theta^3},$$

which will ordinarily be of order n^{-1} at least. With the aid of such an ancillary statistic the loss of accuracy tends to zero for large samples.

The function of the ancillary statistic is analogous to providing a true, in place of an approximate, weight for the value of the estimate. If a number of large samples were available, and if

$$M = S(L)$$

when the summation is taken over all the samples, then M will be the logarithm of the likelihood of θ from the unbiassed sample; but necessarily

$$M' = S(L'), \quad M'' = S(L'')$$

and if $\hat{\theta}$ be the value of θ for which M' vanishes, and $\hat{\theta}_p$ the value for which L_p' vanishes, then with large samples, when $\theta = \hat{\theta}$,

$$L_p' = (\hat{\theta} - \hat{\theta}_p)\, L_p''.$$

Hence $\hat{\theta}$ is given by the equation

$$S\{L''(\hat{\theta} - \hat{\theta}_p)\} = 0,$$

or $$M''\hat{\theta} = S(L_p''\hat{\theta}_p),$$

where $$M'' = S(L'').$$

If we had ignored the ancillary statistic and taken as weights the mean values

$$-\overline{L''} = +\frac{1}{V(\hat{\theta})},$$

the loss of weight in the combined value,

$$S\left(\frac{1}{w'}V(w)\right),$$

is the sum of the contributions to the loss of weight from the several samples. Each contribution is equal to the sampling variance of L'' multiplied by $V(\hat{\theta})$, and this is just the quantity we have found as measuring the loss of information.

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On the idea of frequency. By Mr W. BURNSIDE, Honorary Fellow of Pembroke College.

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A statistical table is in effect a classification of a finite number, N, of objects in respect of a finite number of different classes A, B, C, ... It is assumed that unambiguous rules have been laid down by which it is possible to determine whether any particular one of the objects does or does not belong to any particular one of the classes. The application of these rules to a particular object does not depend on the fact that that object is one of a finite number of N objects, so that the process of classification may be started on a non-finite collection of objects. When the collection of objects is non-finite the process can never end. On the other hand when the collection is finite the process must end; and when completed it will determine how many of the objects belong to any particular class. If in this way it is found that N_1 ($\leqslant N$) of the objects belong to class A, the proper fraction N_1/N is spoken of as the frequency of class A in the collection. In particular cases it may be zero or unity. In general it is a rational proper fraction. For its determination in complicated cases it may be convenient to suppose the collection to be arranged in some special way, but its value is absolutely independent of any such particular arrangement.

The question it is proposed to discuss here is whether this conception of the frequency of a class in a finite collection can be legitimately extended to the case of a non-finite collection. There will probably be many who will wonder why such a question should be asked at all, when the answer must so obviously be in the negative. On the other hand there are writers on theoretical statistics who assume that the answer must be equally obviously in the affirmative. Thus Mr R. A. Fisher[*] writes, "This object is accomplished by constructing a hypothetical infinite population of which the actual data are regarded as constituting a random sample. The law of distribution of this hypothetical population is specified by relatively few parameters, which are sufficient to describe it exhaustively in respect of all qualities under discussion."

It may be noticed that there is nothing in this passage, or in others with which I have been able to compare it, to show whether the infinite population, to be constructed, is or is not enumerable.

The question to be discussed may be approached from two distinct points of view. One may either attempt to construct (from Mr Fisher's standpoint) a non-finite collection in which a given class has a given frequency: or starting from a given non-finite collection one may attempt to determine the frequency of a particular class.

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[*] *Phil. Trans.* vol. 222, p. 311.

If with M white objects we combine an equal number M of black objects, we form a finite collection of white and black objects, in which the frequency of the class of white objects is $\frac{1}{2}$. This is true whatever finite integer M may be. If I understand Mr Fisher aright, he states that starting from this fact, and supposing M to increase without limit, we obtain a definite conception of an infinite collection of white and black objects in which the frequency of the class of white objects is $\frac{1}{2}$. So long as M is finite the statement that the number of white objects is equal to the number of black objects had a definite meaning and its truth can be checked. Has the statement "the non-finite number of white objects is equal to the non-finite number of black objects" a definite meaning, and can it be checked? If not the process of increasing M without limit does *not* lead to a definite conception of a non-finite collection of whites and blacks in which the whites have the frequency $\frac{1}{2}$. Put more shortly Mr Fisher's process involves the assumption that the idea of "equality in number" can be extended from finite to non-finite collections.

Taking now the second point of view, it may be asked what is the frequency of even positive integers in the totality of all positive integers? Here there are no such finite numbers as the N and N_1 in the original definition. On the other hand N and N_1 have definite values for any finite collection of distinct integers however large. Hence a possible way of answering the question is to form N_1/N for a large collection and then to suppose the size of the collection to increase without limit. It is however clear that the limit that N_1/N approaches, if it approaches one at all, will depend on the way in which the successive finite collections are formed. If we take for the collection N the first N integers *in order of magnitude*, the limit of N_1/N is $\frac{1}{2}$. It is perfectly simple to arrange the integers in such an order that the limit of N_1/N is $\frac{1}{3}$ or any other proper fraction; or again to arrange them so that N_1/N oscillates continually as N increases. This method then of attempting to answer the question of what is the frequency of even integers in the totality of all integers, certainly will not lead to a result. Until a unique value, which cannot be disputed, has been found for this frequency is there any reason for supposing that the phrase "the frequency of even integers in the totality of all integers" has any meaning at all?

If the non-finite collection is not enumerable the difficulties of attaching a definite meaning to a frequency would appear even more formidable. For instance—In the totality of real numbers between zero and unity what is the frequency of those lying between ·1 and ·2?

The bursting of soap-bubbles in a uniform electric field. By Professor C. T. R. Wilson and Mr G. I. Taylor, Trinity College.

[*Read* 8 December, 1924.]

(Plates VI and VII.)

The stability of a charged raindrop has been discussed mathematically by Lord Rayleigh. The case of an uncharged drop in a uniform electric field is perhaps of more meteorological importance[*] but a mathematical discussion of the conditions for stability turns out to be very much more difficult in this case, owing to the fact that the drop ceases to be spherical before it bursts. Moreover it does not seem possible to express its geometrical shape by means of any simple mathematical expressions. On the other hand, by using a soap bubble instead of a water drop it was found possible to carry out experiments under well-defined conditions in this case, whereas experiments with Rayleigh's charged drop would be difficult.

The method adopted in the present experiments was to place a soap bubble of measured volume on a horizontal wet aluminium plate. The half bubble thus formed was subjected to the influence of a uniform electric field by placing a second parallel plate above it and maintaining a known potential difference between the two. This is evidently equivalent to experimenting with a complete bubble of twice the volume freely floating in a uniform field. In the former case the film is at right angles to the wet plate along the circle of contact, and the plate, with the half bubble, forms an equipotential surface corresponding with the complete bubble and the equatorial equipotential plane.

The two circular metal plates were mounted horizontal and parallel to one another on insulating supports so that their distance apart could be varied. The lower one was earthed and the upper one was connected to the positive pole of an influence machine with a battery of Leyden jars in parallel. It was found that the potential of the upper plate could be adjusted with considerable delicacy by varying the distance from the knob of one of the jars of a pointed wire connected to earth.

A bubble of measured volume was placed on the middle of the lower plate by means of a pipette. The influence machine was set going and the field was gradually increased. The bubble was seen to elongate as the field increased. When the field had become

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[*] See "Investigation of Lightning Discharges" by C. T. R. Wilson, *Phil. Trans.* A, vol. 221, p. 104 (1921).

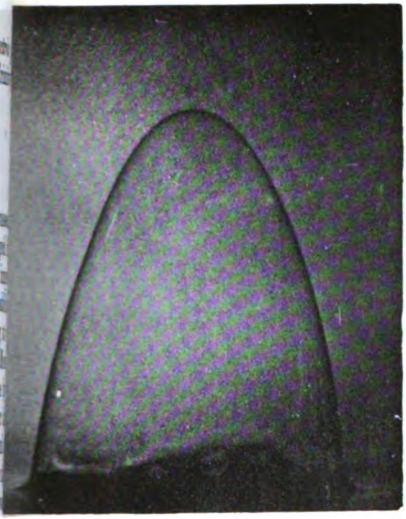

Fig. 1

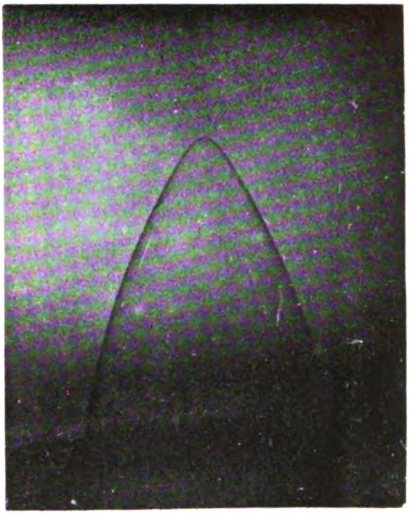

Fig. 2

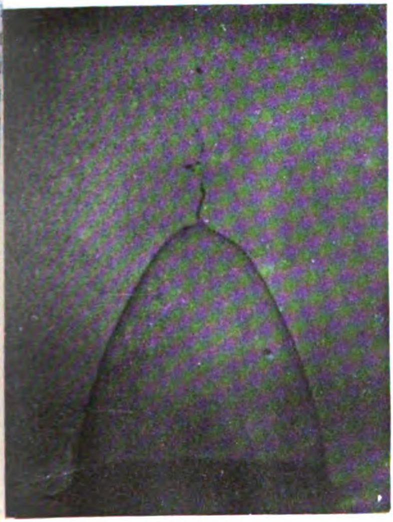

Fig. 3

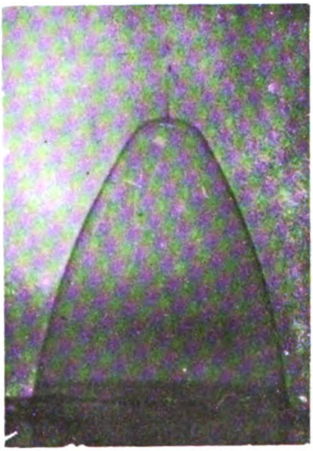

Fig. 4

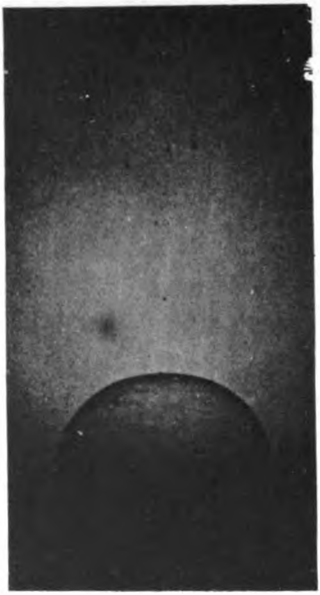

Fig. 5

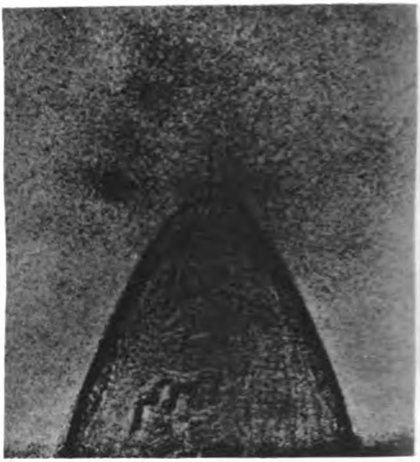

Fig. 6

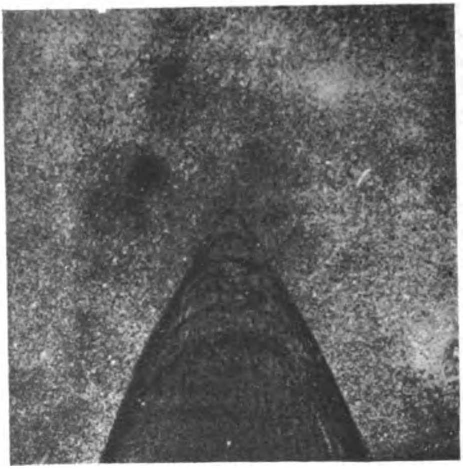

Fig. 7

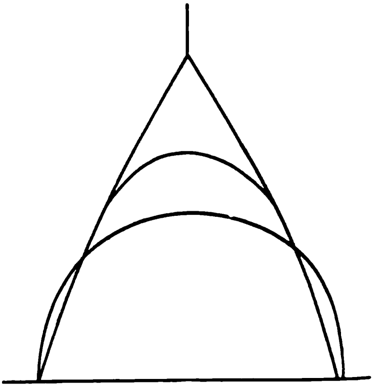

Fig. 8

sufficiently intense the bubble assumed a form roughly resembling the small end of an egg. Photographs of the bubble in this condition were taken by means of a spark from a set of Leyden jars which were independent of those connected to the plate. Fig. 1 is a photograph taken at this stage.

Shortly after the stage represented in Fig. 1 had been reached it was found that quite a small increase in the field produced large changes in the shape of the bubble. While remaining egg-shaped the end rapidly became narrower and then pointed. At this stage it ceased to remain stationary. The end vibrated with great rapidity and as photographs showed, filaments or drops were being thrown off from the end.

The formation of the filaments of this kind has been studied by Zeleny* in the case of drops at the end of a tube, but it is not possible to deduce from his measurements information about isolated water drops.

Fig. 2 shows the bubble in the vibrating stage and apparently just before the filament has been formed. Figs. 3 and 4 show the bubble just after the formation of the filament. In Fig. 3 the drops into which the filament breaks may be seen. At this stage the bubble is flattening very rapidly. The initial stages of the flattening can be seen in Fig. 4, but the flat stage represented in Fig. 5 is rapidly reached and most of the photographs taken in the vibrating stage with exposures timed at random are of this type. The drops into which the filament has broken can be seen in Fig. 5.

In order to follow the rapidly changing shape a rapid succession of exposures was made on the same plate, by allowing a battery of large Leyden jars to discharge through a water resistance and into a smaller single jar with short illuminating spark gap in parallel. Two photographs obtained in this way are shown in Figs. 6 and 7. Fig. 8 is a tracing taken from a photograph which was too faint for reproduction. It is interesting because it shows the almost perfect cone into which the top of the bubble is drawn at the moment when the filament is first formed. In some of Zeleny's experiments the electrified liquid surfaces assumed similar conical forms.

The equilibrium of the bubble requires that

$$T\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\pi\sigma^2=P,$$

where P is the excess of air pressure inside the bubble, σ is the surface density of charge, ρ_1 and ρ_2 the principal radii of curvature at any point and T is the effective surface tension (in this case twice the two surface tensions of soap solution).

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* *Camb. Phil. Soc. Proc.* vol. 18, p. 71 (1914); *Physical Review*, vol. 10, p. 1 (1917).

For bubbles of given shape but different sizes the surface density of charge at corresponding points is proportional to the field but independent of the linear dimensions. The field therefore which produces a given shape should be proportional to

$$\sqrt{T/(\text{linear dimensions})}$$

and in particular the field which causes the bubble, in the present experiments, to become just unstable should be inversely proportional to the square root of the radius of the undistorted bubble.

A number of measurements were made of the field required to burst bubbles of different volumes. The potential difference between the plates was measured by means of a micrometer spark gap using Kaye and Laby's tables for sparking potentials between balls of 5 cm. diameter.

The results are given in the following table:

Volume of Half Drop (cubic centimetres)	Radius, r cm.	F, Field necessary for bursting (volts per cm.)	$F\sqrt{r}$
0·028	0·251	7520	3770
0·056	0·299	7000	3830
0·112	0·377	6060	3720
0·224	0·475	5150	3550
0·56	0·645	4410	3540
0·70	0·694	4410	3670
1·00	0·782	4130	3650
2·50	1·062	3550	3660

As will be seen from the table the product $F \times \sqrt{r}$ is nearly constant. This agrees with the theory outlined above.

It is of interest to find the size of the largest stable water drop in the greatest field that can exist without sparking at normal atmospheric pressure.

Taking this to be 30,000 volts per cm. the radius of the largest soap bubble would be $\frac{(3700)^2}{(30,000)^2} = \cdot015$ cm. If the surface tension of soap solution is taken as 29 C.G.S. units (so that the effective surface tension of the film is 58) and that of water as 75; the largest water drop is $\frac{\cdot015 \times 75}{58} = 0\cdot02$ cm.

The passage of electrons through small apertures. By J. F. Lehmann and T. H. Osgood. (Communicated by Professor Sir E. Rutherford, F.R.S.)

[*Read* 18 May, 1925.]

Several workers* in the past have investigated the ionisation produced in gases by moving electrons over different parts or the whole of their range by enclosing a source of electrons and various electrodes in a chamber containing gas at a pressure of the order of 0·01 mm. of mercury. This method is obviously open to many objections. The use of a high speed mercury vapour pump, however, enables a direct method to be used, in which electrons are accelerated from a glowing filament and passed through a small aperture into an ionisation chamber containing gas at a considerable pressure. The gas streaming through the aperture can be pumped away fast enough to maintain a good vacuum round the filament. In the course of some experiments on the total ionisation produced in gases by slow electrons, the authors found it possible, by using a hole 0·13 mm. in diameter in a copper plate 0·135 mm. thick, to keep a pressure in the neighbourhood of the filament as low as 0·0005 mm., while the pressure in the ionisation chamber was 1 mm. Electrons were accelerated directly from the tungsten filament to the plane anode containing the hole at its centre by any required potential difference V, and it was at first assumed that all those electrons which passed through the hole had an energy eV at the moment of their entry into the ionisation chamber.

Since the values obtained for the total ionisation in air were absurdly small, it was suspected that this assumption was incorrect. To investigate this point, retarding potentials were applied to the beam of electrons inside the ionisation chamber, the whole apparatus being well evacuated for the purpose. It was found that with an accelerating potential between filament and anode of 200 volts, less than 1 per cent. of the electrons entering the ionisation chamber had velocities equivalent to the full drop of 200 volts. Of the remainder, the majority had energies less than that corresponding to a 40 volt drop (Fig. 2). The present experiments were undertaken to investigate this question.

The apparatus used is shown in Fig. 1. It consisted of two glass tubes, 2·6 cm. in diameter, connected to a brass collar by ground joints cemented by hard wax. In the centre of the collar was a shoulder on which rested the washer A carrying the aperture under examination. To permit of easy changing of the aperture, the upper half only of the apparatus was directly connected to the pumping system, the lower surface and outer edge of *A* being channelled to permit the evacuation of the lower part; the lower tube and the collar could then be easily removed and the washer lifted out. The experiments were carried out at a pressure less than 0·0005 mm., when the mean free path of an electron is about a metre.

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* E.g. Kossel, *Ann. d. Phys.* vol. 37, p. 393 (1912); F. Mayer, *Ann. d. Phys.* vol. 45, p. 1 (1914).

The electrons were accelerated from the filament *B* to the anode *A*; some passed through the aperture and were caught and measured in the Faraday cylinder *C*, which was connected to a Dolezalek electrometer with a condenser of from ·1 to ·5 mfd. capacity in parallel. This cylinder was 4 cm. long, 2·2 cm. in diameter, and had an opening 1 cm. in diameter through which the electrons entered. Its distance from *A* could be adjusted by means of the windlass *E*. In order to prevent the accumulation of charges on the walls of the tubes, they were lined with thin copper foil for a distance of 4 cm. from the collar.

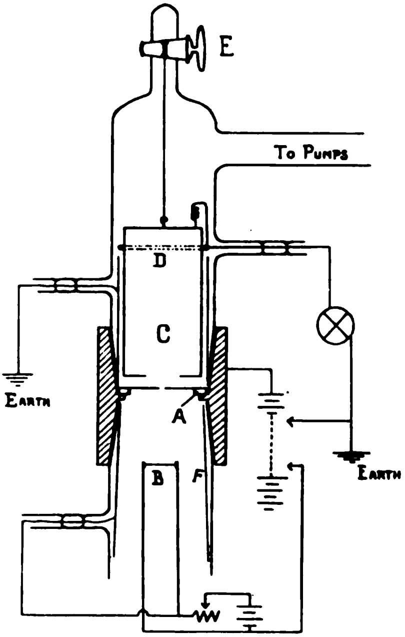

Fig. 1

The velocity of the electrons after passing through *A* was determined by applying retarding fields between the Faraday cylinder and the anode. Any electron with velocity greater than that corresponding to the applied retarding field will strike the inside of *C* and be caught and measured; slower ones will be turned back to *A*. The rate of charge of the electrometer with zero retarding potential was first determined; a definite retarding field was then applied and the rate re-determined. The ratio of these two readings gave the fraction of the total number of electrons passing through *A* which had a velocity greater than that corresponding to the applied retarding potential. Curves in which this ratio is plotted against the retarding field are shown in Figs. 2 and 3

It was necessary to be sure that all the electrons which passed through the anode were collected in the Faraday cylinder when no retarding field was applied. To check this point the height of the cylinder above the anode was varied from 1 mm. to 5 mm., but no change in the rate of charge of the electrometer was observed. It was therefore assumed that up to at least this distance all the electrons were effectively collected. By calculation from the dimensions of the cylinder it can be shown that the loss of electrons, whether secondary ones generated by impact of the primary beam on the brass, or primaries reflected from the walls of the cylinder, through the opening, is less than 1 per cent. of the total beam. This was well within the experimental error. The factor which limited the accuracy of the experiments was the constancy of emission of the filament. To minimise the error due to fluctuations in this, "zero" readings, i.e. with no retarding potential, were taken before and after each reading with any given retarding field.

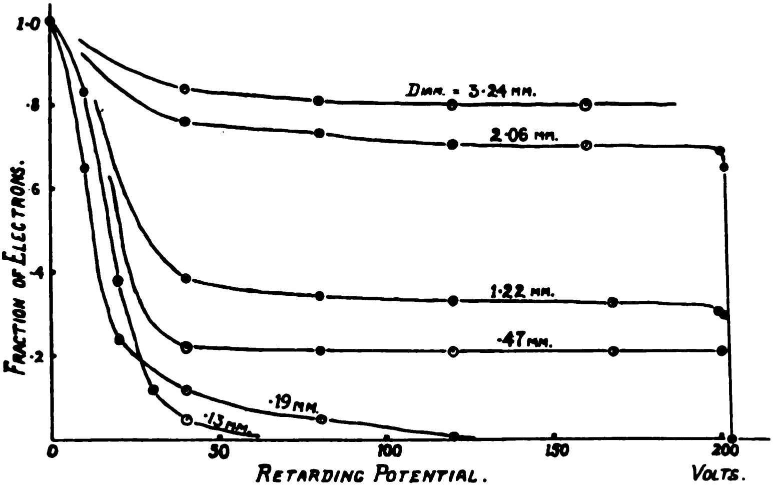

Fig. 2

These zero readings showed variations of the order of 3 per cent., so that the results are subject to an error of this amount.

To render the accelerating field as uniform as possible at the anode, the filament *B* was set at a considerable distance (40 mm.) away; and the surface of the anode kept plane for as great a distance as possible around the aperture. When a capillary tube was used as an aperture it was found that if it projected from the plane of the anode into the filament chamber the percentage of full-velocity electrons passing through it was much less than if its lower end lay in the plane of the anode; showing that the distribution of the lines of force in the accelerating field was an important factor in the experiments. In an endeavour to overcome this difficulty, a grid of copper gauze was placed midway between the filament and the anode; the electrons were accelerated to the grid and then passed through a space of zero potential gradient towards the aperture. This, however, did not improve the homogeneity of the beam. It was also found that the relative positions of the filament, the anode and the shield *F* had a marked effect on the fraction of full-velocity electrons. By applying suitable potentials to the shield this fraction could generally be changed. The optimum potential for this shield varied over a wide range for different positions of the filament. For example, for an accelerating field of 250 volts this potential varied from that of the anode to 300 volts below this.

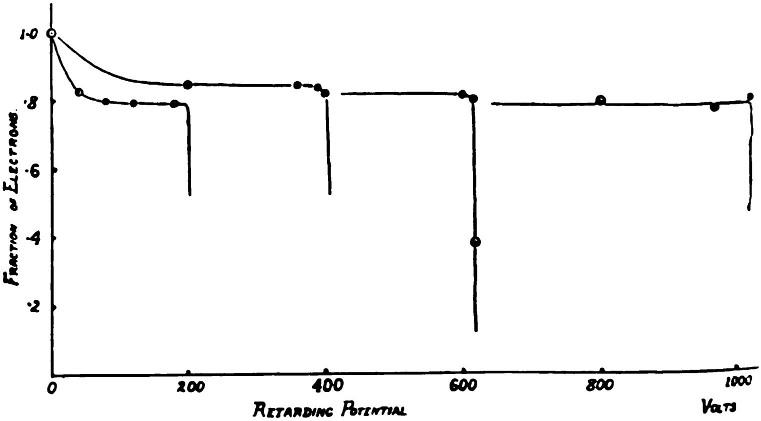

Fig. 3

A series of experiments was carried out to determine the effect of the area of the hole on the velocity of the electrons which passed through it. Holes of various diameters were made in copper foil 0·135 mm. thick. The electrons were accelerated by a potential difference of 204 volts, passed through the hole, and then analysed by means of retarding potentials. The results are shown in Fig. 2, where the fraction of electrons reaching the collecting cylinder is plotted against the applied retarding field. Holes 0·13 mm. and 0·19 mm. in diameter allowed less than 1 per cent. of the electrons to pass through with a velocity equivalent to the accelerating field. A hole 0·47 mm. in diameter allowed 20 per cent. of the electrons to pass through with a velocity of 201 volts. In view of the potential drop along the heated filament (heated by a 4-volt battery) this velocity is in good agreement with the accelerating field of 204 volts. As the diameter of the hole was further increased, the fraction of electrons possessing the full velocity also increased; but even with a 3·24 mm. hole only 80 per cent. of the electrons had the full velocity. The remainder had velocities equivalent to less than 100 volts, the great bulk being below 40 volts. In the case of the largest hole, the magnitude of the accelerating potentials, between 200 and 1000 volts, did not appreciably affect the fraction of full-velocity electrons.

A hole 3·24 mm. in diameter was much too large to use in the ionisation experiments from which the present investigation arose. Various capillary tubes were therefore tested as apertures, for it was known that such an arrangement was used by another worker* on the same subject. The tubes were mounted along the axis of the apparatus normal to the plane of the anode, care being taken that the end of the tube nearer the filament lay in the plane of the anode. Two diameters of copper capillary were used; viz.: 1·25 mm. and 0·40 mm. If care was taken that the filament was central, all tubes, of either diameter and of any length from 5 mm. to 16 mm., gave roughly the same velocity distribution in the electron beam. About 80 per cent. of the electrons retained their full velocity; the remainder had low energies, equivalent to less than one-fifth of the accelerating field. Fig. 3 shows the results obtained with a capillary tube 0·40 mm. in diameter and 16 mm. long. The effect of different accelerating fields is given in the curves for 204, 410, 620 and 1025 volts. In all cases about 80 per cent. of the electrons in the beam had a velocity equal to that due to the accelerating field.

An attempt was made to improve the distribution of electron velocities by inserting between the anode and Faraday cylinder an auxiliary electrode with a central hole 3 mm. in diameter, as shown in Fig. 4. It was possible that the slow electrons might be ejected in random directions from the end of the capillary tube while the fast ones remained in a well-defined beam, and it was hoped that the auxiliary electrode, if kept at the same potential as the capillary, would remove most of these slow ones. This was found not to be the case. A permanent retarding potential of 40 volts was next applied between this electrode and the anode in the hope that the slow electrons would be stopped while the fast ones travelled on. The electrons which passed through the auxiliary electrode were analysed as before, but no improvement in homogeneity was obtained. It appeared that the effects due to the additional hole were simply superimposed on those due to the capillary, so that the net effect gave no improvement in the electron beam.

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* G. A. Anslow, *Science*, Nov. 7, 1924; *Phys. Rev.* April, 1925.

The results of these experiments may be explained qualitatively by assuming that during their passage through the aperture a large number of the electrons strike the walls and produce secondary electrons of comparatively low velocity; also, in the case of holes in thin sheet metal, secondary electrons will be emitted from the filament side of the anode: some of these secondaries, pursuing curved paths in the accelerating field, will pass through the hole.

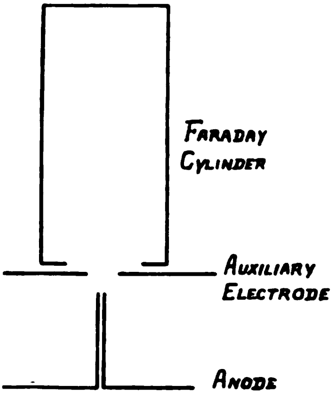

Fig. 4

As a result of the former effect the percentage of full-velocity electrons should increase with the area of the hole, as is shown to be the case in Fig. 2. Considering tubes, it is evident that the opening at one end of a capillary will subtend only a small solid angle at a point on the wall where a primary impact occurs. Therefore in this case the number of secondary electrons emerging with the primary beam should be correspondingly small, and it is to be expected that a capillary tube will give a more homogeneous emergent beam than a hole of the same diameter in a thin sheet. Further, the use of a tube cuts out the secondary electrons from the filament side of the anode, since these travel in a direction inclined to the axis of the tube at the moment when they enter it.

A thorough quantitative explanation would require more information than is at present available as to the direction and velocity of ejection of secondary electrons for a given direction of impact of primary electrons.

SUMMARY.

The velocities of electrons emerging through an aperture in an anode to which they were accelerated have been investigated.

The velocity distribution of electrons passing through holes ranging from 0·13 mm. to 3·24 mm. in diameter in a thin copper sheet is shown in Fig. 2. In this case the accelerating field between filament and anode was 204 volts.

Corresponding results for a capillary tube 0·40 mm. in diameter and 16 mm. long are shown in Fig. 3, where the curves are plotted for different accelerating potentials. Experiments were also made with shorter lengths, down to 5 mm., and with a tube 1·25 mm. in diameter. All the tubes investigated gave the same distribution of velocities of the electrons in the emergent beam, within the limits of error of the experiments.

In conclusion the authors wish to express their thanks to Sir Ernest Rutherford for his interest and advice during the course of the work.

An apparatus for the investigation of the effect of poisonous substances, and mixed vapours on catalytic activity. By F. H. CONSTABLE, B.A., Strathcona Research Student, St John's College.

[*Read* 4 May, 1925.]

An experimental arrangement has been devised in which the composition of the vapour phase passing over the catalyst can be varied at will, till the one vapour may be completely shut off without affecting the flow of the other. The apparatus may also be adapted for the use of two vapours alternately as reactants. At the same time it is necessary to maintain the catalyst at a definite and known temperature, and to maintain a constant rate of flow of the mixed vapours. These results have been obtained with the apparatus shown (Fig. 1).

The design is based on the experience gained in the use of vaporised liquids as reactants in the study of the Catalytic Action of Copper*. A double boiler system was used, together with a condenser and receiver capable of dealing with large rates of flow without causing the loss of the gaseous resultants of the chemical reaction as bubbles forced through the condenser at *K*. Both boilers A_1, and A_2 were electrically heated, and were provided with glass valves at *B* which were opened or closed by the glass rods *b*. These rods very nearly fitted the glass tubes in which they slide, and the joint was made tight by a collar of rubber tubing. The tubes *DE*, provided with taps at *D*, conducted away the vapour when the ground glass surfaces *bB* were in contact. The rod and tube *bB* were arranged sloping slightly downwards, and the connecting tube to the furnace sloping upwards so that liquid could collect on the ground glass surfaces and keep the joint tight without the use of extraneous substances. In addition the superheating wires on the connecting tubes were moved far enough from *B* to let sufficient liquid condense to wet the ground glass without blocking the tube. This factor is important in obtaining accurate results, as drops of liquid condensed and suddenly vaporised cause an apparent increase in gas evolution, followed by a corresponding diminution later. The delivery tubes from the boilers met in a wider tube which carried the vapour into the furnace.

The Reaction Tube was of the same design as before, with the exception that the glass tube containing the catalyst was of considerably smaller bore than that of the tube surrounding it. Thus the catalyst and thermocouple were completely surrounded with the vapour that was to pass over them. This arrangement gives very regular results. The tube holding the thermocouple and catalyst was connected with the condenser *I* by the waxed cork *H*. It was easy to remove the catalyst for inspection by gently warming the waxed cork *H*, and slowly pulling the condenser *I*. On folding back the rubber tube at the end of the furnace *F* the catalyst tube slid out easily. The double surface condenser *I* contained the inner tube asymmetrically arranged in the outer case, so that the liquid could run down the sides of the glass and not drip off the rounded end of the inside tube thus causing the formation of bubbles in *K*.

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* Palmer and Constable, *Proc. Roy. Soc.* 'A,' vol. 106, p. 251 et seq.

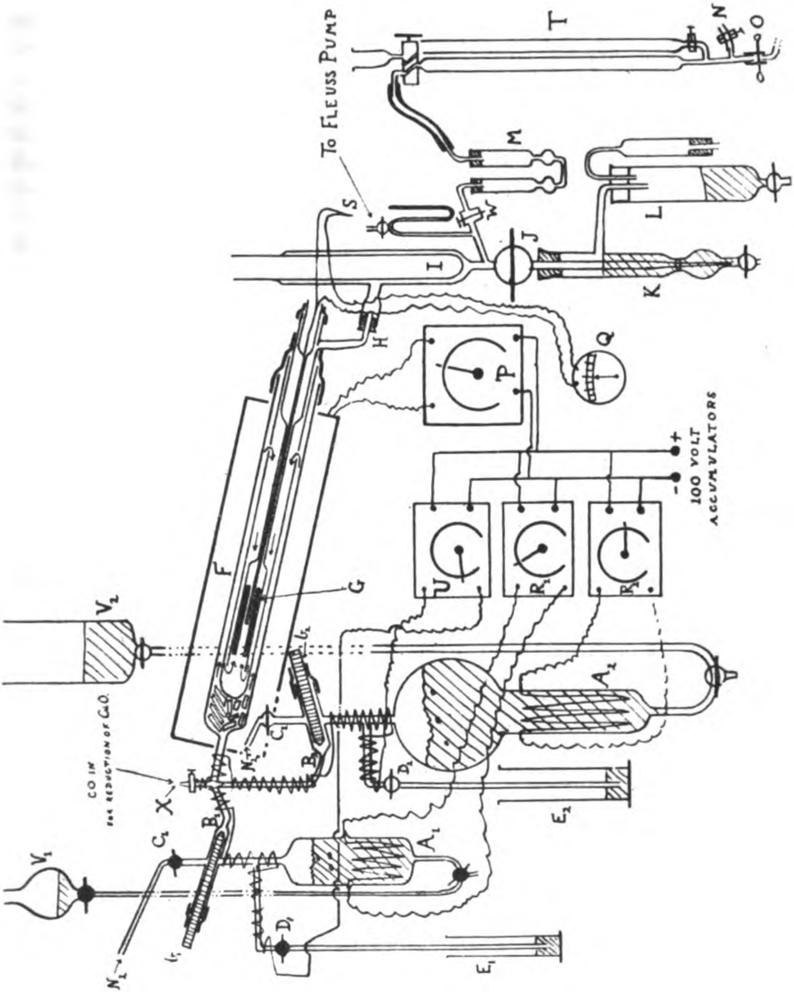

Fig. 1. Apparatus.

The tap at J was specially large to avoid constriction and the consequent formation of gas bubbles which would be lost through the receiver. The loss of gas in this manner may become a serious error with small tubes, more particularly when liquids of different chemical nature are being used. A constant level of liquid is maintained in the receiver K, the excess dripping into L which is guarded by a drying tube. K was also designed as a delivery pipette, so that known volumes of the resultant liquid could be run off for analysis. The tubes M* were used to purify the gaseous reaction products before their volume was measured in the gas burette T. The pipe N supplied water at constant pressure, and was closed by a screw clip. The liquid in the two sides of the burette was levelled by opening the pinch cock O which communicated directly to the sink. Thus the burette may be filled rapidly with water, and the levelling can be performed quickly.

The temperature of the catalyst was read from the galvanometer Q, which was connected with a copper-constantin thermocouple one junction of which was in steam at S, and the other enclosed in a thin glass tube in the catalyst mass. The furnace, which consisted of nichrome ribbon wound on asbestos around a brass tube, was controlled by the resistance P. The other rheostat U controlled the temperature of the glass tubes conveying the vaporised liquid to the reaction tube.

Pure nitrogen gas was supplied to each boiler through the taps C_1 and C_2, and air free liquids through the vessels V_1 and V_2. Both boilers were lagged with asbestos to minimise the effect of draughts on the rate of boiling and to prevent as far as possible losses due to radiation. A table was constructed connecting the rate of evaporation with studs on the resistances R_1 and R_2 for each liquid used. Such data could be obtained for each boiler before using the catalyst by means of the side tubes D_1 and D_2, the calibrated receivers E_1 and E_2 being immersed in cold water; or during the experiments by replacing L by a narrow calibrated tube. Both methods gave good results. The observed composition of the liquid distillate was a check on the accuracy of the measurements, and showed that the presence of the second boiler did not interfere with the rate of evaporation from the first.

If it was desired to use two vapour reactants alternately (as in the following experiments) the rates of boiling of each must be made very nearly equal, otherwise on changing from one vapour stream to the other considerable change in the apparent rate of gas evolution will take place, due to the displacement of air at the cool end of the reaction tube. The required setting of R_1 and R_2 to give equal rates of boiling was known from preliminary experiments. To ensure success with an apparatus of this type it is very necessary to use current supplied at a constant potential, as from accumulators, the variation in potential difference in the electric lighting circuit being sufficient to render the results quite valueless.

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* When ethyl alcohol and a copper catalyst were being used one limb contained chromic acid on pumice, and the other (nearest the burette) damp soda-lime.

*Some results on the poisoning of a copper gauze catalyst by iso-amyl alcohol obtained from fusel oil**.

The catalyst used was a roll of copper gauze on which a layer of pure electrolytic copper had been deposited. To follow the poisoning to the best advantage it was desirable to use a very active preparation, and advantage was taken of Hinshelwood's work showing that very considerable increase can be produced in the catalytic activity of massive copper by alternate oxidation and reduction. The catalyst was oxidised and reduced ten times successively at 320° C., and was very active (cf. Table II).

The amyl alcohol was fractionated through a Dufton column till it showed a boiling range of 131·1–131·2° C. Thus whatever poisonous substances (probably traces of fatty acids) may be present distil with the alcohol, and a constant pressure of poison is maintained over the catalyst during an experiment. The smaller boiler A_1 contained the amyl alcohol, and A_2 the ethyl alcohol, the apparatus being filled with nitrogen. The side tubes D were now opened (the ends of which were sealed under liquid alcohol), the liquids in the boilers heated to boiling, and R_1 and R_2 set so that both alcohols were kept at the boiling point. The apparatus is now ready to work, and may be left in this condition while the final adjustment of the furnace and the measuring instruments is made.

When all was ready, R_2 was adjusted to give a rate of boiling of about 30 c.c. per hour, and the initial temperature reaction-velocity curve taken for ethyl alcohol. As the cooling curve was being completed R_1 was adjusted so that the second boiler vaporised alcohol at the same rate as the first, the amyl alcohol distilling into E_1. P was adjusted so that the catalyst would attain a constant temperature of 220° C. When the milli-voltmeter Q showed satisfactory constancy of temperature the reacting vapours were changed.

After some experience it was found quite unnecessary to shut the taps J and W, as very little sucking back occurred. D_2 was

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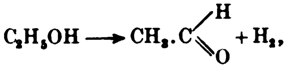
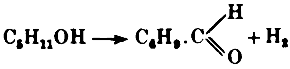
* The reactions catalysed by the copper are

(1) 

$$C_2H_5OH \longrightarrow CH_3.C\begin{matrix}\diagup H\\ \diagdown O\end{matrix} + H_2,$$

(2)  which falls off with time owing to poisoning due to traces of impurity in the amyl alcohol.

$$C_5H_{11}OH \longrightarrow C_4H_9.C\begin{matrix}\diagup H\\ \diagdown O\end{matrix} + H_2$$

opened giving a second way of escape for the alcohol vapour. B_1 was next opened, D_1 shut, followed shortly afterwards by the shutting of B_2. It was found possible to do this without causing any loss of gas at K. The time of commencement of the poisoning was taken as the time at which B_2 was shut. Amyl alcohol vapour was now passing through the apparatus at the same rate as did the ethyl alcohol vapour. Readings of the burette were taken every minute for the first 38 minutes, after which time the activity was reduced to about one-tenth of its initial value. The reacting vapour was again changed to ethyl alcohol, and the reaction-velocity temperature curve again taken. Finally on cooling to 220° C., the amyl alcohol poisoning readings were again taken for

TABLE I. *The poisoning of copper gauze by amyl alcohol, with the intermediate values found for ethyl alcohol.*

Time in mins.	Reaction-velocity observed at 220° C. c.c. per min.	Reaction-velocity calculated (amyl) c.c. per min.	Nature of reactant
0	3·48	3·40	Ethyl alcohol Expt. I, Table II
2	3·16	3·05	Amyl alcohol
4	2·60	2·68	,, ,,
6	2·43	2·35	,, ,,
8	2·03	2·10	,, ,,
10	1·82	1·81	,, ,,
12	1·54	1·58	,, ,,
14	1·43	1·38	,, ,,
16	1·19	1·21	,, ,,
18	1·11	1·07	,, ,,
20	0·904	0·931	,, ,,
22	0·834	0·836	,, ,,
24	0·798	0·714	,, ,,
26	0·641	0·630	,, ,,
28	0·538	0·547	,, ,,
30	0·479	0·480	,, ,,
32	0·448	0·416	,, ,,
34	0·364	0·363	,, ,,
36	0·316	0·317	,, ,,
38	0·258	0·274	Ethyl alcohol Expt. II, Table II
40	—	—	Amyl alcohol
42	0·23	0·209	,, ,,
44	0·18	0·182	,, ,,
46	0·15	0·163	,, ,,
48	0·10	0·145	,, ,,
50	—	—	,, ,,
52	0·10	0·112	Ethyl alcohol Expt. III, Table II

12 minutes. A third reaction-velocity temperature curve was obtained for ethyl alcohol. The results obtained are shown in Table I, together with those calculated from the simple logarithmic decay law

$$\log v = -kt + c.$$

TABLE II. *Ethyl alcohol.*

Experiment I

Temperature °C.	Observed velocity c.c. per min.	Calculated velocity c.c. per min.
220	3·48	3·61
225	4·35	4·35
230	5·25	5·35
235	6·58	6·52
240	7·94	8·01
245	9·55	9·70
250	11·5	11·6
255	13·8	13·7
260	15·9	16·2
265	19·9	19·5

Experiment II

220	0·258	0·255
225	0·303	0·304
230	0·382	0·378
235	0·458	0·462
240	0·577	0·573
245	0·689	0·695
250	0·815	0·832
255	1·00	0·995
260	1·16	1·19
265	1·44	1·41
270	1·71	1·70

Experiment III

220	0·109	0·113
225	0·141	0·137
230	0·170	0·169
235	0·205	0·205
240	0·251	0·252
245	0·308	0·304
250	0·365	0·371
255	0·449	0·445
260	0·541	0·535
265	0·647	0·641
270	0·785	0·771

The observations from which the data of Table II are tabulated are shown in Fig. 2, curves I, II, and III. The curves IV and V, which are included here for convenience, are studies of sintering by heat treatment. The catalyst from curve IV was heated to 550° C. in pure nitrogen, and the curve V was then obtained. Fig. 3 shows the result of plotting $\log_{10}v$ against $1/T$. The results show that the law of Arrhenius holds accurately for the poisoned and sintered catalysts, and further that the temperature coefficient of dehydrogenation is unaltered by poisoning and sintering. The values of the temperature coefficients, and the "heat of activation" deduced are shown in Table III.

TABLE III.

Experiment	Activity of catalyst	Intercept on logarithmic curve ($1/T_1 - 1/T_2 = 2 \times 10^{-4}$)	Heat of Activation in cals. per gram mol.	Temperature coefficient at 250° C.
Curve I	1·00	0·87	19,840	1·433
Curve II	0·07	0·88	20,070	1·439
Curve III	0·03	0·90	20,530	1·450
Curve IV	1·00	0·88	20,070	1·439
Curve V	0·17	0·89	20,390	1·447

Theory of the poisoning of a distribution of centres of activity.

It has been shown* to a first approximation that the centres of activity are distributed over a catalyst surface according to an exponential law.

If n = the total number of centres of activity present, and ϵ = the "heat of activation" on one of these centres, then

$$\delta n = C . e^{h\epsilon} . \delta\epsilon$$

and the relation is restricted between the limits ϵ_1 and ϵ_2, the constants C and h as well as the limits ϵ_1 and ϵ_2 being variables for different substances.

It is well known that the presence of traces of substances which are very strongly adsorbed by the catalyst, or have violent chemical action on it, causes the catalytic activity to fall rapidly. The poisons themselves may be divided into two classes:

(1) "Poisons" which cover the surface practically equally rapidly at all points, e.g. chemical reaction with copper atoms, or the adsorption of large molecules having many points of attachment to the surface.

(2) "Poisons" which selectively attack the more active centres: i.e. with which the rate of covering varies with the nature of the centre covered.

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* Constable, *Proc. Roy. Soc.* 'A,' vol. 108, p. 374. "On the Mechanism of Catalytic Decomposition."

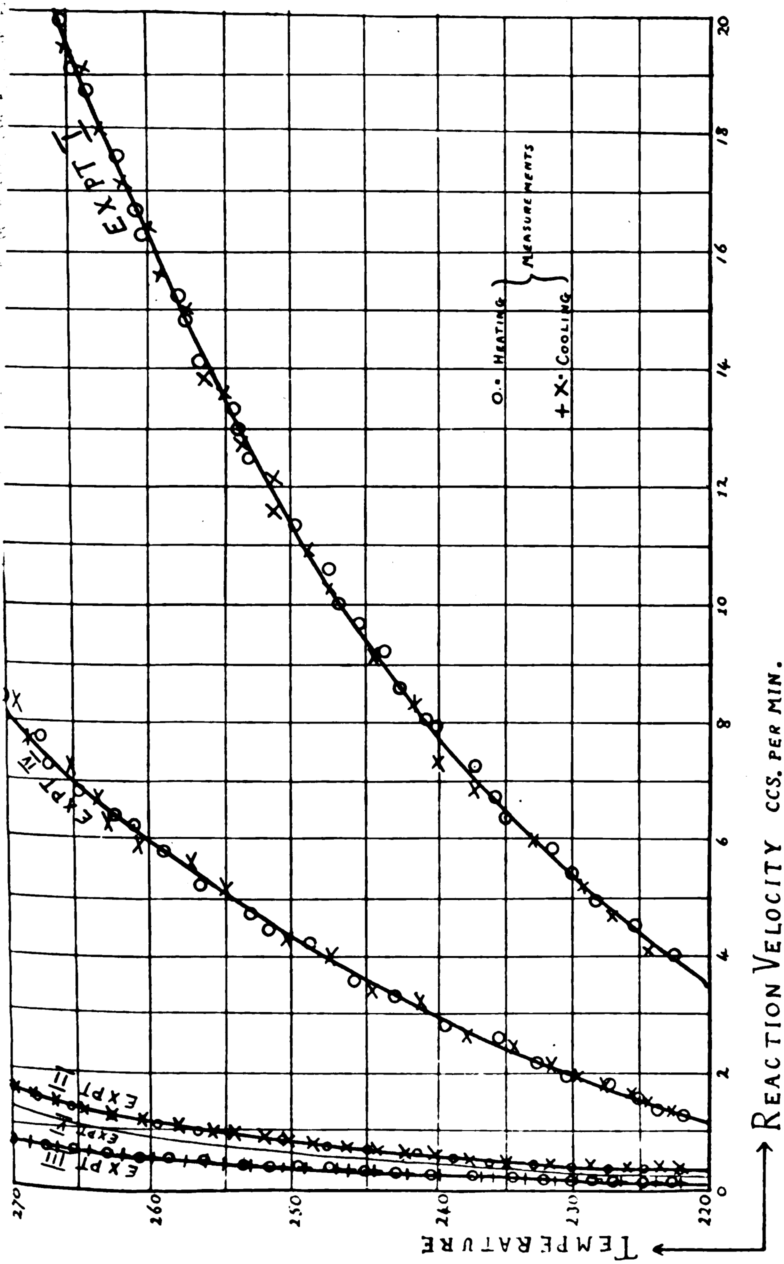

Fig. 2. The Observations.

The equations for the decay of the reaction-velocity caused by poisons of the first class have been already worked out*. The logarithm of the reaction-velocity decreases as a linear function of the time. It is interesting to work out the equations for "poisons" of the second class.

The centres of activity differ from each other in the extent of their strain energy, and consequently in the magnitude of their external field. The lowering of the "heat of activation" on a reaction centre has been taken as a measure of the strain energy and external field peculiar to the group of atoms.

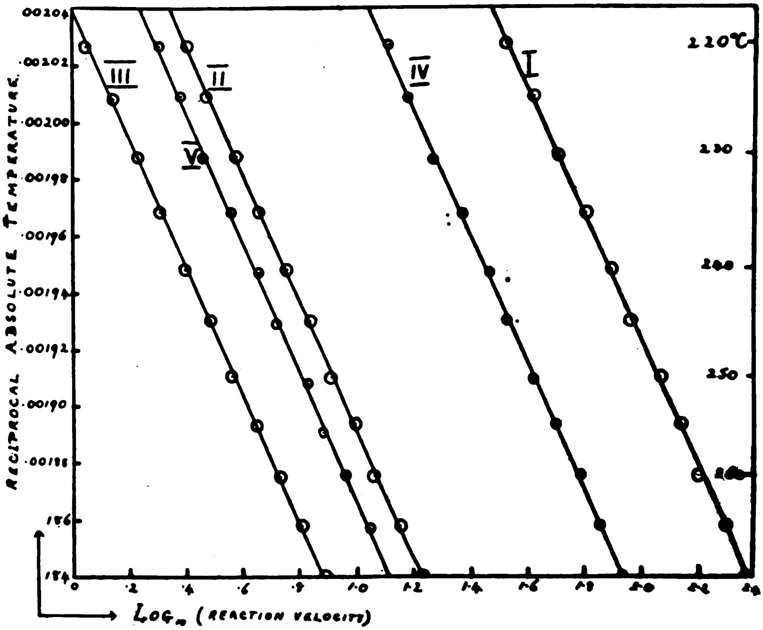

Fig. 3. Showing the Temperature Coefficient unchanged by poisoning and sintering.

If a = the value of ϵ for the uncatalysed reaction, then $(a - \epsilon)$ is a measure of the strain in a centre.

Let $\int$ be the number of centres of activity with "heat of activation" ϵ. If the rate of covering of reaction centres of a given type is proportional to the external field shown on each we have

$$-\frac{d\int}{dt} = k(a - \epsilon) . \int,$$

where k is a constant, since the number that decay depends not only on the external field shown on each centre but also on the number of centres to be covered.

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* See Palmer and Constable, *Proc. Roy. Soc.* 'A,' vol. 107, p. 263.

Hence $$\log \int = -k(\alpha - \epsilon)t + C.$$
But when $$t = 0, \quad \int_0 = C.e^{h\epsilon}.$$
Therefore $$\int = C.e^{h\epsilon}.e^{-(\alpha-\epsilon)kt}$$
or $$\delta n = C.e^{h\epsilon}.e^{-(\alpha-\epsilon).kt}.\delta\epsilon \qquad \text{......(1).}$$

This equation represents the number (δn) of centres with "heat of activation" ϵ after time t. The rate of reaction on one of thes centres is $\frac{1}{\sigma_\epsilon}.e^{-\epsilon/RT}$, where σ_ϵ is the mean life of a molecule on these centres. Thus the reaction-velocity at the catalyst surface

$$\begin{aligned} v &= \int_{\epsilon_1}^{\epsilon_2} \frac{C.e^{h\epsilon}.e^{-(\alpha-\epsilon)kt}.e^{-\epsilon/RT}}{\sigma_\epsilon}.\delta\epsilon \\ &= \int_{\epsilon_1}^{\epsilon_2} \frac{Ce^{-\alpha kt}.e^{\epsilon(h+kt-1/RT)}}{\sigma_\epsilon}.\delta\epsilon \\ &= \left[\frac{Ce^{-\alpha kt}.e^{\epsilon(h+kt-1/RT)}}{\{h+kt-1/RT\}.\sigma_\epsilon}\right]_{\epsilon_1}^{\epsilon_2}, \end{aligned}$$

assuming σ_ϵ to vary only slowly with ϵ.

$$\therefore\ v = \frac{C}{(h+kt-1/RT)\,\sigma_\epsilon}\{e^{h\epsilon_1}.e^{-\epsilon_1/RT}.e^{-(\alpha-\epsilon_1)kt} - e^{h\epsilon_2}.e^{-\epsilon_2/RT}\} \qquad \text{......(2),}$$

since α is the upper limit of ϵ on the surface. Further, since experiment shows that only a very small portion of a surface is catalytically active*, the great majority of the chemical action usually takes place on the centres of activity with the lowest ϵ, thus the quantity $(h + kt - 1/RT)$ must be decidedly negative for most catalyst surfaces†. Hence the term $e^{h\epsilon_2}.e^{-\epsilon_2/RT}$ becomes negligible compared with the first, though the catalyst may be almost completely poisoned.

If v_t = the reaction velocity after time t, then

$$v_t = \frac{C}{\{h+kt-1/RT\}\,\sigma_\epsilon}\{e^{h\epsilon_1}.e^{-\epsilon_1/RT}.e^{-(\alpha-\epsilon_1)kt}\} \text{......(3).}$$

But $$v_0 = \frac{C}{(h-1/RT)\,\sigma_\epsilon}\{e^{h\epsilon_1}e^{-\epsilon_1/RT}\}.$$

Therefore $$v_t = \frac{v_0 . e^{-(\alpha-\epsilon_1)kt}}{1 + \dfrac{kt}{(h-1/RT)}} \qquad \text{......(4),}$$

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* See H. S. Taylor, *J. Phys. Chem.* vol. 28, p. 911. (1924.)

† I.e. kt must be considerably less than $|\,h - 1/RT\,|$ hence $1\Big/1 + \frac{kt}{(h-1/RT)}$ does not vary much during the poisoning. See "Mechanism of Catalytic Decomposition" *loc. cit.*

and it has been shown from general considerations that the term

$$\left(1+\frac{kt}{(h-1/RT)}\right)$$

does not differ much from unity.

The poisoning of homogeneous reaction centres with heat of activation ϵ_1 is given by the equation

$$-\frac{d\int_\epsilon}{dt}=k\,(a-\epsilon_1)\int_\epsilon$$

or $$v_t=v_0\,.\,e^{-(a-\epsilon_1)kt},$$

since $$v\propto\int_\epsilon \qquad \ldots\ldots(5).$$

The relations (4) and (5) differ only by the presence of a term which is very close to unity, and varies only slowly with t. Thus an exponential distribution of centres of activity behaves in the same manner whether all types of centre are covered equally fast, or the active centres are covered faster than the inactive. During selective poisoning the exponentially distributed centres of activity can therefore behave as if they constituted a surface of centres having the least value of the heat of activation in the distribution.

When very special relations exist between the constants one may hope to detect some deviations from the logarithmic law of poisoning. In these cases, owing to the small value of $|h-1/RT|$ (the physical meaning of which is that the rate of reaction on the centres falls only slowly as the value of the heat of activation increases) the correcting term becomes of considerable magnitude as the poisoning progresses, and the initially rapid rate of fall of the activity according to the logarithmic law is considerably retarded as the time proceeds.

Equation (3) is also of considerable interest. It shows the reaction-velocity as a function of both the temperature of the surface, and the time measured from the commencement of the reaction. As the temperature rises the reaction-velocity also changes, the rate of reaction increasing if the poisoning is not very rapid, though the apparent temperature coefficient of the reaction is less than on a clean surface. With faster rates of poisoning the reaction-velocity may remain constant, or even decrease, depending on the rate of rise of temperature of the surface. The velocity obtained on cooling must always be less than that obtained during a previous heating.

This equation gives quantitative explanation of the forms of the reaction-velocity temperature curves obtained during the comparison of the rates of dehydrogenation of various alcoholic substances by copper*. It also shows that the temperature coefficient of chemical reaction remains unaltered after poisoning except in special cases.

It is a great pleasure to thank Prof. T. M. Lowry, F.R.S., for his stimulating interest; Dr W. H. Mills, F.R.S., for the continued use of his laboratory; and Dr W. G. Palmer for his continued interest in this work.

SUMMARY.

(1) An apparatus has been devised in which the composition of the vapour phase passing over the catalyst can be varied at will, or two vapours can be used alternately as reactants.

(2) By using ethyl alcohol, and amyl alcohol from fusel oil alternately as reactants, the poisoning of a very active copper gauze catalyst has been followed and the temperature coefficient of the dehydrogenation of ethyl alcohol by the copper has been measured during the poisoning.

(3) The activity of the catalyst, poisoned by amyl alcohol, fell in accordance with the simple logarithmic law

$$\log v = -kt + c.$$

(4) Confirmation was obtained that the rates of dehydrogenation of ethyl and amyl alcohols were equal.

(5) The temperature coefficient of dehydrogenation on the catalyst was unchanged throughout the poisoning, though the reaction velocity was reduced 30 times.

(6) The temperature coefficient was also shown to be unchanged during sintering, the reaction-velocity having been reduced 10 times.

(7) Equations are derived for the rate of poisoning of an exponential distribution of centres of activity when the poison attacks the more active centres selectively.

(8) The relation deduced

$$v_t = \frac{v_0 . e^{-(a-\epsilon_1).kt}}{1 + \dfrac{kt}{(h - 1/RT)}}$$

cannot usually be distinguished (except with special relations between the constants) from the simple logarithmic law of poisoning.

(9) Thus the exponential distribution normally poisons like a surface composed of homogeneous centres, the temperature coefficient being unchanged by the poisoning.

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* *Proc. Roy. Soc.* 'A,' vol. 107, p. 258, Fig. 1.

(10) The equation

$$v_t = \frac{C}{(h + kt - 1/RT) . \sigma_\epsilon} \{e^{h\epsilon_1} . e^{-(\epsilon-\epsilon_1)kt} . e^{-\epsilon/RT}\}$$

describes the reaction-velocity as a function of time, and temperature, and gives adequate explanation of previous experimental results on the forms of the heating and cooling curves with "poisonous" alcohols.

(11) Frenkel has shown $\sigma_\epsilon = \tau_0 . e^{\mu_0/RT}$, where τ_0 is the vibration frequency of the molecule perpendicular to the surface, and μ_0 is the heat of desorption of one gram molecule. Thus if the heat of desorption is not small compared with the energy necessary to cause chemical disruption of the molecule, the previous equation becomes

$$v_t = \frac{C}{\{h + kt - 1/RT\} \tau_0} \{e^{h\epsilon_1} . e^{-(\epsilon-\epsilon_1)kt} . e^{-\epsilon+\mu_0/RT}\},$$

the form of the equation being unaltered.

On Pedal Quadrics in Non-Euclidean Hyperspace. By Mr J. P. GABBATT, Peterhouse.

[*Received* 7 March, *read* 4 May, 1925.]

1. *Introductory.* The following are well-known theorems of elementary geometry: Given any euclidean plane triangle, $A_0A_1A_2$, and any pair of points, X, Y, isogonally conjugate q. $A_0A_1A_2$; then the orthogonal projections of X, Y on the sides of $A_0A_1A_2$ lie on a circle, the *pedal circle* of the point-pair*. If either of the points X, Y describe a (straight) line, m, then the other describes a conic circumscribing $A_0A_1A_2$, and the pedal circle remains orthogonal to a fixed circle, J†; thus the pedal circles in question are members of an ∞^2 linear system of circles of which the circle J and the line at infinity constitute the Jacobian. In particular, if the line m meet A_jA_k at L_i $(i, j, k = 0, 1, 2)$, then the circles on A_iL_i as diameter, which are the pedal circles of the point-pairs A_i, L_i, are coaxial‡; the remaining circles of the coaxial system being the director circles of the conics, inscribed in the triangle $A_0A_1A_2$, which touch the line m. If M_i denote the orthogonal projection on m of A_i, and N_i the orthogonal projection on A_jA_k of M_i, then the three lines M_iN_i meet at a point (Neuberg's theorem)§, viz. the centre of the circle J. Analogues for three-dimensional space of most of these theorems are also known‖. (1·1)

The writer has previously¶ considered the conics, in a non-euclidean plane, which replace the pedal circles. The object of the present paper is to discuss the corresponding figures in higher space. The following theorems, exactly analogous to those of (1·1), will be proved: Given any simplex, $A_0A_1 \ldots A_n$, or briefly $[A]$, in non-euclidean space of n dimensions, and any pair of points, X, Y, isogonally conjugate q. $[A]$, then (4·1) the orthogonal projections of X, Y on the walls of $[A]$ lie on a quadric $(n-1)$-fold of revolution about the line XY; the *pedal quadric* of the point-pair X, Y. If either of the points X, Y describe a (straight) line, m, then (4·2) the other describes an n-ic curve (1-fold) containing all the vertices of $[A]$; and the pedal quadrics are members of an ∞^n linear system of order-quadrics, of which the Jacobian is an $(n+1)$-ic $(n-1)$-

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* Steiner, *Annales de Math.* 19, (1828–9), 37–64.

† Substantially due to T. Lemoyne, *Nouv. Ann.* (4) 4, (1904), 400–402.

‡ Bodenmiller; see Gudermann, *Grundriss d. anal. Sphärik*, Köln, (1830), 138.

§ Neuberg, *Nouv. Correspondance math.* 2, (1876), 189; *ibid.* 4, (1878), 379–382; *Arch. d. Math. u. Phys.* (3) 3, (1902), 89–93.

‖ See Servais, *Bull. Sc. Acad. r. de Belgique* (5) 8, (1922), 50–66; *Mathesis* 36, 1922), 87–89.

¶ *Proc. Camb. Phil. Soc.* 21, (1923), 340–342.

fold. In particular, if the line m meet at L_i that wall, a_i, of $[A]$ which does not contain A_i, then (4·3) the director quadrics (2·7) of the $n+1$ point-pairs A_i, L_i (which are also the pedal quadrics of those point-pairs) are such that each quadric contains all the common points of the remaining n quadrics; thus the $n+1$ quadrics are members of an ∞^{n-1} linear system of order quadrics: the remaining quadrics of the system being the director quadrics of those quadrics, inscribed in $[A]$, which touch every flat containing m. If M_i denote the orthogonal projection of A_i on any flat, and N_i the orthogonal projection of M_i on a_i; then (5·2) the $n+1$ lines M_iN_i all meet ∞^{n-2} linear $(n-2)$-folds, and therefore determine an $(n-1)$-fold of class $n-1$, which touches every flat containing any of the lines M_iN_i. (1·2)

The leading propositions of (1·2) appear as special cases of more general theorems. Thus, if the class-quadrics of any ∞^1 linear system which includes the Absolute be termed *confocal*, and if the class-quadrics of any confocal system which includes a quadric inscribed to a given simplex be termed *infocal* to that simplex; then (3·3) the pedal quadric (q. $[A]$) of any infocal quadric (to $[A]$) may be specified; when the given quadric breaks up into a (necessarily isogonally conjugate) point-pair, then its pedal quadric is that previously (1·2) specified. Given any $r+1$ infocal class-quadrics, not linearly related; then (3·6) any class-quadric of the ∞^r linear system determined by the $r+1$ quadrics is also infocal, and the corresponding pedal quadrics constitute an ∞^r linear system of order-quadrics.

The pedal quadric of any (infocal) quadric, Σ, is (3·3) the absolute reciprocal of a quadric, Σ', confocal with Σ. If $n=2$, then (3·8) the pedal quadric of Σ' is also the absolute reciprocal of Σ; but this is not generally true if $n>2$. For all values of n, the quadric Σ is (3·9) the reciprocal of its own pedal quadric if and only if Σ is inpolar to each of a certain group of $n+1$ point-hyperspheres; if a pair of points satisfy the condition, then (4·4) the two points of the pair are correspondents on the Jacobian of the $n+1$ point-hyperspheres, an $(n+1)$-ic $(n-1)$-fold which the writer has termed[*] the Wallacian of the simplex of reference.

2. *A Theorem of Rohn.* K. Rohn has proved[†] a theorem of which the dual may be stated as follows: Given an ∞^1 linear system of class-quadrics in space of n dimensions; if a variable simplex be self-polar to a fixed quadric of the system, and if n walls of the simplex respectively envelope n fixed quadrics of the system, then the remaining wall of the simplex also envelopes a fixed quadric of the system.

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[*] *Proc. Camb. Phil. Soc.* 21, (1923), 763–771.
[†] *Abh. sächs. Ges. Wiss.* 31, (1909), 335–367.

Rohn's proof is confined to the case in which two quadrics of the system possess a common self-polar simplex; for complete generality we must dispense with this assumption*. Then, let the class-quadrics Ω, Σ be respectively expressed by

$$\Omega \equiv \omega_\xi^2 = 0,$$
$$\Sigma \equiv \sigma_\xi^2 = 0.$$

Let a variable simplex be self-polar to the quadric Ω, let n walls, respectively expressed by $\eta_x = 0$, $\zeta_x = 0$, ..., $\phi_x = 0$ of the simplex respectively envelope the class-quadrics expressed by $\mu\Omega + \Sigma = 0$, $\nu\Omega + \Sigma = 0$, ..., $\pi\Omega + \Sigma = 0$; let the remaining wall of the simplex be expressed by $\psi_x = 0$. Let $\omega_\iota\omega_\xi$, $\sigma_\iota\sigma_\xi$ be denoted by Ξ_ι, $\overline{\Xi}_\iota$ respectively; let the determinants $(\text{HZ} \ldots \Psi)$, $(\overline{\text{H}}\text{Z} \ldots \Psi)$, ..., $(\text{HZ} \ldots \overline{\Psi})$ be denoted by D, $D^{(\eta)}$, ..., $D^{(\psi)}$ respectively; and let $\Delta + \tau\Theta + \ldots$ denote the discriminant of $\Omega + \tau\Sigma$. Then

$$D \equiv (\eta\zeta \ldots \psi) . \Delta,$$
$$D^{(\eta)} + D^{(\zeta)} + \ldots + D^{(\psi)} \equiv (\eta\zeta \ldots \psi) . \Theta. \qquad (2{\cdot}1)$$

Now, since the simplex is self-polar to the quadric Ω, therefore η_ι is proportional to the cofactor of H_ι in the determinant D. Multiplying each term of the proportion by $\mu\text{H}_\iota + \overline{\text{H}}_\iota$, adding for all values of ι, and remembering that $\mu\omega_\eta^2 + \sigma_\eta^2 = 0$, we see that

$$\mu D + D^{(\eta)} = 0.$$

Similarly $$\nu D + D^{(\zeta)} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\pi D + D^{(\phi)} = 0. \qquad (2{\cdot}2)$$

Specifying ρ by means of the equation

$$\rho D + D^{(\psi)} = 0, \qquad (2{\cdot}3)$$

adding all the left-hand terms and using (2·1), we have

$$(\mu + \nu + \ldots + \pi + \rho)\,\Delta + \Theta = 0. \qquad (2{\cdot}4)$$

Reversing the process (2·2), we see from (2·3) that the wall $\psi_x = 0$ of the simplex envelopes the class-quadric expressed by $\rho\Omega + \Sigma = 0$. Thus we have the dual of Rohn's theorem as follows: If a variable simplex be self-polar to the (fixed) class-quadric expressed by $\Omega = 0$, if $\Sigma = 0$ express any other (fixed) class-quadric, and if n walls of the simplex respectively envelope the n (fixed) class-quadrics expressed by $\mu\Omega + \Sigma = 0$, $\nu\Omega + \Sigma = 0$, ..., $\pi\Omega + \Sigma = 0$; then the remaining wall of the simplex envelopes the (fixed) class-quadric expressed by $\rho\Omega + \Sigma = 0$, where ρ is such that, if $\Delta + \tau\Theta + \ldots$ be the discriminant of $\Omega + \tau\Sigma$, then

$$(\mu + \nu + \ldots + \pi + \rho)\,\Delta + \Theta = 0. \qquad (2{\cdot}5)$$

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* See Segre, *Encyk. d. math. Wiss.* III C 7, 863–4, footnote 288.

In the sequel, the quadric $\Omega = 0$ will be identified with the Absolute of non-euclidean geometry, and any two class-quadrics of an ∞^1 linear system $\lambda\Omega + \Sigma = 0$ will be termed *confocal*. A simplex which is self-polar to the Absolute (being such that any two of its walls are perpendicular to each other) will be termed a *right simplex*. Thus the theorem (2·5) may be enunciated: If a variable right simplex be such that n of its walls respectively envelope n (fixed) quadrics of a confocal system, then the remaining wall also envelopes a (fixed) quadric of the system. (2·6)

Applying (2·6) to the case in which the n given quadrics are identical, and remembering that the common point of any n walls of a right simplex is the absolute pole of the remaining wall, we have the theorem: If n variable tangent flats of a quadric be mutually perpendicular, then the locus of the common point of the n flats is a quadric, viz. the absolute reciprocal of a quadric confocal with the given quadric. This locus may be termed the *director quadric* of the given quadric. In the euclidean case it is a hypersphere; but not (in general) otherwise. (2·7)

3. *Pedal Quadrics in General.* If the Absolute class-quadric be expressed by the equation

$$\Omega \equiv \omega_\xi^2 = 0,$$

then the class-quadric expressed by the equation

$$\Sigma \equiv \sigma_\xi^2 = 0$$

will be confocal with a quadric inscribed to the base-simplex, if and only if the value of the quotient σ_{ii}/ω_{ii} be the same for all values of i. Σ is then termed a quadric *infocal* to the base-simplex, or, if no ambiguity thereby arises, simply an *infocal quadric*. If the constant value of σ_{ii}/ω_{ii} be denoted by κ, then the inscribed quadric confocal with Σ is expressed by

$$\Sigma - \kappa\Omega = 0.$$

Clearly an infocal quadric which touches one wall of the base-simplex must touch all the remaining walls. (3·1)

Now, of n mutually perpendicular flats, let $n - 1$ touch the infocal quadric Σ, and let the remaining one touch the inscribed quadric confocal with Σ. Then the locus of the common point of the n flats is a quadric, S', viz. (2·5) the absolute reciprocal of that expressed by

$$\Sigma' \equiv \Sigma + (\kappa - \Theta/\Delta)\,\Omega = 0.$$

S' will be termed the *pedal quadric*, and Σ' the *polar-pedal quadric* (q. the base-simplex) of Σ. Σ' is confocal with Σ, and therefore infocal to the base-simplex. The relation between Σ and Σ' is clearly $(1-1)$. (3·2)

If α_i denote that wall of the base-simplex which does not contain the vertex A_i, if A_i' denote the absolute pole of α_i, and

M any point of the section by a_i of S'; then the flat a_i and $n-1$ tangent flats from M to Σ are all mutually perpendicular. The line MA_i' is the common join of the $n-1$ tangent flats, which therefore all touch the tangent cone from the point A_i' to Σ; thus (2·7) the section by a_i of S' is the director quadric $(n-2)$-fold of the section by a_i of the tangent cone from A_i' to Σ. We have therefore the following theorem: *Let a_i denote any wall of a given simplex, and A_i' the absolute pole of a_i. Let Σ denote a class-quadric infocal to the simplex, Σ_i the quadric $(n-2)$-fold which is the section, by a_i, of the tangent cone from A_i' to Σ, and Q_i the director quadric $(n-2)$-fold, in a_i, of Σ_i $(i = 0, 1, \ldots, n)$. Then the $n+1$ quadrics Q_i are all contained by a quadric $(n-1)$-fold, S', the* pedal quadric *of Σ q. the simplex; and S' is the absolute reciprocal of a quadric confocal with Σ.* (3·3)

It should be noted that any quadric, Σ, inscribed in a given simplex has a pedal quadric, $\bar{S}'$, q. that simplex; and that $\bar{S}'$ is also the director quadric of $\bar{\Sigma}$. (3·4)

In the euclidean case, since the sections of any pedal quadric by the $n+1$ walls of the simplex of reference all become $(n-2)$-fold hyperspheres, therefore every pedal quadric becomes an $(n-1)$-fold hypersphere. (3·5)

Given any $r+1$ class-quadrics (of general position) infocal to the same simplex, $[A]$; it follows from (3·1) that every class-quadric of the ∞^r linear system which they determine is also infocal to $[A]$, and therefore possesses a pedal quadric q. $[A]$. Moreover the covariant Θ, and therefore $\kappa - \Theta/\Delta$, are linear in the coefficients of Σ. Thus, if any ∞^r linear system of infocal class-quadrics be denoted by $[\Sigma]$, and the polar-pedal quadrics of the system $[\Sigma]$ by $[\Sigma']$; then (3·2) the quadrics $[\Sigma']$ constitute an ∞^r linear system related to the system $[\Sigma]$, the polar-pedal quadric of each quadric of $[\Sigma]$ being the corresponding quadric of $[\Sigma']$. Reciprocating absolutely, we have the theorem: *Given an ∞^r linear system of infocal class-quadrics; then the pedal quadrics of the system constitute an ∞^r linear system of order-quadrics.* (3·6)

Since the polar-pedal quadric, Σ', of any (infocal) quadric, Σ, is also infocal, therefore Σ' possesses a polar-pedal quadric, Σ'', expressed (2·5, 3·2) by the equation

$$\Sigma'' \equiv \Sigma - (n-2)(\kappa - \Theta/\Delta)\,\Omega = 0.$$

If $n = 2$, then the quadrics Σ, Σ'' are identical, i.e. each of the quadrics Σ, Σ' is the absolute reciprocal of the pedal quadric of the other.

If $n > 2$, then Σ, Σ'' are not in general identical. Only in two special cases does identity occur, viz. (i) when Σ is the Absolute, Ω, (ii) when $\Theta = \kappa\Delta$. In both these cases, Σ is the absolute reciprocal of its own pedal quadric.

It is easy to interpret the condition (ii) geometrically. For let u_{ij} denote the cofactor of ω_{ij} in the determinant Δ, so that the Absolute order-quadric is expressed by the equation $U \equiv u_x^2 = 0$. Then the equation

$$K_i \equiv \omega_{ii}.U - \Delta x_i^2 = 0,$$

expresses the tangent cone to the Absolute from the point A_i' (3·3); i.e. K_i is the point-hypersphere determined by A_i' ($i = 0, 1, \ldots, n$). The class-quadric $\Sigma \equiv \sigma_x^2 = 0$ is inpolar to K_i if

$$\omega_{ii}.\Theta - \sigma_{ii}.\Delta = 0.$$

Σ is thus inpolar to all the point-hyperspheres K_i ($i = 0, 1, \ldots, n$) if and only if

$$\sigma_{00}/\omega_{00} = \sigma_{11}/\omega_{11} = \ldots = \sigma_{nn}/\omega_{nn} = \Theta/\Delta.$$

Remembering that in (ii) κ denotes the common value of the quotients σ_{ii}/ω_{ii}, we see that all those class-quadrics (and no others) which are inpolar to all the point-hyperspheres K_i are infocal to the base-simplex and satisfy the condition $\Theta = \kappa\Delta$. Summing up: *Let Σ denote any quadric infocal to a given simplex, $[A]$, and Σ' the absolute reciprocal of the pedal quadric, q. $[A]$, of Σ; then Σ' is also infocal to $[A]$.* (3·7)

In two dimensions, Σ is the absolute reciprocal of the pedal quadric (conic), q. $[A]$, of Σ'. (3·8)

In higher space, no two distinct quadrics are such that each is the absolute reciprocal of the pedal quadric of the other. A quadric Σ, other than the Absolute, possesses a pedal quadric (q. $[A]$), of which Σ is itself the absolute reciprocal, if and only if Σ be inpolar to each of the point-hyperspheres determined by the absolute poles of the walls of $[A]$. (3·9)

4. *Pedal Quadrics of Point-Pairs.* A point-pair $X \equiv (x_0, x_1, \ldots, x_n)$, $Y \equiv (y_0, y_1, \ldots, y_n)$ constitutes an infocal class-quadric if and only if (3·1) the value of the quotient x_iy_i/ω_{ii} is the same for all values of i; the points X, Y are then *isogonal conjugates* q. the base-simplex. The pedal quadric, as specified (3·3), of the point-pair X, Y is the absolute reciprocal of a quadric confocal with the point-pair, i.e., is a quadric which meets the Absolute at its common points with the absolute polar flats of X, Y, and is thus a quadric of revolution about the line XY. The quadric Σ_i (3·3) degenerates into the pair of points, X_i, Y_i, which are the orthogonal projections of X, Y on the wall a_i of the simplex. The quadric Q_i, which now contains the pair of points X_i, Y_i, remains the director quadric of that point-pair; i.e., Q_i is now the locus of the common point of $n-1$ mutually perpendicular flats [linear $(n-2)$-folds in a_i] of which any number contain X_i and the remainder contain Y_i. Thus from (3·3) we obtain the following: *The orthogonal projections, on the walls of a simplex, $[A]$, of a pair*

of points, X, Y, isogonally conjugate q. $[A]$ lie on a quadric of revolution about the line XY; the pedal quadric, q. $[A]$, of the point-pair X, Y. (4·1)

Now let us consider the system of class-quadrics, each member of which (i) is infocal to $[A]$ and (ii) touches every flat which contains a given (straight) line of general position. The restriction (i) involves the satisfaction of n linear conditions, while (ii) involves the satisfaction of $\frac{1}{2}n(n-1)$ further linear conditions. Since $\frac{1}{2}n(n+3) - n - \frac{1}{2}n(n-1) = n$ additional conditions are required for the complete determination of a quadric, therefore the class-quadrics in question constitute an ∞^n linear system. Clearly if one of a pair of points isogonally conjugate q. $[A]$ lies on the given line, then the point-pair is a class-quadric of the system; also the second point of the pair lies on the isogonal transform, q. $[A]$, of the given line, viz. on an n-ic curve (1-fold) which contains all the vertices of $[A]$. Using (3·6), we have the theorem: *If one of a pair of points isogonally conjugate q. a simplex, $[A]$, describe a (straight) line, then the other describes an n-ic curve containing all the vertices of $[A]$; and the pedal quadric, q. $[A]$, of the point-pair is a member of a certain ∞^n linear system of order-quadrics, of which the remaining members are the pedal quadrics of the remaining class-quadrics, infocal to $[A]$, which touch every flat containing the given line. All these pedal quadrics have a common Jacobian $(n-1)$-fold, which is of order $n+1$.* In the euclidean case, the Jacobian breaks up (3·5) into a hypersphere and the flat at infinity counted $n-1$ times. (4·2)

It is obvious that any vertex of $[A]$, and any point of the opposite wall, constitute a point-pair isogonally conjugate q. $[A]$. It may be shown, in the same manner as in (4·2), that the quadrics, inscribed in $[A]$, which touch every flat containing a given line of general position, constitute an ∞^{n-1} linear system of class-quadrics; of this system the $n+1$ isogonally conjugate point-pairs made up of any vertex of $[A]$ and the meet of the given line with the opposite wall are members. Using (3·4), we see that: *If A_i denote any vertex, and a_i the opposite wall, of a simplex $[A]$, and if a given (straight) line meet a_i at L_i $(i = 0, 1, \ldots, n)$: then the $n+1$ pedal quadrics, q. $[A]$, of the point-pairs A_i, L_i are also the director quadrics of those point-pairs, and are members of an ∞^{n-1} linear system of order-quadrics; thus each of the $n+1$ pedal quadrics contains all the common points of the other n. The remaining quadrics of the ∞^{n-1} system are the director quadrics of the remaining class-quadrics, inscribed to $[A]$, which touch every flat containing the given line.* (4·3)

Applying (3·9) to the case of a point-pair, we have (in general) the following: *The pedal quadric, q. $[A]$, of a pair of points, X, Y (necessarily isogonally conjugate q. $[A]$), breaks up into a pair of* *planes (viz., the absolute polars of X, Y) if and only if X, Y be corresponding points on the Jacobian of the $n+1$ point-hyperspheres determined by the absolute poles of the walls of $[A]$.* (4·4)

5. *The Extension of Neuberg's Theorem.* From (4·3), or from the consideration that μ_i is the polar flat of A_i q. the point-hypersphere determined by the point M[*], may be derived the theorem: *If M denote any point, and μ_i the flat containing M and perpendicular to the line A_iM; then the linear $(n-2)$-folds common to the pairs of flats α_i, μ_i all meet ∞^{n-2}*[†] *(straight) lines, which lie on an $(n-1)$-fold of order $n-1$ $(i=0, 1, \ldots, n)$*[‡]. (5·1)

Let M_i, μ denote the pole, polar of μ_i, M respectively. Then, reciprocating (5·1) absolutely, we obtain the following: *If μ denote any flat, M_i the orthogonal projection on μ of A_i, and N_i the orthogonal projection on α_i of M_i; then the lines M_iN_i all meet ∞^{n-2} linear $(n-2)$-folds, and there is an $(n-1)$-fold of class $n-1$ which touches every flat containing any of the lines M_iN_i $(i=0, 1, \ldots, n)$.* (5·2)

The last theorem is an analogue of that of Neuberg (1·1); for if $n=2$, then the lines M_iN_i meet at a point. But in this case they may be more fully specified. For the $n+1$ linear $(n-2)$-folds of (5·1) are now three points, L_i, in line; L_i being the absolute pole of the line M_iN_i. The director conic, S_i, of the point-pair A_i, L_i contains M; therefore the absolute reciprocal, Σ_i, of S_i touches the absolute polar, μ, of M. But (3·4) S_i is also the pedal conic of the point-pair A_i, L_i; therefore Σ_i is the polar-pedal conic of that point-pair. By the reciprocity in two dimensions (3·8), the point-pair A_i, L_i constitutes the polar-pedal conic of Σ_i. Thus, if α_i' denote the absolute polar of A_i, then the line-pair α_i', M_iN_i constitutes the pedal conic of Σ_i; and since Σ_i is a member of the ∞^2 linear system of infocal class-conics which touch μ, therefore (4·2) the line-pair α_i', M_iN_i is a member of the related ∞^2 linear system of order-conics. Thus the line M_iN_i is the join of a pair of corresponding points on the Jacobian of the latter system. For the case of two dimensions, we have therefore a complete analogue of the Neuberg-Lemoyne theorem as follows: *If A_i denote a vertex of a triangle, $[A]$, in a non-euclidean plane, μ any line in the plane, M_i the orthogonal projection on μ of A_i, and N_i the orthogonal projection on A_jA_k of M_i, and if J denote the Jacobian of the pedal conics, q. $[A]$, of those conics (infocal to $[A]$) which touch the line μ; then the lines M_iN_i are joins of pairs of corresponding points on J, and meet at a point $(i, j, k = 0, 1, 2)$.* (5·3)

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[*] See Schläfli, *J. f. Math.* 65, (1866), 189–197; Berzolari, *Rend. Circ. Mat. Palermo*, 20, (1905), 229–247; Brusotti, *ibid.* 248–255.

[†] Instead of ∞^{n-3}.

[‡] The plane euclidean case of this theorem is discussed by R. T. Robinson, *Math. Gazette*, 12, (1925), 377–383.

The temperature distribution in a transformer or other laminated core of rectangular cross section in which heat is generated at a uniform rate. By J. D. Cockcroft, B.A., St John's College.

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1. *Introduction.*

The problem of the temperature distribution in transformer or other laminated cores in which heat is generated at a uniform rate, does not appear to have been treated by exact mathematical methods. A good deal of work of a semi-empirical nature has been done by Vidmar*, Kuchler† and others, the object of which has been to enable designers of transformers to obtain an estimate of the maximum temperature likely to prevail in the core.

This paper deals with the case of a typical transformer core of rectangular cross section, constructed of thin sheets of laminated iron. One dimension of the core is large compared with the other two, so that the problem can be treated two-dimensionally. Heat is generated in the core at a uniform rate by eddy currents induced by the alternating magnetic induction in the core. The thermal conductivity in the direction of the laminations is about seventy times that across the laminations. The core is immersed in oil for cooling purposes and the cooling conditions at the surface are assumed to be represented by a linear relation between the surface temperature drop and the heat flow perpendicular to the surface.

Experimental figures are given for the rate of generation of heat, thermal conductivities and the *actual* cooling conditions. The effect of the deviation of the latter from the linear relation is discussed.

The results obtained express the temperature at any point in the cross section in the form of an infinite series which is very rapidly convergent. In general only the first term of the series needs to be evaluated and tables are given enabling this to be very rapidly carried out. Isothermals are drawn for one or two standard cases.

2. *Core with uniform oil temperature.*

A laminated core of rectangular cross section ($2a \times 2b$) is considered. Axes X and Y are taken through the centre of the section as in Fig. 1, with the X axis parallel to the direction of the laminations. The length of the core in the Z direction is great compared

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[*] Vidmar, "The heat radiation problem of the Transformer," *Electrot. u. Maschinenbau*, vol. 39, p. 68. Feb. 6, 1921.

[†] Kuchler, "The heating of oil cooled transformers," *Electrot. Zeits.* vol. 44, pp. 54, 59. Jan. 18, 1923.

with a or b so that the problem is treated two dimensionally. The thermal conductivities in the X and Y directions are K and K', where K/K' is of the order of seventy.

The rate of heat generation in the core is W units per unit volume.

The surface thermal resistance is R, i.e. the surface temperature drop is R times the heat flow perpendicular to the surface.

The first case considered is that in which the oil temperature around the faces of the core is uniform.

The differential equation of heat conduction to be satisfied from $X = +a$ to $X = -a$, $Y = +b$ to $Y = -b$ is

$$K\frac{\partial^2 T}{\partial x^2} + K'\frac{\partial^2 T}{\partial y^2} + W = 0 \qquad \text{......(1).}$$

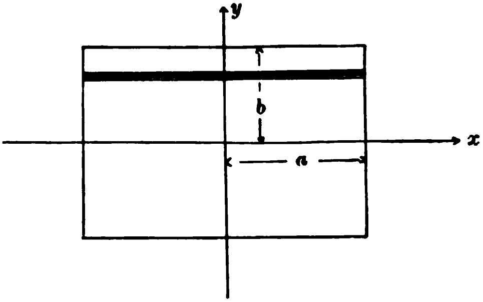

Fig. 1

The boundary conditions are

$$T - T_0 = -\ RK\frac{\partial T}{\partial x}, \quad x = +a,$$

$$T - T_0 = +\ RK\frac{\partial T}{\partial x}, \quad x = -a,$$

$$T - T_0 = -\,R'K'\frac{\partial T}{\partial y}, \quad y = +b,$$

$$T - T_0 = +\,R'K'\frac{\partial T}{\partial y}, \quad y = -b.$$

T_0 is the oil temperature assumed constant at the faces. R' is the surface thermal resistance at the Y faces, taken different from R for greater generality.

We transform equation (1) by writing

$$\left.\begin{aligned} T - T_0 &= \frac{W}{2K}\psi \\ y &= \sqrt{\frac{K'}{K}}\cdot y_1 \end{aligned}\right\} \qquad \text{......(2)}$$

and obtain
$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial {y_1}^2}+2=0 \qquad \ldots\ldots(3).$$

The boundary conditions now become

$$\psi=\mp \quad RK \quad \frac{\partial\psi}{\partial x}=\mp\mu\frac{\partial\psi}{\partial x}, \qquad x=\pm a \quad \ldots(4\,a),$$

$$\psi=\mp R'\sqrt{KK'}\,\frac{\partial\psi}{\partial y_1}=\mp\mu'\frac{\partial\psi}{\partial y_1}, \quad y_1=\pm b_1 \quad \ldots(4\,b),$$

writing
$$\mu=RK, \quad \mu'=R'\sqrt{KK'}$$
$$b_1=\sqrt{\frac{K}{K'}}\,b.$$

We now make the further transformation

$$\psi=\phi+(a^2-x^2)+2a\mu \qquad \ldots\ldots(5).$$

Equation (3) then becomes

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial {y_1}^2}=0 \qquad \ldots\ldots(6),$$

whilst the boundary conditions become

$$\phi=-\mu\frac{\partial\phi}{\partial x}, \quad x=+a \qquad \ldots\ldots(7\,a),$$

$$\phi=+\mu\frac{\partial\phi}{\partial x}, \quad x=-a \qquad \ldots\ldots(7\,b),$$

$$\phi=x^2-a^2-2a\mu-\mu'\frac{\partial\phi}{\partial y_1}, \quad y_1=+b_1 \quad \ldots\ldots(7\,c),$$

$$\phi=x^2-a^2-2a\mu+\mu'\frac{\partial\phi}{\partial y_1}, \quad y_1=-b_1 \quad \ldots\ldots(7\,d).$$

We assume
$$\phi=\sum_{n=1}^{n=\infty}A_n\cos\alpha_n x\cosh\alpha_n y_1,$$

where the coefficients α_n are constants. This form satisfies (6).

To satisfy (7 *a*) and (7 *b*) we must have

$$\sum_{n=1}^{n=\infty}A_n\cos\alpha_n a\cosh\alpha_n y_1=+\mu\alpha_n\sum_{n=1}^{n=\infty}A_n\sin\alpha_n a\cosh\alpha_n y_1,$$

so that for all n we must have

$$\tan\alpha_n a=\frac{1}{\mu\alpha_n} \qquad \ldots\ldots(8).$$

The roots of equation (8) therefore give α_n. By the usual graphical method of solving this equation we see at once that $\alpha_n a$ approaches $(n-1)\pi$ as n becomes large.

We have still to satisfy (7 *c*) and (7 *d*). Both these equations lead to the same result

$$\sum_1^\infty A_n (\cosh \alpha_n b_1 + \mu' \alpha_n \sinh \alpha_n b_1) \cos \alpha_n x = x^2 - a^2 - 2a\mu \quad \ldots(9).$$

This equation has to hold for all values of X from $-a$ to $+a$. We therefore expand $(x^2 - a^2 - 2a\mu)$ in a Fourier series over this interval.

It is easily shown, using equation (8), that for $m \neq n$,

$$\int_{-a}^{+a} \cos \alpha_m x \cos \alpha_n x \, dx = 0,$$

whilst

$$\int_{-a}^{+a} \cos^2 \alpha_n x \, dx = a \left\{1 + \frac{\sin 2\alpha_n a}{2\alpha_n a}\right\} \quad \ldots\ldots(10)$$

and

$$\int_{-a}^{+a} (x^2 - a^2 - 2a\mu) \cos \alpha_n x \, dx = -\frac{4 \sin \alpha_n a}{\alpha_n^3} \quad \ldots(11).$$

We therefore find that

$$A_n = -4a^2 \frac{\sin \alpha_n a}{(\alpha_n a)^3} \frac{1}{1 + \dfrac{\sin 2\alpha_n a}{2\alpha_n a}} \frac{1}{\cosh \alpha_n b_1 + \mu' \alpha_n \sinh \alpha_n b_1} \quad (12).$$

Thus

$$\psi = (a^2 - x^2) + 2a\mu + \sum_1^\infty A_n \cos \alpha_n x \cosh \alpha_n y_1, \text{ from (5)}$$

and $T = T_0 + \dfrac{W}{2K} \psi$

$$= T_0 + WaR + \frac{W}{2K} \left\{(a^2 - x^2) + \sum_1^\infty A_n \cos \alpha_n x \cosh \alpha_n y_1\right\}.$$

The final solution is therefore

$$T = T_0 + WaR + \frac{W}{2K} \left\{ (a^2 - x^2) - 4a^2 \sum_1^\infty \frac{\sin \alpha_n a}{(\alpha_n a)^3} \frac{1}{1 + \dfrac{\sin 2\alpha_n a}{2\alpha_n a}} \right.$$

$$\left. \times \frac{\cos \alpha_n x \cosh \sqrt{\dfrac{K}{K'}} . \alpha_n y}{\cosh \sqrt{\dfrac{K}{K'}} \alpha_n b + R' \sqrt{KK'} \sinh \sqrt{\dfrac{K}{K'}} \alpha_n b} \right\} \quad \ldots\ldots(13).$$

Note. (1) For b infinite, the solution reduces to that of the simple one dimensional case.

(2) If R is zero, i.e., if there is no temperature drop at the surface, then equation (8) for α_n becomes

$$\tan \alpha_n a = \infty$$

or

$$\alpha_n a = (2n-1)\frac{\pi}{2},$$

so that (13) becomes

$$T = T_0 + \frac{W}{2K}\left\{(a^2 - x^2) - \frac{32a^2}{\pi^3}\left(\frac{\cosh\sqrt{\frac{K}{K'}}\frac{\pi y}{2a}}{\cosh\sqrt{\frac{K}{K'}}\frac{\pi b}{2a}}\cos\frac{\pi x}{2a}\right.\right.$$

$$\left.\left. - \frac{1}{3^3}\frac{\cosh\sqrt{\frac{K}{K'}}\frac{3\pi y}{2a}}{\cosh\sqrt{\frac{K}{K'}}\frac{3\pi b}{2a}}\cos\frac{3\pi x}{2a} + \dots\right)\right\} \quad \text{......(14).}$$

This particular case is analytically similar to the problem of the distribution of shear stress in a rectangular bar*.

It will be shown later that the series obtained for T in (13) is convergent throughout the range required.

3. *Core with non uniform oil temperature.*

We consider a case in which the oil temperature across the four faces is defined as follows

$$\begin{aligned} T_0 &= T_1 + f_1(x), && \text{when } y_1 = +b_1, \\ T_0 &= T_1 + f_2(x), && \quad ,, \quad y_1 = -b_1, \\ T_0 &= T_1 + f_3(y_1), && \quad ,, \quad x = +a, \\ T_0 &= T_1 + f_4(y_1), && \quad ,, \quad x = -a \qquad \text{......(15),} \end{aligned}$$

where T_1 is a constant.

We have to add to the solution already obtained a function T' such that

$$\left.\begin{aligned} &\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y_1^2} = 0, \\ \text{and} \quad & T' - T_0 = \mp\mu\frac{\partial T'}{\partial x}, \quad x = \pm a, \\ & T' - T_0 = \mp\mu\frac{\partial T'}{\partial y_1}, \quad y_1 = \pm b_1 \end{aligned}\right\} \qquad \text{......(16),}$$

where T_0 is now expressed by (15).

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* Love, *Elasticity*, p. 322.

By precisely similar methods to those used in deriving the results of section 2, we find

$$T = T_1 + \sum_1^\infty \left\{\frac{(P_n + Q_n)\cosh \alpha_n y_1}{2(\cosh \alpha_n b_1 + \mu'\alpha_n \sinh \alpha_n b_1)} + \frac{(P_n - Q_n)\sinh \alpha_n y_1}{2(\sinh \alpha_n b_1 + \mu'\alpha_n \cosh \alpha_n b_1)}\right\}\cos \alpha_n x$$
$$+ \sum_1^\infty \left\{\frac{(R_n + S_n)\cosh \beta_n x}{2(\cosh \beta_n a + \mu\beta_n \sinh \beta_n a)} + \frac{(R_n - S_n)\sinh \beta_n x}{2(\sinh \beta_n a + \mu\beta_n \cosh \beta_n a)}\right\}\cos \beta_n y_1 \quad \text{......(17)},$$

where α_n and β_n are the roots of the equations

$$\tan \alpha_n a = \frac{1}{\mu\alpha_n},$$
$$\tan \beta_n b_1 = \frac{1}{\mu'\beta_n},$$

and P_n, Q_n, R_n, S_n are the coefficients in the expansions of $f_1(x)$, $f_2(x)$, etc., i.e.

$$\left.\begin{aligned} f_1(x) &= \sum_1^\infty P_n \cos \alpha_n x \\ f_2(x) &= \sum_1^\infty Q_n \cos \alpha_n x \end{aligned}\right\}, \quad \text{from } x = -a \text{ to } x = +a,$$

$$\left.\begin{aligned} f_3(y_1) &= \sum_1^\infty R_n \cos \beta_n y_1 \\ f_4(y_1) &= \sum_1^\infty S_n \cos \beta_n y_1 \end{aligned}\right\}, \quad \text{from } y_1 = -b_1 \text{ to } y_1 = +b_1.$$

The coefficients obtained for any actual function are more complex than in an expansion in integral multiples of $\frac{\pi x}{2a}$ but they can be evaluated if necessary. Thus if $f_1(x)$ constant $= C$, then

$$P_n = 2C\frac{\sin \alpha_n a}{\alpha_n a}\,\frac{1}{1 + \sin 2\alpha_n a/2\alpha_n a}.$$

It may easily be verified that (17) satisfies equations (16).

The case where the oil temperature is symmetrical about the centre leads to a simplification of (17) for we may then write

$$P_n = Q_n,$$
$$R_n = S_n,$$

and obtain

$$T' = T_1 + \sum_1^\infty \left\{ \frac{P_n \cosh \alpha_n y_1 \cos \alpha_n x}{\cosh \alpha_n b_1 + \mu' \alpha_n \sinh \alpha_n b_1} + \frac{R_n \cosh \beta_n x \cos \beta_n y_1}{\cosh \beta_n a + \mu \beta_n \sinh \beta_n a} \right\} \quad \text{......(18).}$$

The complete solution is in either case

$$T = T' + WaR + \frac{W}{2K} \left\{ (a^2 - x^2) + \sum_1^\infty A_n \cos \alpha_n x \cosh \alpha_n y_1 \right\} \quad (19).$$

4. *A core with central oil duct.*

Oil ducts are generally introduced to assist the cooling of the cores, a free circulation of oil being maintained in the direction perpendicular to the plane of the section shown in Fig. 2.

The oil temperature in the duct can be assumed to be known with sufficient exactitude from experimental data in any particular case. Since it will differ in general from the oil temperature outside the duct and since further the oil in the duct will be moving it is advisable to take a special thermal surface resistance R'' for the duct surfaces. The temperature in the core will be symmetrical about the centre line of the duct, so that it will only be necessary to consider one half of the core.

Let the temperature across the duct be represented by

$$f_2(x) = T_1 + \sum_1^\infty Q_n \cos \alpha_n x,$$

and let the temperature of the oil outside be considered constant at T_1. The boundary conditions to be satisfied are then

$$\left.\begin{aligned} T - f_2(x) &= R'' \sqrt{KK'} \frac{\partial T}{\partial y_1}, & y_1 &= 0 \\ &= \mu'' \frac{\partial T}{\partial y_1}, & & \\ T - T_1 &= -\mu' \frac{\partial T}{\partial y_1}, & y_1 &= b_1 \\ T - T_1 &= \mp \mu \frac{\partial T}{\partial x}, & x &= \pm a \end{aligned}\right\} \quad \text{......(20).}$$

The analytical treatment follows the same lines as sections 2 and 3, but terms of type $\sinh \alpha_n \left(y_1 - \frac{b_1}{2} \right)$ are introduced on account of the asymmetry about the centre line of the half core.

The final result obtained is

$$T = T_1 + WaR + \frac{W}{2K}(a^2 - x^2) + \sum_1^\infty \left\{ C_n \cosh \alpha_n \left(y_1 - \frac{b_1}{2} \right) + D_n \sinh \alpha_n \left(y_1 - \frac{b_1}{2} \right) \right\} \cos \alpha_n x \quad \ldots\ldots(21),$$

where

$$-C_n = \frac{\dfrac{W}{2K}\left(\dfrac{2 \sinh \dfrac{\alpha_n b_1}{2} + \alpha_n (\mu' + \mu'') \cosh \dfrac{\alpha_n b_1}{2}}{1 + \dfrac{\sin 2\alpha_n a}{2\alpha_n a}} \, \dfrac{4a^2 \sin \alpha_n a}{(\alpha_n a)^3} \right) - Q_n \left(\sinh \dfrac{\alpha_n b_1}{2} + \mu' \alpha_n \cosh \dfrac{\alpha_n b_1}{2} \right)}{\sinh \alpha_n b_1 (1 + \alpha_n^2 \mu' \mu'') + \alpha_n (\mu' + \mu'') \cosh \alpha_n b_1},$$

$$D_n = \frac{\dfrac{W}{2K}\left(\dfrac{\alpha_n (\mu' - \mu'') \sinh \dfrac{\alpha_n b_1}{2}}{1 + \dfrac{\sin 2\alpha_n a}{2\alpha_n a}} \, \dfrac{4a^2 \sin \alpha_n a}{(\alpha_n a)^3} \right) - Q_n \left(\cosh \dfrac{\alpha_n b_1}{2} + \mu' \alpha_n \sinh \dfrac{\alpha_n b_1}{2} \right)}{(1 + \alpha_n^2 \mu' \mu'') \sinh \alpha_n b_1 + \alpha_n (\mu' + \mu'') \cosh \alpha_n b_1} \quad \ldots\ldots(22).$$

It may easily be verified that this expression satisfies equations (20).

The case of a non uniform oil temperature round the faces of the core can be solved but the algebra becomes so complex that it is not included.

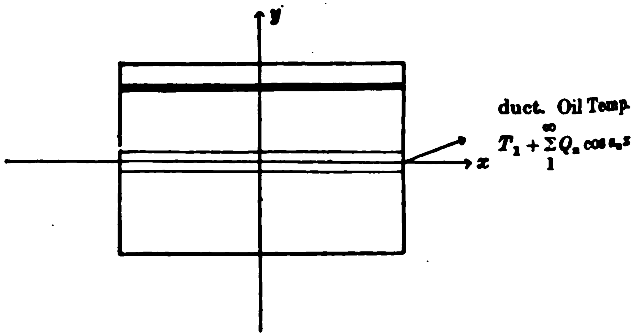

Fig. 2

5. *The effect of the variation of surface thermal resistance with heat flow.*

So far we have considered only the ideal case in which the surface conditions are represented accurately by equations of type

$$T - T_0 = \mp RK \frac{\partial T}{\partial x}, \quad x = \pm a,$$

$$T - T_0 = \mp R'K' \frac{\partial T}{\partial y}, \quad y = \pm b,$$

where R and R' are constants. Actually however the surface thermal resistance varies with both heat flow and oil temperature. In Fig. 3 $(T - T_0)$ is plotted against $-K\frac{\partial T}{\partial x}$, i.e. against the heat flow for oil temperatures 40° C. and 60° C. from experimental data. It is evident that the relation is non linear. It is impossible to take this departure from linearity into account in the analysis so that it remains to make the best approximation possible to the surface conditions.

If we take a straight line such as AB in Fig. 3, which lies as nearly as possible to all the points on the curve likely to be relevant in any particular case, to represent the surface conditions the analytical expression for the boundary conditions becomes

$$T - T_0 = T_2 \mp R_1 K \frac{\partial T}{\partial x}, \quad x = \pm a,$$

where T_2 gives the intersection of AB with the Y axis and R_1 represents the slope of AB. In other words we assume there is a fixed temperature drop T_2 at the surface and a mean surface thermal resistance R_1. Thus, if the surface temperature drop is not likely to exceed 30 degrees and the mean oil temperature is 40° C. we take as the surface conditions

$$T - 40 = 3 \mp 100K \frac{\partial T}{\partial x}, \quad x = \pm a \qquad \text{......(23)}$$

for the X faces.

Since the faces of the core parallel to the direction of the laminations will in general be much cooler than the X faces, a surface temperature relation using the same constants as (23) for the Y faces might be too much in error. In this case it may be possible to get a closer approximation by using a different value of the mean surface thermal resistance, keeping the same fixed surface drop T_2, for the Y faces, i.e. we take the conditions for the Y faces to be represented by

$$T - T_0 = T_2 \mp R_2 K' \frac{\partial T}{\partial y}, \quad y = \pm b.$$

We should get a closer approximation by changing T_2 as well, but the resulting complication in the numerical work makes this inadvisable.

The effect of this modification in the surface conditions on the results of the preceding section is to substitute

$(T_0 + T_2)$ for T_0 in (13), $(T_1 + T_2)$ for T_1 in (18) and (21),
R_1 and R_2 for R and R' in (13), (18), (21) and (22).

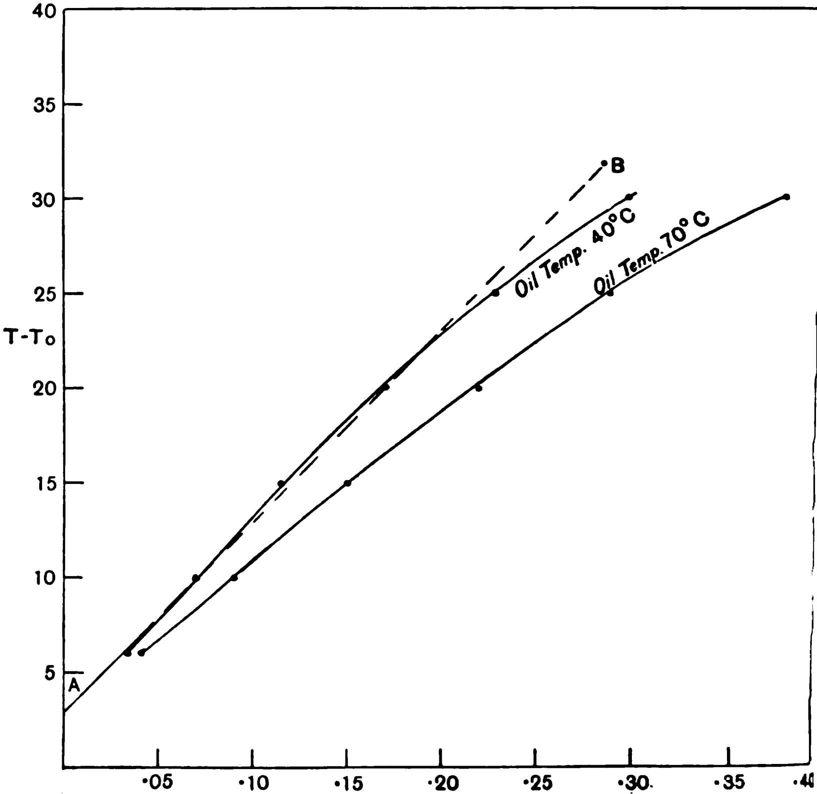

Fig. 3

6. *Numerical results.*

We have in general to work out numerically such results as

$$T = T_1 + WaR + \frac{W}{2K}\left\{(a^2 - x^2) + \sum_1^\infty A_n \cosh \alpha_n y_1 \cos \alpha_n x\right\}$$

$$+ \sum_1^\infty \left\{P_n \frac{\cosh \alpha_n y_1 \cos \alpha_n x}{\cosh \alpha_n b_1 + \mu' \alpha_n \sinh \alpha_n b_1} + R_n \frac{\cosh \beta_n x \cos \beta_n y_1}{\cosh \beta_n a + \mu \beta_n \sinh \beta_n a}\right\} \quad \ldots\ldots(24),$$

where

$$A_n = -4a^2 \frac{\sin \alpha_n a}{(\alpha_n a)^3} \frac{1}{1 + \dfrac{\sin 2\alpha_n a}{2\alpha_n a}} \frac{1}{\cosh \alpha_n b_1 + \mu' \alpha_n \sinh \alpha_n b_1},$$

and P_n, R_n are empirically determined coefficients depending on the oil temperature.

In considering the convergence of the series

$$\sum_1^\infty A_n \cosh \alpha_n y_1 \cos \alpha_n x$$

we may take the worst possible case in the interval considered to be given by $y_1 = b_1$, $x = 0$. The nth term of the series then becomes

$$-4a^2 \frac{\sin \alpha_n a}{(\alpha_n a)^3} \frac{1}{1 + \dfrac{\sin 2\alpha_n a}{2\alpha_n a}} \frac{1}{1 + \mu' \alpha_n \tanh \alpha_n b_1}.$$

For large values of n we find easily that

$$\alpha_n a \approx (n-1)\pi + \frac{a}{RK(n-1)\pi},$$

so that
$$\sin \alpha_n a \approx \frac{a}{RK(n-1)\pi},$$

and
$$\frac{1}{1 + \mu' \alpha_n \tanh \alpha_n b_1} \approx \frac{a}{\mu'(n-1)\pi}.$$

Hence $A_n \cos \alpha_n x \cosh \alpha_n y$, is at most of order $1/(n-1)^5$.

The series is easily proved to be uniformly convergent in the separate variables in the interval and may be differentiated term by term.

On substituting numerical values for the different constants it is found that the contribution of the second term of the series is in general less than 0·5 per cent. of that of the first term, so that in all practical cases, we have only to calculate A_1.

K for ordinary silicon sheet iron may be taken as 0·5 watts, cm.$^{-1}$, deg.$^{-1}$. K' varies with the pressure to which the laminations are subjected, but we may take K/K' as 77·5 in the numerical work.

W may be taken as 0·02 watts per c.c. R_1 varies between 80 and 100 (deg. watts^{-1} cm.^{2}), R_2 between 60 and 140 according to the method of approximation to the surface conditions adopted.

$$\frac{y_1}{y} = \frac{b_1}{b} = \sqrt{\frac{K}{K'}} = 8{\cdot}8,$$

α_n is defined by
$$\tan \alpha_n a = \frac{1}{R_1 K \alpha_n},$$

and β_n by
$$\tan \beta_n b_1 = \frac{1}{R_2 \sqrt{KK'} \beta_n}.$$

In Table I the first root α_1 is given for four typical values of a and four values of R_1.

TABLE I.

α_1

R_1	a = 50	25	10	5
80	·01864	·0288	·0485	·069
90	·0179	·0274	·0455	·0656
100	·0172	·0262	·0434	·0628
120	·0162	·0242	·0390	·0570

TABLE II.

	a	$R_1=80$. b/a = $\frac{1}{4}$	$\frac{1}{2}$	1	2
$R_2=60$	50	1560	204	3·39	$9{\cdot}35.10^{-4}$
	25	952	202	8·5	$1{\cdot}52.10^{-2}$
	10	480	177	21·2	0·296
	5	281	145	32·7	1·57
$R_2=80$	50	1530	200	3·32	$9{\cdot}16.10^{-4}$
	25	915	197	8·3	$1{\cdot}48.10^{-2}$
	10	460	170	20·1	0·281
	5	270	137	30·8	1·48

	a	$R_1=90$. b/a = $\frac{1}{4}$	$\frac{1}{2}$	1	2
$R_2=60$	50	1880	267	5·20	$2{\cdot}01.10^{-3}$
	25	1130	260	12·80	$3{\cdot}1.10^{-2}$
	10	580	232	31·8	0·568
	5	326	177	42·9	2·39
$R_2=80$	50	1810	256	5·00	$1{\cdot}93.10^{-3}$
	25	1105	252	12·4	$3{\cdot}01.10^{-2}$
	10	556	222	30·1	0·546
	5	315	167·5	40·4	2·25

	a	$R_1=100$. b/a = $\frac{1}{4}$	$\frac{1}{2}$	1	2
$R_2=60$	50	2100	318	7·38	$3{\cdot}67.10^{-3}$
	25	1290	320	18·0	$5{\cdot}6.10^{-2}$
	10	660	276	41·6	0·924
	5	366	205	53·4	3·39
$R_2=80$	50	2070	314	7·27	$3{\cdot}61.10^{-3}$
	25	1260	312	17·6	$5{\cdot}51.10^{-2}$
	10	640	265	39·9	0·872
	5	354	196	50·6	3·22

Table II gives A_1 for values of R_1 of 80, 90, 100 and R_2 of 60 and 80 for values of a 50, 25, 10 and 5, and ratios b/a of 2, 1, 0·5 and 0·25.

From Table II numerical values for the first series of terms in (24) can be obtained quickly. The convergence of the series of

Core 100 cms. × 100 cms. Oil Temp. 40° C.

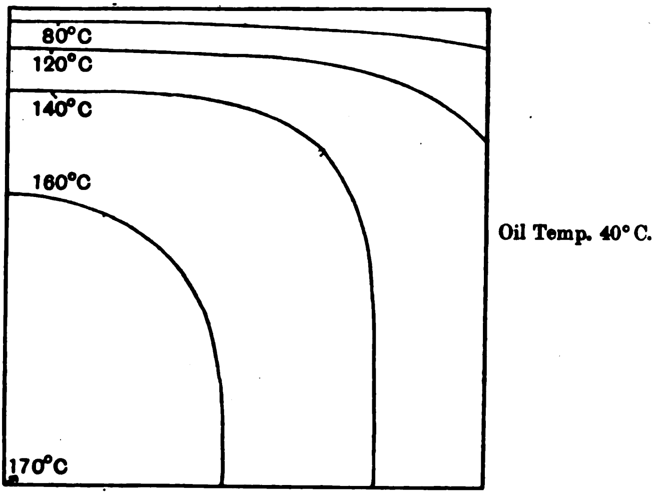

Fig. 4

Core 20 cms. × 20 cms. Oil Temp. 40° C.

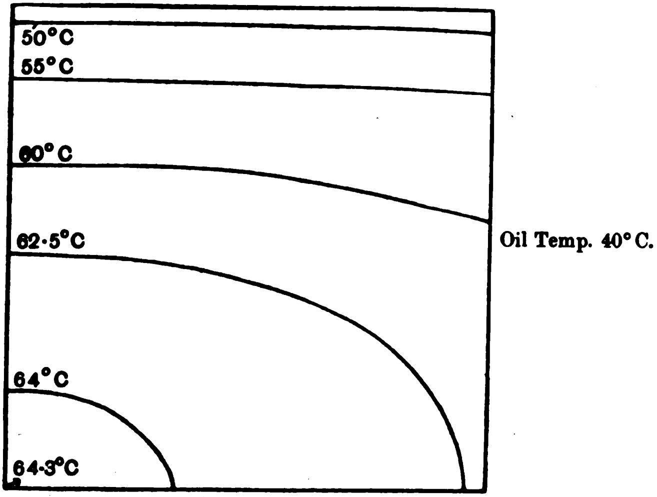

Fig. 5

terms depending on the variation of the oil temperature depends entirely on the coefficients P_n and R_n. Generally it should be possible to approximate by taking only one or two terms in this series.

Isothermals are drawn for two cores in Figs. 4 and 5 from the results of section 2.

Fig. 4 is drawn for a core 1 metre square. The heat loss is 0·02 watts per c.c. The surface thermal resistance is assumed to be constant at 80 (deg., watts^{-1}, cm.^{2}) along the faces perpendicular to the laminations and constant at 140 along the faces parallel to the laminations. The oil temperature is assumed to be constant at 40° C. round the core. The isothermals are plotted from the expression

$$T - T_0 = 0{\cdot}02\,\{6500 - x^2 - 3{\cdot}14 \cos 0{\cdot}01864x \cosh 0{\cdot}164y\}.$$

Fig. 5 corresponds to a core 20 cms. square with the same cooling conditions and heat loss as the 1 metre core. R_1 is taken constant at 100 and R_2 at 100. The method of section 5 is applied and the isothermals plotted from the expression

$$T - T_0 = 3 + 0{\cdot}02\,\{1100 - x^2 - 38{\cdot}2 \cos 0{\cdot}0434x \cosh 0{\cdot}382y\}.$$

It is hoped that experimental figures will be available shortly to compare with the above results.

The thanks of the author are due to Mr B. G. Churcher of the Research Department of Messrs Metropolitan-Vickers for suggesting the problem and for supplying the experimental data; and to Mr Jeffreys for his interest in the investigation.

The theory of the influence of magnetic fields on the stopping power of gases for α-particles. By R. de L. Kronig[*]. (Communicated by Mr R. H. Fowler.)

[*Received* 25 May, *read* 20 July, 1925.]

The purpose of this note is to calculate, on the basis of the assumptions which underlie the theory of the stopping of α-particles by matter, the influence of space quantization on this phenomenon. The results arrived at may be capable of an experimental test and will serve to illuminate these assumptions and to indicate their limitations.

The stopping of α-particles can be explained, as is well known, by assuming that the particle transfers energy to the electrons of the atoms which it passes, giving rise to excitation or ionization[†]. By using classical mechanics in the computation of this energy and by taking into account the forces under which the electrons in the atoms move, a fair agreement can be obtained between the theoretical and the experimental value for the stopping power of substances with low atomic number. For impacts in which the shortest distance between the particle and the electron at the time of closest approach is small compared to V/ω, where V is the velocity of the particle and ω the frequency of the electron in its orbit, the electron may be regarded as free. Then the energy transferred is[‡]

$$\Delta E = \frac{2N^2e^4}{mV^2}\frac{1}{p^2+\lambda^2}, \quad \lambda = \frac{Ne^2}{mV^2}, \qquad \ldots\ldots(1)$$

where Ne is the charge, V the velocity of the particle, m the electronic mass and p the shortest distance between the original position of the electron and the path of the α-particle. For collisions where p is not small compared to V/ω the binding forces must be taken into account and will reduce the energy transferred.

Bohr in his papers[§] treats the electrons of the atom as if bound elastically to equilibrium positions with the frequencies appearing in the dispersion formula. He finds for the loss per cm. of energy T of the α-particle due to an electron in the atom with vibration frequency n_0

$$-\frac{dT}{dx} = \frac{4\pi LN^2e^4}{mV^2}\log\frac{\gamma V}{\lambda n_0}, \qquad \ldots\ldots(2)$$

where L is the number of atoms per cm.^{3} and γ a constant, 1·123.

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[*] Wm. Bayard Cutting Traveling Fellow, Columbia University.

[†] J. J. Thomson, *Phil. Mag.* vol. 23, p. 449. (1912.)

[‡] See J. J. Thomson, *loc. cit.*

[§] N. Bohr, *Phil. Mag.* vol. 25, p. 10 (1913); vol. 30, p. 581 (1915).

R. H. Fowler* recently has made the corresponding calculations for a substance consisting of atoms with one electron in a circular orbit of quantum number $n = 1$. The result is of the same form as that obtained by Bohr, but n_0 represents now the angular velocity of the electron in its orbit and $\gamma = 2{\cdot}43$.

In his calculation Fowler takes into account the random orientation and phase of the atoms by averaging over all directions of the normal of the unperturbed orbit and all values of the initial phase. If, however, a magnetic field of sufficient strength be applied, space quantization will set in, and the question arises what effect this will have on the theoretical stopping power, if one adheres strictly to the mechanical model. Now only those positions of the normal in which it is parallel or opposite to the magnetic field will be allowed. Taking this into account one finds by the method of Fowler for the average loss of energy $\overline{\Delta E}$ per impact, when the particle flies parallel or perpendicular to the field

$$\overline{\Delta E} = \frac{2N^2e^4}{mV^2}\frac{1}{p^2+\lambda^2}\Psi\left(\frac{pn_0}{V}\right),$$

where $\Psi_{\parallel}(\sigma) = \tfrac{1}{2}A'^2 + \tfrac{1}{2}A^2 + B^2 + 3\sigma AB,$

$$\Psi_{\perp}(\sigma) = \tfrac{1}{4}A'^2 + \tfrac{1}{2}B'^2 + \tfrac{3}{4}A^2 + \tfrac{1}{2}B^2 + \tfrac{3}{2}\sigma B(A - 2C).$$

Here $$A = \int_0^\infty \frac{\cos\sigma t\,dt}{(1+t^2)^{\frac{3}{2}}},\quad B = \int_0^\infty \frac{t\sin\sigma t\,dt}{(1+t^2)^{\frac{3}{2}}},$$

$$C = \int_0^\infty \frac{t^2\cos\sigma t\,dt}{(1+t^2)^{\frac{3}{2}}},$$

and A' and B' are obtained from A and B by substituting 2σ for σ. In both cases $\overline{\Delta E}$ takes the value (1) for very swift collisions, as it should. For the loss of energy T per cm. one obtains expressions of the form (2), where

$$\log\gamma = \lim_{\epsilon\to 0}\left[\log\epsilon + \int_\epsilon^\infty \Psi(\sigma)\frac{d\sigma}{\sigma}\right].$$

In the integration the relations

$$\int (A^2+B^2)\frac{d\sigma}{\sigma} = -\frac{AB}{\sigma},\quad \int\frac{B^2}{\sigma}d\sigma = \tfrac{1}{2}(B^2 - A^2)$$

are useful, and remembering that

$$\lim_{\sigma\to 0} A(\sigma) = 1,\quad B(\sigma) \sim -\sigma(\log\tfrac{1}{2}\sigma + 0{\cdot}5772)\quad(\sigma\to 0),$$

one finds $$\gamma_{\parallel} = 3{\cdot}56,\quad \gamma_{\perp} = 2{\cdot}00,$$

a difference which would give rise to several per cent. difference in the range of the particle.

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* *Proc. Camb. Phil. Soc.* 1925.

Now stopping experiments in atomic hydrogen are not possible, and it is probable that helium, whose magnetic moment parallel to the field is always zero, has actually zero moment so that the concept of space quantization would lose its significance here. For molecular hydrogen, however, Glaser[*] has recently found something akin to space quantization for fields of several thousand gauss and at pressures of 300 mm. and less. This would permit making stopping experiments in a long solenoid, and these might be well worth while trying as the hydrogen molecule would seem to be even more anisotropic than the hydrogen atom. For several other gases a similar effect was observed.

From a theoretical view-point the question arises, whether one should expect such a difference in the stopping power, i.e. whether the strict adherence to the model is at all permissible. In a recent paper of Heisenberg[†] the idea has been formulated that the reaction of an atomic system to radiation when passing from the degenerate case of no magnetic field to the non-degenerate case of space quantization in a magnetic field, changes only by quantities of the order (frequency of Larmor-precession) divided by (orbital frequency). The idea is applied by Heisenberg to the Ornstein-Burger summation rules for multiplets and their Zeeman-components and to the polarization of the resonance radiation, by Fowler[‡] to the intensities in band spectra and by the author[§] to the derivation of a permanence theorem for the intensities of the Zeeman-components in weak and strong magnetic fields. The index of refraction of a gas also shows itself to be independent of the direction of the light beam relative to the magnetic field, disregarding quantities of the order of the above-mentioned ratio, so that the gas in spite of space quantization reacts toward the radiation as if composed of isotropic oscillators. It seems not unlikely therefore that a similar consideration may apply to the stopping of α-particles. This would mean that the use of the model must be regarded only as an approximate method of computing the stopping power. A measure for the approximation with which the model may be employed for taking into account the binding forces in the atom would be for instance the difference between the orbital frequency and the corresponding spectral frequency. From this standpoint the justification for the original method of Bohr in making use of elastically bound electrons becomes even more obvious, the similarity with the case of the index of refraction being evident.

The validity of equation (2) for the stopping power would not

[*] *Ann. d. Phys.* vol. 75, p. 459. (1924.)

[†] W. Heisenberg, *Zs. f. Phys.* vol. 31, p. 617. (1925.)

[‡] R. H. Fowler, *Phil. Mag.* vol. 49, p. 1272 (1925). See also G. H. Dieke, *Nature*, vol. 115, p. 875. (1925.)

[§] *Zs. f. Phys.* vol. 31, p. 885. (1925.)

be seriously affected by our knowledge of the approximateness of the model, for the introduction of the atomic forces in the calculation is simply a means of finding a limit for the distance beyond which an orbital electron has little influence on the α-particle. A change in this limit has only a small effect on the result, as is apparent from the way γ enters into the formula, while for very close impacts the electron can be considered as free and the mechanical calculation will very likely hold.

On Hausdorff's proof of the extended Riesz-Fischer theorem. By Mr S. Pollard, Trinity College.

[*Received* 8 June, *read* 20 July, 1925.]

1. The extended Riesz-Fischer theorem, in the special case of the trigonometrical orthogonal system, states that, *if α and β are real numbers such that*

$$\alpha > 0, \quad \beta > 0,$$
$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then
$$\sum_{k=-\infty}^{\infty} |c_k|^{\alpha} < \infty \qquad \text{......(1)}$$

implies, whenever $\alpha \leqslant 2$, the existence of a solution of the system of integral equations,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt \quad (k = 0, \pm 1, \pm 2, \ldots) \quad \text{......(2)},$$

such that
$$\int_{-\pi}^{\pi} |f(t)|^{\beta}\, dt < \infty \qquad \text{......(3)},$$

and further
$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^{\beta}\, dt\right]^{\frac{1}{\beta}} \leqslant \left[\sum_{k=-\infty}^{\infty} |c_k|^{\alpha}\right]^{\frac{1}{\alpha}} \qquad \text{......(4)}.$$

When $\alpha = 2$, this reduces to the original Riesz-Fischer theorem.

2. Hausdorff, who first proved this result*, obtains it in two stages.

In the first he supposes that all but a finite number of the c's are zero. When this is so, as is well known, the equations (2) are satisfied by

$$f(t) = \sum_{k=-\infty}^{\infty} c_n e^{ikt},$$

since the series on the right contains only a finite number of non-zero terms, and all that has to be proved is the inequality (4).

This he establishes by means of the Schwarz-Hölder inequalities combined with the theory of relative maxima and minima.

In the second stage he deduces (3) in its general form from the special case just described.

3. This he accomplishes by means of the mechanism of Hellinger integration as applied by Riesz to determine when a function $F(x)$ is an integral of a function $f(x)$ such that

$$\int |f(x)|^{\beta}\, dx < \infty.$$

---

* Hausdorff, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen," *Math. Zeitschrift*, 16 (1923), 163–169.

The object of this memoir is to show that this employment of a special mechanism can be avoided, as it is possible to proceed very simply from the special case to the general case by a direct passage to the limit.

As Hausdorff observes, (1) implies, since $a \leqslant 2$,

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty,$$

so that the Riesz-Fischer theorem in its original form yields the existence of a solution $f(t)$ of the equations (2) such that

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt < \infty. \qquad \text{......(5).}$$

If we write, for finite positive integral values of n,

$$f_n(t) = \sum_{k=-n}^{n} c_k e^{ikt},$$

then it follows from (5) that $f_n(t)$ converges on the mean towards $f(t)$, i.e.

$$\int_{-\pi}^{\pi} |f(t) - f_n(t)|^2 \, dt \to 0,$$

and this implies the existence of a sequence $\{n_r\}$ of positive integers such that

$$f_{n_r}(t) \to f(t) \qquad \text{......(6),}$$

almost everywhere in $(-\pi, \pi)$.

The special case gives

$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_{n_r}(t)|^{\beta} \, dt\right]^{\frac{1}{\beta}} \leqslant \left[\sum_{k=-n_r}^{n_r} |c_k|^{a}\right]^{\frac{1}{a}} \qquad \text{......(7).}$$

Now it is known that, if, in a measurable set E,

$$\phi_p(x) \geqslant 0$$

and
$$\phi_p(x) \to \phi(x),$$

almost everywhere in E, then

$$\underline{\lim} \int_E \phi_p(x)\, dx \geqslant \int_E \phi(x)\, dx.$$

Apply this to (7). We have immediately, by (6),

$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^{\beta} \, dt\right]^{\frac{1}{\beta}} \leqslant \left[\sum_{k=-\infty}^{\infty} |c_k|^{a}\right]^{\frac{1}{a}},$$

which is the result required.

On the representation of the modular group of order $\frac{1}{2}p(p^2-1)$ as a group of linear substitutions on $\frac{1}{2}(p-1)$ symbols, when p is a prime of the form $4n+3$. By Mr W. BURNSIDE, Honorary Fellow of Pembroke College.

[*Received* 16 March, *read* 4 May 1925]

The actual form of the set of linear substitutions on 3 symbols, which gives a representation of the simple group of order 168, has been known for many years*.

In these *Proceedings* (vol. 20, pp. 247–9) I have determined the form of the set of linear substitutions on 5 symbols which gives a representation of the simple group of order 660.

In the following pages I shew how to set up a set of linear substitutions on $\frac{1}{2}(p-1)$ symbols which gives a representation of the modular group of order $\frac{1}{2}p(p^2-1)$, whatever prime p may be of the form $4n+3$.

An incidental result is perhaps worth calling attention to. If p is a prime congruent to 3 (mod. 4), g a primitive root of $g^{\frac{1}{2}(p-1)} \equiv 1$ (mod. p); $i_1, i_2, \ldots, i_m$, those integers between 0 and p for which $1+g^i$ and $1-g^i$ are both quadratic residues; $j_1, j_2, \ldots, j_n$ those integers between 0 and p, for which $1+g^j$ and $1-g^j$ are both quadratic non-residues; and α any $\frac{1}{2}(p-1)$th root of unity, except 1; then when $p \equiv 3$ (mod. 8)

$$\left(1-2\sum_1^m \alpha^{i_t}+2\sum_1^n \alpha^{j_t}\right)\left(1-2\sum_1^m \alpha^{-i_t}+2\sum_1^n \alpha^{-j_t}\right)=p;$$

while if $p \equiv 7$ (mod. 8)

$$\left(1+2\sum_1^m \alpha^{i_t}-2\sum_1^n \alpha^{j_t}\right)\left(1+2\sum_1^m \alpha^{-i_t}-2\sum_1^n \alpha^{-j_t}\right)=p.$$

Let p be a prime of the form $4n+3\;(=2s+1)$ and let g be a primitive root of the congruence

$$g^s \equiv 1 \text{ (mod. } p).$$

It is known† that if P is an operation of order p of the modular group G, of order $\frac{1}{2}p(p^2-1)$, there are operations S and A of G, such that

$$SPS^{-1}=P^g, \quad S^s=E,$$

$$A^{-1}SA=S^{-1}, \quad A^2=E.$$

---

* It is given, for instance, in Klein's *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, vol. 1, p. 705; and in Weber's *Lehrbuch der Algebra*, vol. 2, § 131. See also Burnside, *Theory of Groups*, 2nd edition, p. 311.

† Burnside, *Theory of Groups*, p. 436.

Denoting by $\{S, P\}$ the metacyclical group of order $\frac{1}{2}p(p-1)$ generated by S and P, the operations of the $p+1$ sets

$$\{S, P\},\quad A\{S, P\},\quad AP^iA\{S, P\},\quad i = 1, 2, \dots, p-1$$

are easily seen to be all distinct; and therefore every operation of G occurs once and just once in these sets. It follows that if, in any representation of G as a group of linear substitutions, the substitutions corresponding to P, S and A are known, the substitution corresponding to each operation of G can be written down.

Now G admits* a representation as an irreducible group Γ of linear substitutions on s symbols, in which the characteristics of every operation of order p are either $\frac{1}{2}(-1+\sqrt{-p})$ or $\frac{1}{2}(-1-\sqrt{-p})$, the characteristic of S is zero, and that of every other operation of G, except E, is either -1, 0 or 1.

If w is a primitive pth root of unity, and

$$\lambda = \sum_{i=0}^{s-1} w^{g^i} = \tfrac{1}{2}(-1+\sqrt{-p}),$$

$$\bar{\lambda} = \sum_{i=0}^{s-1} w^{-g^i} = \tfrac{1}{2}(-1-\sqrt{-p}),$$

the multipliers of P in Γ are $w, w^g, w^{g^2}, \dots, w^{g^{s-1}}$. If then that form of Γ is taken in which P is completely reduced, the form of P is

$$x_0' = wx_0,\quad x_1' = w^g x_1,\ \dots,\quad x_i' = w^{g^i}x_i,\ \dots,\quad x'_{s-1} = w^{g^{s-1}}x_{s-1}.$$

Since $SPS^{-1} = P^g$, the form of S must be

$$x_0' = s_0x_1,\quad x_1' = s_1x_2,\ \dots,\quad x'_{s-1} = s_{s-1}x_0,$$

and since $\quad S^s = E,\quad s_0s_1\dots s_{s-1} = 1.$

Hence, without changing the form of P, S may be taken in the form

$$x_0' = x_1,\quad x_1' = x_2,\ \dots,\quad x'_{s-1} = x_0.$$

It remains to determine the form of A. Suppose it is

$$x_i' = \sum_{j=0}^{s-1} a_{ij}x_j \quad (i = 0, 1, \dots, s-1).$$

Then SA is
$$x_i' = \sum_{j=0}^{s-1} a_{i+1,j}x_j,$$

and AS^{-1} is
$$x_i' = \sum_{j=0}^{s-1} a_{i,j}x_{j-1}.$$

Since SA and AS^{-1} are the same substitution

$$a_{i+1,j} = a_{i,j+1}$$

for all values of i and j, so that

$$a_{i,j} = a_{0,i+j},$$

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* *loc. cit.* p. 502.

where $i+j$ is reduced mod. s. Hence A is of the form

$$x_i' = \sum_{j=0}^{s-1} a_{i+j}x_j \quad (i = 0, 1, \ldots, s-1)$$

and A^2 is
$$x_i' = \sum_{k=0}^{s-1} x_k \sum_{j=0}^{s-1} a_{i+j}a_{j+k}.$$

The condition $A^2 = E$ becomes

$$\sum_{i=0}^{s-1} a_i^2 = 1, \quad \sum_{i=0}^{s-1} a_i a_{i+k} = 0 \quad (k \neq 0).$$

Since the Hermitian form that is invariant for Γ and $\overline{\Gamma}$ is $\sum_0^{s-1} x_i \bar{x}_i$, it follows at once that the a's are real. Now the coefficients in the substitutions of Γ are* rational functions of w; and therefore the a's are real rational functions of w. It is therefore legitimate to put

$$a_{2i} = b_0(w^{g^i} + w^{-g^i}) + b_1(w^{g^{i+1}} + w^{-g^{i+1}}) + \ldots + b_{s-1}(w^{g^{i-1}} + w^{-g^{i-1}})$$
$$(i = 0, 1, 2, \ldots, s-1),$$

the b's being rational numbers; and it is easily verified that the relations

$$\sum_0^{s-1} a_i^2 = 1, \quad \sum_0^{s-1} a_i a_{i+k} = 0,$$

become
$$p\sum_0^{s-1} b_i^2 - 2\left(\sum_0^{s-1} b_i\right)^2 = 1,$$

$$p\sum_0^{s-1} b_i b_{i+k} - 2\left(\sum_0^{s-1} b_i\right)^2 = 0.$$

The characteristic of A, χ_A, is given by

$$\chi_A = \sum_{i=0}^{s-1} a_{2i} = -\sum_{i=0}^{s-1} b_i.$$

From this point it is necessary to deal separately with the cases $p \equiv 7$ (mod. 8) and $p \equiv 3$ (mod. 8). In the first χ_A is -1, and 2 is a quadratic residue of p.

The characteristic of AP^{g^j} is given by

$$\chi_{AP^{g^j}} = \sum_{i=0}^{s-1} a_{2i} w^{g^{i+j}}$$
$$= \sum_{i=0}^{s-1} w^{g^{i+j}} \{b_0(w^{g^i} + w^{-g^i}) + \ldots + b_t(w^{g^{i+t}} + w^{-g^{i+t}}) + \ldots$$
$$+ b_{s-1}(w^{g^{i-1}} + w^{-g^{i-1}})\}.$$

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* Burnside, *Proc. Lond. Math. Soc.* vol. 3 (second series), p. 246.

That of $AP^{g^{j+1}}$ is given by

$$\chi_{AP^{g^{j+1}}} = \sum_{i=0}^{s-1} w^{g^{i+j+1}} \left(b_0 \left(w^{g^i} + w^{-g^i}\right) + \dots \right.$$
$$\left. + b_{t+1} \left(w^{g^{i+t+1}} + w^{-g^{i+t+1}}\right) + \dots + b_{s-1} \left(w^{g^{i-1}} + w^{-g^{i-1}}\right)\right).$$

The coefficient of b_t in $\chi_{AP^{g^j}}$ is the same as that of b_{t+1} in $\chi_{AP^{g^{j+1}}}$. It is sufficient therefore to calculate χ_{AP}, as then $\chi_{AP^{g^j}}$ can be obtained by a cyclical permutation of the suffixes.

Since 2 is a quadratic residue, the coefficient of b_0 in χ_{AP} is $\lambda + \frac{1}{2}(p-1)$; and the coefficient of each other b is either $\lambda + \bar{\lambda}$, 2λ or $2\bar{\lambda}$.

The coefficient of b_t is

$$\sum_{i=0}^{s-1} w^{g^i} \left(w^{g^{i+t}} + w^{-g^{i+t}}\right).$$

The condition that this coefficient should be $\lambda + \bar{\lambda}$ is that one of the two numbers $1 + g^t$ and $1 - g^t$ should be a quadratic residue and the other a non-residue; i.e. that their product should be a non-residue, say

$$1 - g^{2t} \equiv -g^{2t'} \pmod{p}.$$

Since 2 is a quadratic residue, there must be a value of t, say m, for which

$$g^{2m} \equiv 2,$$

and then
$$t' = 0.$$

If $t' = 0$, the congruence may be written

$$1 - g^{-2m} \equiv g^{-2m}.$$

Hence the coefficient of b_m is $\lambda + \bar{\lambda}$, and that of b_{-m} is either 2λ or $2\bar{\lambda}$.

When t' is not zero the congruence may be written in the forms

$$\begin{aligned} 1 - g^{2(t-t')} &\equiv -g^{-2t'}, \\ 1 - g^{-2t} &\equiv g^{2t'-2t}, \\ 1 - g^{2(t'-t)} &\equiv g^{-2t}. \end{aligned}$$

It follows that when the coefficient of b_t $(t \neq m)$ is $\lambda + \bar{\lambda}$, there is a number t' such that the coefficient of $b_{t-t'}$ is also $\lambda + \bar{\lambda}$, and those of b_{-t} and $b_{t'-t}$ are either 2λ or $2\bar{\lambda}$. These four b's are distinct unless $t' = 2t$, which leads at once to a contradiction. It follows at once that of the $s - 1$ b's other than b_0, there are $1 + \frac{s-3}{2} \left(= \frac{p-3}{4}\right)$ whose coefficients are $\lambda + \bar{\lambda}$ and an equal number whose coefficients are either 2λ or $2\bar{\lambda}$. Now the sum of the coefficients of all the b's, except b_0, is $\frac{1}{2}(p-5)$ λ's and $\frac{1}{2}(p-1)$ $\bar{\lambda}$'s. Hence of the $\frac{1}{4}(p-3)$ b's whose coefficients are either 2λ or $2\bar{\lambda}$, $\frac{1}{8}(p+1)$ have coefficient

$2\bar{\lambda}$ and $\frac{1}{8}(p-7)$ have coefficient 2λ. Let b' denote any one of the b's other than b_0 whose coefficient in χ_{AP} is $\lambda+\bar{\lambda}$, b'' any one whose coefficient is 2λ, and b''' any one whose coefficient is $2\bar{\lambda}$. Then

$$\chi_{AP} = b_0 \tfrac{1}{2}(p-1) + b_0\lambda + (\lambda+\bar{\lambda})\Sigma_0 b' + 2\lambda\Sigma_0 b'' + 2\bar{\lambda}\Sigma_0 b''',$$

and

$$\chi_{AP^{\sigma^j}} = b_j \tfrac{1}{2}(p-1) + b_j\lambda + (\lambda+\bar{\lambda})\Sigma_j b' + 2\lambda\Sigma_j b'' + 2\bar{\lambda}\Sigma_j b''',$$

where $\Sigma_j b'$ is formed from $\Sigma_0 b'$ by increasing the suffix of each b which there occurs by j; and $\Sigma_j b''$, $\Sigma_j b'''$ are formed in the same way from $\Sigma_0 b''$ and $\Sigma_0 b'''$

Suppose first that $\chi_{AP^{\sigma^j}}$ is rational. Then

$$b_j + 2\Sigma_j b'' - 2\Sigma_j b''' = 0$$

and $$\chi_{AP^{\sigma^j}} = b_j \tfrac{1}{2}(p-1) - \Sigma_j b' - (\tfrac{1}{2}b_j + \Sigma_j b'' + \Sigma_j b''').$$

Now $$\Sigma_j b' + \Sigma_j b'' + \Sigma_j b''' = 1 - b_j,$$

so that $$b_j = \frac{2}{p}(\chi_{AP^{\sigma^j}} + 1).$$

Next suppose that $\chi_{AP^{\sigma^j}}$ is λ. Then

$$b_j + 2\Sigma_j b'' - 2\Sigma_j b''' = 1$$

and $$b_j \tfrac{1}{2}(p-1) - \Sigma_j b' - 2\Sigma_j b''' = 0,$$

giving $$b_j = \frac{1}{p}.$$

Finally if $\chi_{AP^{\sigma^j}}$ is $\bar{\lambda}$,

$$b_j + 2\Sigma_j b'' - 2\Sigma_j b''' = -1$$

and $$b_j = \frac{1}{p}.$$

Since all rational characteristics are 0, 1 or -1, each b is either

$$0,\ \frac{1}{p},\ \frac{2}{p} \text{ or } \frac{4}{p}.$$

The supposition that $\chi_{AP^{\sigma^j}}$ is λ involves the relation

$$\frac{1}{p} + 2\Sigma_j b'' - 2\Sigma_j b''' = 1.$$

There are $\frac{1}{8}(p-7)$ b's in $\Sigma_j b''$ and the greatest value of each is $\frac{4}{p}$. The greatest value of the right-hand side in the last equation is $1 - \frac{6}{p}$. Hence $\chi_{AP^{\sigma^j}}$ cannot be λ, and for each value of j

$$b_j + 2\Sigma_j b'' - 2\Sigma_j b'''$$

is either 0 or -1. Adding the s equations of this form, and remembering that $\sum\limits_{0}^{s-1} b_j = 1$, the sum of the left-hand sides is -1. Hence for just one value of j, which without loss of generality may be taken to be zero, the right-hand side is -1.

The b's therefore satisfy the equations

$$b_0 + 2\Sigma_0 b'' - 2\Sigma_0 b''' = -1,$$

$$b_j + 2\Sigma_j b'' - 2\Sigma_j b''' = 0 \quad (j = 1, 2, \ldots, s-1).$$

The first of these equations may be written

$$-2\Sigma_0 b'' + 2\Sigma_0 b''' = \frac{p+1}{p}.$$

Now the greatest possible value of the left-hand side of this equation is given by taking each b in $\Sigma_0 b''$ zero and each b in $\Sigma_0 b'''$ equal to $\frac{4}{p}$. Since in the latter sum there are $\frac{1}{8}(p+1)$ b's, this greatest value is $\frac{p+1}{p}$. This then is the only way in which the first equation can be satisfied.

Since $$b_0 + \Sigma_0 b' + \Sigma_0 b'' + \Sigma_0 b''' = 1,$$

$$\Sigma_0 b' = \frac{p-3}{2p}.$$

There are $\frac{1}{4}(p-3)$ b's in $\Sigma_0 b'$, and therefore the average value of each is $\frac{2}{p}$. Now it has been seen that if b_t occurs in $\Sigma_0 b'$, then b_{-t} occurs in either $\Sigma_0 b''$ or $\Sigma_0 b'''$, and therefore b_0 occurs in either $\Sigma_t b''$ or $\Sigma_t b'''$. The relation

$$b_t + 2\Sigma_t b'' - 2\Sigma_t b''' = 0$$

in which every term in the two sums, except the b_0 term, is a multiple of 4, cannot therefore be satisfied unless b_t is $\frac{2}{p}$. The only b's then that can satisfy the conditions are those given by

$$b_0 = \frac{1}{p}.$$

Every b in $$\Sigma_0 b' = \frac{2}{p}.$$

Every b in $$\Sigma_0 b'' = 0.$$

Every b in $$\Sigma_0 b''' = \frac{4}{p}.$$

When $p \equiv 3 \pmod{8}$, $\chi_A = 1$ and 2 is not a quadratic residue. It may be shewn, as in the former case, that if the coefficient of b_t in χ_{AP} is $\lambda + \bar{\lambda}$, there is a number t' (which in this case cannot be zero) such that the coefficient of $b_{t-t'}$ is $\lambda + \bar{\lambda}$, and the coefficients of b_{-t} and $b_{t'-t}$ are either 2λ or $2\bar{\lambda}$. It follows that of the $\frac{1}{2}(p-3)$ b's, other than b_0, $\frac{1}{4}(p-3)$ have $\lambda + \bar{\lambda}$ for coefficient, and an equal number have either 2λ or $2\bar{\lambda}$. In this case, since 2 is a non-residue, the sum of the coefficients of all the b's except b_0, is $\frac{1}{2}(p-3)$ λ's and $\frac{1}{2}(p-3)$ $\bar{\lambda}$'s. Hence $\frac{1}{8}(p-3)$ b's have 2λ for coefficient and an equal number have $2\bar{\lambda}$. If then b', b'', b''' are used in the same sense as in the previous case,

$$\chi_{AP} = b_0 \tfrac{1}{2}(p-1) + b_0\bar{\lambda} + (\lambda + \bar{\lambda})\Sigma_0 b' + 2\lambda\Sigma_0 b'' + 2\bar{\lambda}\Sigma_0 b''',$$

$$\chi_{AP^{\sigma^j}} = b_j \tfrac{1}{2}(p-1) + b_j\bar{\lambda} + (\lambda + \bar{\lambda})\Sigma_j b' + 2\lambda\Sigma_j b'' + 2\bar{\lambda}\Sigma_j b'''.$$

If $\chi_{AP^{\sigma^j}}$ is rational, $\quad b_j + 2\Sigma_j b''' - 2\Sigma_j b'' = 0$

and $$b_j = \frac{2}{p}(\chi_{AP^{\sigma^j}} - 1).$$

If $\chi_{AP^{\sigma^j}}$ is λ, $\quad b_j + 2\Sigma_j b''' - 2\Sigma_j b'' = -1,$

and $$b_j = -\frac{3}{p}.$$

If $\chi_{AP^{\sigma^j}}$ is $\bar{\lambda}$, $\quad b_j + 2\Sigma_j b''' - 2\Sigma_j b'' = 1,$

and $$b_j = -\frac{3}{p}.$$

It follows that each b is either 0, $-\frac{2}{p}$, $-\frac{3}{p}$ or $-\frac{4}{p}$. With these values it is impossible to satisfy the relation

$$b_j + 2\Sigma_j b''' - 2\Sigma_j b'' = 1,$$

so that for one value of j, which may be taken as zero,

$$b_0 + 2\Sigma_0 b''' - 2\Sigma_0 b'' = -1$$

and $$b_j + 2\Sigma_j b''' - 2\Sigma_j b'' = 0 \quad (j = 1, 2, \ldots, s-1).$$

The first of these equations may be written

$$2\Sigma_0 b'' - 2\Sigma_0 b''' = \frac{p-3}{p},$$

and it can be satisfied only by taking each b in $\Sigma_0 b''$ to be zero, and each b in $\Sigma_0 b'''$ to be $-\frac{4}{p}$. It may then be shewn, exactly as in the previous case, that each b in $\Sigma_0 b'$ must be $-\frac{2}{p}$. In this case then the only values of the b's which can satisfy the conditions are those given by

$$b_0 = -\frac{3}{p}.$$

Each b in $\quad \Sigma_0 b' = -\frac{2}{p}.$

Each b in $\quad \Sigma_0 b'' = 0.$

Each b in $\quad \Sigma_0 b''' = -\frac{4}{p}.$

In both cases the s equations which the s b's must satisfy are cyclical, and their determinant is necessarily different from zero. They must have unique solutions; and therefore the sets of values of the b's, which have been determined indirectly, must be the unique solutions.

Since $\chi_A = \pm 1$ and $\chi_A = -\sum_0^{s-1} b_i$, the conditions already found that A shall be of order 2 take the form

$$\sum_0^{s-1} b_i^2 = \frac{3}{p},$$

$$\sum_0^{s-1} b_i b_{i+k} = \frac{2}{p}, \quad k \neq 0.$$

Now the group of linear substitutions Γ is known to exist, and the values already found for the b's are the only possible ones. They therefore necessarily satisfy the above conditions. The form of these conditions may be modified as follows. If a is any sth root of unity, except 1,

$$\sum_{i=0}^{s-1} b_i^2 + \sum_{k=1}^{s-1} a^k \sum_{i=0}^{s-1} b_i b_{i+k} = \frac{3}{p} + \frac{2}{p}\sum_{k=1}^{s-1} a^k = \frac{1}{p}$$

or

$$\sum_0^{s-1} a^i b_i \,.\, \sum_0^{s-1} a^{-i} b_i = \frac{1}{p}.$$

If

$$\Sigma_0 b'' = b_{i_1} + b_{i_2} + \ldots + b_{i_m},$$

$$\Sigma_0 b''' = b_{j_1} + b_{j_2} + \ldots + b_{j_n},$$

then when $p \equiv 7 \pmod{8}$, $m = \frac{1}{8}(p-7)$, $n = \frac{1}{8}(p+1)$

$$b_0 + 2(b_{i_1} + b_{i_2} + \ldots + b_{i_m}) - 2(b_{j_1} + b_{j_2} + \ldots + b_{j_n}) = -1.$$

$$b_k + 2(b_{i_1+k} + b_{i_2+k} + \ldots + b_{i_m+k}) \\ - 2(b_{j_1+k} + b_{j_2+k} + \ldots + b_{j_n+k}) = 0 \quad (k = 1, 2, \ldots, s-1)$$

and

$$\sum_0^{s-1} a^k b_k (1 + 2a^{-i_1} + 2a^{-i_2} + \ldots \\ + 2a^{-i_m} - 2a^{-j_1} - 2a^{-j_2} - \ldots - 2a^{-j_m}) = -1.$$

Hence whatever sth root of unity, except 1, α may be

$$(1 + 2\alpha^{i_1} + 2\alpha^{i_2} + \dots + 2\alpha^{i_m} - 2\alpha^{j_1} - 2\alpha^{j_2} - \dots - 2\alpha^{j_n})$$
$$(1 + 2\alpha^{-i_1} + \dots + 2\alpha^{-i_m} - 2\alpha^{-j_1} - \dots - 2\alpha^{-j_n}) = p.$$

When $p \equiv 3 \pmod{8}$, $m = n = \frac{1}{8}(p-3)$, and the corresponding result is

$$(1 - 2\alpha^{i_1} - 2\alpha^{i_2} - \dots - 2\alpha^{i_m} + 2\alpha^{j_1} + 2\alpha^{j_2} + \dots + 2\alpha^{j_n})$$
$$(1 - 2\alpha^{-i_1} - 2\alpha^{-i_2} - \dots - 2\alpha^{-i_m} + 2\alpha^{-j_1} + 2\alpha^{-j_2} + \dots + 2\alpha^{-j_n}) = p.$$

It is interesting to verify the last results in particular cases.

If $p = 23$ we may take $g = 2$. Then it is found that

$$\Sigma_0 b'' = b_3 + b_{10}, \quad \Sigma_0 b''' = b_2 + b_5 + b_7.$$

The values of the b's in ascending order are therefore

$$\tfrac{1}{23}, \tfrac{2}{23}, \tfrac{4}{23}, 0, \tfrac{2}{23}, \tfrac{4}{23}, \tfrac{2}{23}, \tfrac{4}{23}, \tfrac{2}{23}, \tfrac{2}{23}, 0.$$

The coefficient a_0 in the substitution A is, after reduction,

$$a_0 = -\tfrac{1}{23}[2 + w + w^{-1} - 2(w^4 + w^{-4}) + 2(w^8 + w^{-8})$$
$$- 2(w^9 + w^{-9}) - 2(w^{10} + w^{-10}) + 2(w^{11} + w^{-11})].$$

Moreover

$$1 + 2\sum_1^m \alpha^{i_t} - 2\sum_1^n \alpha^{j_t} = 1 - 2\alpha^2 + 2\alpha^3 - 2\alpha^5 - 2\alpha^7 + 2\alpha^{10},$$

and it is actually found that, if α is an 11th root of unity,

$$(1 - 2\alpha^2 + 2\alpha^3 - 2\alpha^5 - 2\alpha^7 + 2\alpha^{10})$$
$$(1 - 2\alpha^{-2} + 2\alpha^{-3} - 2\alpha^{-5} - 2\alpha^{-7} + 2\alpha^{-10}) = 23.$$

If $p = 19$ we may take $g = 4$ and find

$$\Sigma_0 b'' = b_1 + b_2, \quad \Sigma_0 b''' = b_3 + b_5.$$

The values of the b's are

$$-\tfrac{3}{19}, 0, 0, -\tfrac{4}{19}, -\tfrac{2}{19}, -\tfrac{4}{19}, -\tfrac{2}{19}, -\tfrac{2}{19}, -\tfrac{2}{19},$$

and the coefficient a_0 after reduction is

$$a_0 = \tfrac{1}{19}(2 - w - w^{-1} + 2(w^3 + w^{-3}) + 2(w^4 + w^{-4})$$
$$- 2(w^2 + w^{-2}) - 2(w^7 + w^{-7})).$$

Further

$$1 - 2\sum_1^m \alpha^{i_t} + 2\sum_1^n \alpha^{j_t} = 1 - 2\alpha - 2\alpha^2 + 2\alpha^3 + 2\alpha^5,$$

and it may be verified that, if α is a 9th root of unity,

$$(1 - 2\alpha - 2\alpha^2 + 2\alpha^3 + 2\alpha^5)(1 - 2\alpha^{-1} - 2\alpha^{-2} + 2\alpha^{-3} + 2\alpha^{-5}) = 19.$$

On groups of linear substitutions which contain irreducible metacyclical subgroups. By Mr W. Burnside, Honorary Fellow of Pembroke College.

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If G is a group of finite order which contains an operation P of prime order p, permutable only with its own powers, the order of G must, by Sylow's theorem, be of the form $(1 + kp)\,ps$, where s is a factor of $p - 1$. The greatest subgroup of G, which contains self-conjugately $\{P\}$, the subgroup generated by P, must be a metacyclical subgroup $\{S, P\}$, where

$$S^{-1}PS = P^g, \quad S^s = E,$$

while g is a primitive root of the congruence $g^s \equiv 1$ (mod. p).

If $s = p - 1$, P is conjugate with all its powers, and the characteristic of P in each irreducible representation of G is rational[*]. It is proposed here to consider the case when s is less than $p - 1$: so that, if $p - 1 = st$, t is greater than unity.

Let ω be a primitive pth root of unity, and

$$\lambda_1 = \omega + \omega^g + \ldots + \omega^{g^{s-1}},$$

and let $\lambda_1, \lambda_2, \ldots, \lambda_t$ be the t values that λ_1 takes when ω is replaced by any other pth root of unity. Then in each irreducible representation of G, the characteristic, χ_P, of P is of the form

$$\chi_P = a_0 + a_1\lambda_1 + a_2\lambda_2 + \ldots + a_t\lambda_t,$$

where each a is either zero or a positive integer; and in some representations χ_P is t-valued[†].

Let Γ_1 be an irreducible representation of G in which χ_P is t-valued. Then in the family of representations to which Γ_1 belongs there must be a set of t irreducible representations Γ_1, $\Gamma_2, \ldots, \Gamma_t$, in which χ_P takes its t values[‡].

Summing for these t representations,

$$\Sigma\chi_P\chi_{P^{-1}} = \Sigma\,(a_0 + a_1\lambda_1 + \ldots + a_t\lambda_t)\,(a_0 + a_1\bar{\lambda}_1 + \ldots + a_t\bar{\lambda}_t)$$
$$= \frac{1}{t}\left[\left(ta_0 - \sum_1^t a_i\right)^2 + p\sum_{i \neq j}(a_i - a_j)^2\right].$$

Since χ_P is t-valued, $a_1, a_2, \ldots, a_t$ cannot be all equal, and therefore the least value which the right hand side of this equation can take is $p - s$. Now for the identical representation $\chi_P\chi_{P^{-1}}$ is unity, while for all irreducible representations[§]

$$\Sigma\chi_P\chi_{P^{-1}} = p.$$

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[*] Burnside, *Theory of Groups*, § 221.

[†] *Loc. cit.* § 221.

[‡] *Loc. cit.* § 234.

[§] *Loc. cit.* § 218.

If the family to which Γ_1 belongs contained nt members, $\Sigma\chi_P\chi_{P^{-1}}$ for all members of the family could not be less than $n(p-s)$, which is equal to or greater than p if n is greater than unity. Hence n must be unity. Similarly if there were another family of irreducible representations in which χ_P is irrational, $\Sigma\chi_P\chi_{P^{-1}}$ for this family would be not less than $p-s$, so that this again is not possible. It follows that there is just one family of irreducible representations in which χ_P is irrational, and that in this family the characteristics of all operations of G, except those of order p, are rational*.

The least value of the degree of these irreducible representations, in which χ_P is irrational, is s. When the degree of Γ_1 is s, and the group of linear substitutions is thrown into the form in which P is completely reduced, the substitution corresponding to P is

$$x_0' = \omega x_0,\ \ x_1' = \omega^g x_1,\ \ x_2' = \omega^{g^2} x_2,\ \ldots,\ x'_{s-1} = \omega^{g^{s-1}} x_{s-1}.$$

Since
$$S^{-1}PS = P^g$$
the substitution corresponding to S^{-1} must be of the form

$$x_0' = c_0 x_1,\ x_1' = c_1 x_2,\ \ldots,\ x'_{s-1} = c_{s-1} x_0,$$

and since
$$S^s = E,$$
$$c_0 c_1 \ldots c_{s-1} = 1.$$

The variables may therefore be so chosen, without changing the form of P, that S^{-1} is the permutation

$$(x_0 x_1 x_2 \ldots x_{s-1}).$$

If γ is a primitive root of the congruence

$$\gamma^{p-1} \equiv 1 \pmod{p},$$

so that
$$g \equiv \gamma^t \pmod{p},$$
the form of the subgroup $\{P, S\}$ in Γ_i $(i = 2, 3, \ldots, t)$ is got by writing ω^{γ^i} for ω in the above substitutions.

In every irreducible representation except $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$, χ_P is rational. If in such a representation P has a unit multipliers, and b ω-multipliers, then

$$\chi_P = a - b,\quad \chi_E = a + b(p-1),$$

and such a representation may be denoted by $\Gamma_{a,b}$. The result of compounding† the representations Γ_i and $\Gamma_{a,b}$ is given by

$$\Gamma_i\Gamma_{a,b} = \sum_{j=1}^{t} c_j\Gamma_j + \Sigma d\Gamma_{a',b'}.$$

Since P has the same characteristic on either side

$$(a-b)\lambda_i = \Sigma c_j\lambda_j + R,$$

where R is a rational integer.

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* *Loc. cit.* §§ 234, 235.

† *Loc. cit.* § 218.

Hence

$$c_1 = c_2 = \ldots = c_{i-1} = c_i - a + b = c_{i+1} = \ldots = c_t,$$

where each c is zero or a positive integer. Suppose first that $b - a$ is positive. Then $c_j\ (j \neq i)$ is equal to or greater than $b - a$. Now if Γ_j occurs k times in $\Gamma_i \Gamma_{a,b}$, then $\Gamma'_{a,b}$ (the representation inverse to $\Gamma_{a,b}$) occurs* k times in $\Gamma_i \Gamma_{j'}$. But if $j \neq i$, P has no unit multiplier in $\Gamma_i \Gamma_{j'}$; and therefore P has no unit multiplier in $\Gamma'_{a,b}$. Hence if $b - a$ is positive, a must be zero, and each such representation $\Gamma_{0,\beta}$ occurs at least β times in $\Gamma_i \Gamma_{j'}$. This clearly involves the inequality

$$s^2 > (p-1)\Sigma\beta^2.$$

If $a - b$ is positive, c_i is not less than $a - b$, and $\Gamma_{a,b}$ occurs at least $a - b$ times in $\Gamma_i \Gamma_{i'}$. Now in $\Gamma_i \Gamma_{i'}$ the subgroup $\{S, P\}$ has just s unit-representations and they are all distinct. Hence in $\Gamma_i \Gamma_{i'}$ no irreducible representation in which P has a unit multiplier can occur more than once. It follows that if $a - b$ is positive it is unity; and that each irreducible representation for which $a - b$ is unity occurs just once in $\Gamma_i \Gamma_{i'}$. If there are m such representations Γ_{b_i+1,b_i} $(i = 1, 2, \ldots, m)$, the inequality

$$s^2 \geqslant m + p \sum_i b_i$$

follows by considering the degree of $\Gamma_i \Gamma_{i'}$. The sign of equality can only occur when $t = 2$, $p \equiv 3 \pmod{4}$. With the notation so introduced, the relations

$$\Sigma \chi_P \chi_{P^{-1}} = p, \quad \Sigma \chi_P \chi_E = 0,$$

where the summations apply to all irreducible representations, take the form

$$p = p - s + m + \Sigma\beta^2,$$

$$0 = -s + m + p\Sigma b_i - (p-1)\Sigma\beta^2,$$

or

$$s - m = \Sigma b_i = \Sigma\beta^2.$$

Combining these results with the previous inequalities, the first of which gives

$$\Sigma\beta^2 < \frac{s}{t},$$

it follows that

$$m > s - \frac{s}{t},$$

$$\Sigma b_i - m = s - 2m < s\left(\frac{2}{t} - 1\right).$$

If $t \geqslant 3$, at least two of the m b_i's must be zero.

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* *Loc. cit.* § 219.

If $t = 2$ and s is even

$$m > \frac{s}{2} \geqslant \frac{s}{2} + 1,$$

$$\Sigma b_i - m = s - 2m \leqslant -2,$$

and again at least two of the b_i's are zero.

If $t = 2$, and $p \equiv 3 \pmod{4}$, so that s is odd, the second inequality may become the equality

$$s^2 = m + p\Sigma b_i.$$

In this case
$$\Sigma b_i = \frac{s-1}{2}$$

and
$$\Sigma b_i - m \leqslant \frac{s-1}{2} - \frac{s}{2} \leqslant -\frac{1}{2}.$$

Here not more than one of the b_i's is necessarily zero.

It follows that, except when $t = 2$, $p \equiv 3 \pmod{4}$, there is necessarily a unit-representation of G distinct from the identical representation, and that in this representation P has the characteristic 1.

Hence putting aside the case $t = 2$, $p \equiv 3 \pmod{4}$, G of order $(1 + kp)\,ps$ has a self-conjugate subgroup H of order $(1 + kp)\,p$, in which P is not conjugate with any of its powers. Such a group has* a self-conjugate subgroup K of order $1 + kp$, and every operation of H which does not belong to K is of order p.

Consider now the representation of H given by Γ_1. It is not necessarily irreducible, and in it $\Sigma\chi_Q\chi_{Q^{-1}}$ taken for the operations of K is $(1 + kp)\,r$, the extreme values of r being 1 and s^2. Each of the other $(1 + kp)(p - 1)$ operations of H is of order p, and the value of $\chi_P\chi_{P^{-1}}$ for any one of them is $\lambda_i\bar{\lambda}_i$ for some value of i from 1 to t. Hence $\Sigma\chi_P\chi_{P^{-1}}$ for all the operations of H of order p is $(1 + kp)(p - 1)\frac{p - s}{t}$. Now $\Sigma\chi_s\chi_{s^{-1}}$ for all the operations of H must be a multiple of $(1 + kp)\,p$, i.e.

$$(1 + kp)\,r + (1 + kp)\,s\,(p - s) = m\,(1 + kp)\,p,$$

which can only be satisfied by

$$r = s^2, \quad m = s.$$

Hence in the representation Γ_1 of G, every operation of K corresponds to the identical substitution. This is equivalent to stating that the only distinct linear substitutions that occur in Γ_1 are those which correspond to the subgroup $\{S, P\}$.

The result thus arrived at may be stated as follows:

Theorem. If a group G of linear substitutions of finite order, and of degree s, contains the irreducible metacyclical subgroup $\{S, P\}$ generated by

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* *Loc. cit.* § 243.

$$P \sim x_i' = \omega^{g^i} x_i \quad (i = 0, 1, 2, \ldots, s-1),$$
$$S \sim (x_0 x_1 \ldots x_{s-1}),$$

where ω is a primitive pth root of unit, s a factor of $p-1$, and g a primitive root of the congruence $g^s \equiv 1 \pmod{p}$, and if in G the substitution P is permutable only with its own powers, then G contains no other substitutions than those of $\{S, P\}$, unless

$$s = \tfrac{1}{2}(p-1) \text{ and } p \equiv 3 \pmod{4}.$$

The exceptional cases actually exist, and a form of the corresponding groups has been determined in the previous paper.

A theoretical study of the stopping power of hydrogen atoms for α-particles. By Mr R. H. FOWLER, Trinity College.

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§ 1. *Introduction and Summary.* Bohr's original (1913) theory of the stopping power of light atoms for α-particles seems to invite further study. It is well-known that in this theory, which exhibits a general agreement with experiment, the loss of energy by the α-particle is calculated on purely classical principles, as the energy transferred to electrons initially at rest, and elastically bound to atoms. A large part of the loss so calculated (about one-half) comes from transfers to electrons in distant atoms of amounts less than the energy required to transfer those atoms to their next higher stationary state. This has created a serious difficulty according to more recent views of the quantum theory* in which it has been presumed that any effective interaction must always leave the atom in a stationary state, and has given rise to the investigations of Henderson†, which however do not agree with experiment‡. In some recent speculations Bohr has suggested a possible way of escape from this difficulty, of great theoretical interest, to which he has been led by somewhat similar difficulties in the theory of radiation. He suggests that there may be reason to expect that energy may be only statistically conserved in just such a type of interaction by the atomic switches, while the classical calculation for the α-particle ought to retain its validity.

As this calculation of the stopping power has again become of theoretical importance, it seems desirable to re-examine it critically as a purely mechanical calculation. From this point of view it is more proper to calculate the mechanical transfer of energy to the electrons in their presumed orbits than to elastically bound electrons at rest. It has been suggested to me therefore by Prof. Bohr that I should calculate the mechanical transfer of energy from an α-particle to an electron moving in its normal circular orbit in a hydrogen-like atom. Such a calculation is recorded in this paper. It is a straightforward application of perturbation theory, which must be carried to a second approximation, for, as is easily seen, the average first order change of energy must be zero. It is worked out of course only for the case in which the shortest distance p

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* Bohr, *Zeit. fur Phys.* vol. 13, p. 117 (1923); English, *Proc. Camb. Phil. Soc. Suppl.*

† Henderson, *Phil. Mag.* vol. 44, p. 680. (1922).

‡ Fowler, *Proc. Camb. Phil. Soc.* vol. 21, p. 521. (1923.)

between the atomic nucleus and the α-particle is large compared to the dimensions of the orbit.

The results of the calculation can be simply stated. In Bohr's original calculations the rate of loss of energy by the α-particle is given by the equation

$$-\frac{dT}{dx}=\frac{4\pi LN^2e^4}{mV^2}\log\frac{\gamma mV^3}{2\pi Ne^2\omega}. \qquad \text{......(1)}$$

In this equation $-dT/dx$ is the rate of loss of energy by an α-particle of charge Ne and velocity V, moving through a gas containing L atoms with a single electron (mass m, charge $-e$) per c.c. The frequency of these elastically bound electrons is ω, and γ is a constant, defined by a certain integral, whose numerical value is 1·123. In the present calculations, *the result is of exactly the same form*. No new variables are introduced, as might perhaps have been anticipated on dimensional grounds; in equation (1) ω is now the orbital frequency of revolution and γ a constant, similarly determined, of value 2·42. The reason for the greater γ, and consequently greater stopping power, appears to be connected essentially with the fact that in elliptical motion a change of energy entails a change of frequency, whereas in harmonic oscillations it does not. This gives rise, as it were, to a new degree of freedom in the motion which provides an additional vehicle for the transfer of energy.

Equation (1) may be expected to undergo the obvious modifications when the atoms contain more than one electron. It is only proposed to call attention to the leading features of the new experimental results of Gurney* for He and H_2 in connection with it. For these systems, each of *two* equivalent electrons, (1) can be put in the form

$$\frac{V_0^4-V^4}{X}=\frac{32\pi LN^2e^4}{mM}\log\frac{\gamma m\bar{V}^3}{2\pi Ne^2\omega}, \qquad \text{......(1)}'$$

where V_0 and V are the velocities at the ends of a short interval of length X, $\bar{V}$ is a mean value of V for X, and M is the mass of the α-particle. Between the velocities $2{\cdot}0\times10^9$ and $1{\cdot}95\times10^9$, corresponding to a distance of 1 cm. through air at N.T.P., Gurney's results give for H_2 the absolute value

$$(V_0^4-V^4)/X=5{\cdot}4\times10^{35}.$$

It is reasonable to suppose that the ω's will be in approximately the same ratios as the corresponding negative energies of the circular orbits. The negative energy of H_2 is about 2 (13·5) + 3·3 volts, so that ω has 1·12 times its value for the normal orbit in the H-atom. Inserting numerical values in (1)′ we find

$$6{\cdot}4\times10^{35}$$

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[*] Gurney, *Proc. Roy. Soc.* A. 107, 332, 340. (1925.)

to compare with Gurney's value. If we take $\gamma = 1$ instead of 2·42 we find instead $5{\cdot}5 \times 10^{35}$, in almost exact agreement. A possible meaning of this will be discussed in the next paragraph.

If we compare the ratios of the stopping powers of H_2 and He, we are on surer ground, for no absolute values are required, and Gurney's observations may be directly compared with the ratio of the logarithms in (1)′ with

$$\omega\,(H_2) = 1{\cdot}12\,\omega(H);\;\; \omega\,(He) = 2{\cdot}9\,\omega(H).$$

[The negative energy of the two orbits in the helium atom is ½ (54·2 + 24·5) = 39·35 volts.] We find the following rough values:

Ratio of stopping powers, H_2/He.

$\overline{V}$, cm./sec.	Obs.	Calc. $\gamma = 2{\cdot}42$	Calc. $\gamma = 1$
$1{\cdot}98 \times 10^9$	1·19	1·17	1·20
$1{\cdot}68 \times 10^9$	1·22	1·18	1·22
$1{\cdot}26 \times 10^9$	1·30	1·22	1·27
$1{\cdot}0 \times 10^9$	1·40	1·26	1·34

The calculated values of the ratio are in excellent agreement with the observations for the same value of γ which gives agreement for the absolute value in H_2.

This general agreement between theory and experiment is really satisfactory, for there is some reason to expect that the value of γ ought to be taken smaller than that calculated here. The mechanical theory allows of some mean transference of energy however slow the collision, but very slow collisions should probably be strictly adiabatic with an absolutely zero transfer. It is not difficult to determine roughly from § 5 the point at which the integral giving γ should be stopped in order to make $\gamma = 1$. It turns out that we should stop at a value of $2\pi p\omega/V$ equal to about 0·6. This means that as soon as the time taken by the α-particle to describe a circle of radius p reaches about ⅗ths of the time of revolution of the electron in its orbit, the transfer of energy may be expected to cease completely.

§ 2. *The variables of the perturbation problem.* The following sections contain a summary of the calculations, as should allow of a simple verification of the numerical result. Since the initial orbit is circular, but the inclination of all values, the most suitable variables in which to describe the perturbations are the canonical variables

$$J,\, -\xi,\, Q;\;\; v,\, \eta,\, \gamma.$$

These variables, and the associated formulae are conveniently described by Kramers*, and can be summarily treated here. J is the total impulse coordinate, which alone occurs in the energy of the unperturbed system and reduces to the total moment of momentum P for a circular orbit. Q is the moment of momentum about the shortest distance p from the nucleus of the atom to the path of the α-particle; γ is the longitude of the ascending node, and $v + \gamma$ is the "mean longitude," so that v is the mean longitude of the electron in its own orbital plane measured from the ascending node. ξ and η are a type of rectangular coordinates of the electrical centre of the orbit and vanish for a circle. If X and Y are the rectangular coordinates of the electron in its perturbed orbit, with the direction of the ascending node for X-axis and a line $\frac{1}{2}\pi$ forwards in the direction of motion for Y-axis, then to the first order in ξ and η (see Kramers)

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* Kramers, Heliumatom, *Zeit. fur Phys.* vol. 13, p. 326. (1923.)

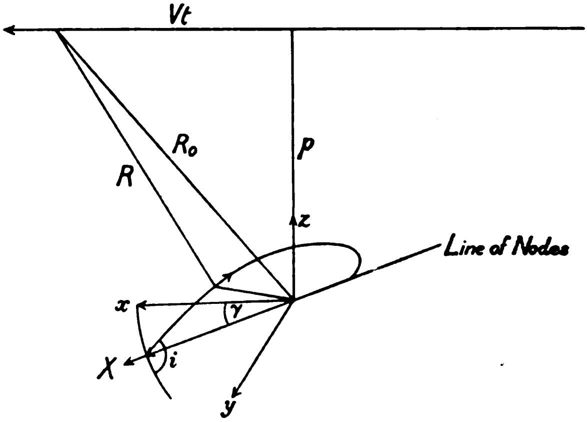
$$X = \frac{J^2}{Eem}\left[\cos v + \frac{\xi}{\sqrt{J}}(1 + \sin^2 v) - \frac{\eta}{\sqrt{J}}\sin v \cos v\right], \quad (2)$$

$$Y = \frac{J^2}{Eem}\left[\sin v - \frac{\xi}{\sqrt{J}}\sin v \cos v + \frac{\eta}{\sqrt{J}}(2 - \sin^2 v)\right], \quad (3)$$

where E is the nuclear charge. The unperturbed energy is

$$H_0 = -\frac{E^2e^2m}{2J^2}. \qquad \text{......(4)}$$

The geometry of the coordinate system will be clear from the figure.



The inclination of the orbital plane is i. The coordinates x, y, z of the electron are given by

$$\left.\begin{aligned} x &= X\cos\gamma - Y\cos i\sin\gamma,\\ y &= X\sin\gamma + Y\cos i\cos\gamma,\\ z &= Y\sin i. \end{aligned}\right\} \qquad \text{......(5)}$$

We require in addition only one more geometrical relation namely

$$Q = P\cos i,$$

which neglecting terms of the second order reduces here to

$$Q = J\cos i \qquad \text{......(6)}$$

since $\xi^2 + \eta^2 = 2(J - P)$.

The perturbation function is the field of the α-particle acting on the electron, which may be written

$$H_1 = \frac{-Ne^2}{R} = \frac{-Ne^2}{R_0} - \frac{Ne^2(xVt + pz)}{R_0^3}, \quad (R_0^2 = p^2 + V^2t^2), \ (7)$$

where Ne is the charge on the α-particle. Since p is large compared to the dimensions of the orbit, the approximate expanded form may be used. The term $-Ne^2/R_0$ is irrelevant. The selected variables satisfy the Hamiltonian equations with a Hamiltonian function $H_0 + H_1$, where

$$\begin{aligned} H_1 = -\frac{NeJ}{mE}\frac{1}{R_0^3}\Bigg[Vt\bigg\{&\left(\cos v + \frac{\xi}{\sqrt{J}}(1+\sin^2 v) - \frac{\eta}{\sqrt{J}}\sin v\cos v\right)J\cos\gamma \\ &- \left(\sin v - \frac{\xi}{\sqrt{J}}\sin v\cos v + \frac{\eta}{\sqrt{J}}(2-\sin^2 v)\right)Q\sin\gamma\bigg\} \\ &+ p\left\{\sin v - \frac{\xi}{\sqrt{J}}\sin v\cos v + \frac{\eta}{\sqrt{J}}(2-\sin^2 v)\right\}(J^2 - Q^2)^{\frac{1}{2}}\Bigg]. \end{aligned}$$
$$\text{......(8)}$$

The unperturbed motion is

$$\left.\begin{aligned} &J = J_0, \ v = n_0t + \alpha = v_0(t), \\ &\xi = 0, \quad \eta = 0, \\ &Q = Q_0, \gamma = \gamma_0. \end{aligned}\right\} \qquad \text{......(9)}$$

Changes of energy after the interaction is over may be obtained directly from changes of J.

§ 3. *The first approximations.* In calculating first approximations the values (9) may be substituted in H_1 after differentiation, *but not in H_0*. We find

$$\dot{J} = \frac{-\partial H_1}{\partial V}, \quad J = J_0 + J_1,$$

$$\begin{aligned} J_1 = -\frac{NeJ_0}{mE}\int_{-\infty}^{t}\frac{d\tau}{R_0^3}\,[V\tau\,\{J_0\cos\gamma_0\sin v_0(\tau) + Q_0\sin\gamma_0\cos v_0(\tau)\} \\ - p\,(J_0^2 - Q_0^2)^{\frac{1}{2}}\cos v_0(\tau)], \text{...(10)} \end{aligned}$$

$$\dot{v} = \frac{E^2e^2m}{J^3} + \frac{\partial H_1}{\partial J} = n_0 - 3n_0\frac{J_1}{J_0} + \frac{\partial H_1}{\partial J}, \quad v = v_0(t) + v_1,$$

$$\begin{aligned} v_1 = \frac{-3n_0}{J_0}\int_{-\infty}^{t} J_1\,d\tau - \frac{Ne}{mE}\int_{-\infty}^{t}\frac{d\tau}{R_0^3} \\ \times [V\tau\,\{2J_0\cos\gamma_0\cos v_0(\tau) - Q_0\sin\gamma_0\sin v_0(\tau)\} \\ + p\,\{(J_0^2 - Q_0^2)^{\frac{1}{2}} + J_0^2(J_0^2 - Q_0^2)^{-\frac{1}{2}}\}\sin v_0(\tau)], \text{...(11)} \end{aligned}$$

where J_1 is given by (10). Next

$$\dot{\xi} = \frac{\partial H_1}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial H_1}{\partial \xi},$$

$$\xi = \xi_1 = \frac{Ne\sqrt{J_0}}{mE}\int_{-\infty}^{t}\frac{d\tau}{R_0^3}$$

$$\times \left[V\tau\left\{\tfrac{1}{2}J_0\cos\gamma_0\sin 2v_0(\tau) + \tfrac{1}{2}Q_0\sin\gamma_0(3+\cos 2v_0(\tau))\right\}\right.$$

$$\left. - \tfrac{1}{2}p(J_0^2 - Q_0^2)^{\frac{1}{2}}(3+\cos 2v_0(\tau))\right], \quad \text{......(12)}$$

$$\eta = \eta_1 = \frac{Ne\sqrt{J_0}}{mE}\int_{-\infty}^{t}\frac{d\tau}{R_0^3}$$

$$\times \left[V\tau\left\{\tfrac{1}{2}J_0\cos\gamma_0(3-\cos 2v_0(\tau)) + \tfrac{1}{2}Q_0\sin\gamma_0\sin 2v_0(\tau)\right\}\right.$$

$$\left. - \tfrac{1}{2}p(J_0^2 - Q_0^2)^{\frac{1}{2}}\sin 2v_0(\tau)\right]. \quad \text{......(13)}$$

$$\dot{Q} = -\frac{\partial H_1}{\partial \gamma}, \quad Q = Q_0 + Q_1,$$

$$Q_1 = -\frac{NeJ_0}{mE}\int_{-\infty}^{t}\frac{V\tau\,d\tau}{R_0^3}\left[J_0\sin\gamma_0\cos v_0(\tau) + Q_0\cos\gamma_0\sin v_0(\tau)\right]. \quad \text{......(14)}$$

$$\dot{\gamma} = \frac{\partial H_1}{\partial Q}, \quad \gamma = \gamma_0 + \gamma_1,$$

$$\gamma_1 = \frac{NeJ_0}{mE}\int_{-\infty}^{t}\frac{d\tau}{R_0^3}\left[V\tau\sin\gamma_0\sin v_0(\tau) + \frac{pQ_0}{(J_0^2 - Q_0^2)^{\frac{1}{2}}}\sin v_0(\tau)\right]. \quad \text{......(15)}$$

Equations (10) to (15) describe fully the first order variation. The chief point of interest in the algebra is to devise simple means by which the results can be checked by independent calculations. Such checks can be simply applied in the limiting case of very swift collisions, when the action becomes impulsive and the change of velocity of the electron must lie along the shortest distance to the track of the α-particle and be asymptotically equal to

$$\frac{2Ne^2}{mpV} \quad \text{......(16).}$$

This is determined by the usual elementary calculation. Taking now equations (2) and (3) and the differentiated equations for $\dot{X}$ and $\dot{Y}$ in unperturbed elliptical motion, and considering the coplanar problem in which $\gamma_0 = 0$, $Q_0 = 0$, we can insert in these equations the limiting values of the changes in J, v, ξ and η from (10) to (13) and verify that at the moment of impact (closest approach) $\delta X = \delta y = \delta \dot{X} = 0$, and $\delta \dot{Y}$ is given by (16). This gives confidence in the correctness of the preceding work.

To obtain values of the stopping power we have to calculate ΔE_0 and average it over all values of γ_0, i_0 and α. We have to the first order

$$\Delta E_0 = \frac{E^2e^2m}{J_0^3} J_1(\infty),$$

and it is easily seen that $\overline{J_1(\infty)} = 0$.

§ 4. *The second approximations.* The change of energy therefore comes from the second order terms, but it is only necessary to calculate these for J. For we have

$$E_0 + \Delta E_0 = -\frac{E^2e^2m}{2(J_0 + J_1(\infty) + J_2(\infty))^2},$$

$$\overline{\Delta E_0} = \frac{E^2e^2m}{J_0^3}(\overline{J_2(\infty)} - \tfrac{3}{2}\overline{\{J_1(\infty)\}^2}/J_0). \quad \text{...(17)}$$

For the second approximation to J we have

$$\dot{J} = -\frac{NeJ}{mER_0^3}\left[Vt\left\{\left(\sin v - \frac{\xi}{\sqrt{J}}\sin 2v + \frac{\eta}{\sqrt{J}}\cos 2v\right)J\cos\gamma + \left(\cos v - \frac{\xi}{\sqrt{J}}\cos 2v - \frac{\eta}{\sqrt{J}}\sin 2v\right)Q\sin\gamma\right\} - p\left\{\cos v - \frac{\xi}{\sqrt{J}}\cos 2v - \frac{\eta}{\sqrt{J}}\sin 2v\right\}(J^2 - Q^2)^{\frac{1}{2}}\right]. \quad \text{......(18)}$$

In (18) the coefficients of ξ and η may be given unperturbed values, and the other terms values from (10) to (15); then

$$\overline{J_2(\infty)} = \int_{-\infty}^{+\infty} \overline{\dot{J}}\,dt, \quad \text{......(19)}$$

and this very much simplifies the calculations, which can be sufficiently illustrated by the first two terms in (18).

Consider first the second term (in ξ). This contributes

$$\frac{NeJ_0}{mE}\int_{-\infty}^{+\infty}\frac{dt}{R_0^3}Vt\sqrt{J_0}\sin 2v_0(t)\cos\gamma_0 \cdot \frac{Ne\sqrt{J_0}}{mE}\int_{-\infty}^{t}\frac{d\tau}{R_0^3} \times [\tfrac{1}{2}V\tau J_0\cos\gamma_0\sin 2v_0(\tau) + (\quad)\sin\gamma_0 + (\quad)].$$

The averaging process with respect to γ_0, i_0 and α consists obviously in the operation

$$\frac{1}{8\pi^2}\int_0^{2\pi}d\alpha\int_0^{\pi}\sin i_0\,di_0\int_0^{2\pi}\dot{J}\,d\gamma_0.$$

Only the square terms—here $\cos^2\gamma_0$—can therefore contribute and it reduces at once, on averaging with respect to γ_0, to

$$\frac{N^2e^2V^2J_0^3}{4m^2E^2}\int_{-\infty}^{+\infty}\frac{t\,dt}{R_0^3}\int_{-\infty}^{t}\frac{\tau\,d\tau}{R_0^3}\sin 2v_0(t)\sin 2v_0(\tau). \quad \text{...(19')}$$

The i_0-integration has here no effect. There is still the α-integration, which again allows only square terms to survive, and we are left with

$$\frac{N^2e^2V^2J_0^3}{8m^2E^2}\int_{-\infty}^{+\infty}\frac{tdt}{R_0^3}\int_{-\infty}^{t}\frac{\tau d\tau}{R_0^3}\left(\sin 2n_0t\sin 2n_0\tau+\cos 2n_0t\cos 2n_0\tau\right). \qquad \ldots\ldots(20)$$

Consider the form of the repeated integrals

$$\int_{-\infty}^{+\infty}Edt\int_{-\infty}^{t}Ed\tau,\quad \int_{-\infty}^{+\infty}Odt\int_{-\infty}^{t}Od\tau,$$

where E and O are even and odd functions of t. $\int_{-\infty}^{t}Od\tau$ is itself *even*, and therefore the second integral always vanishes. Again $\int_{-\infty}^{t}Ed\tau=\int_{0}^{\infty}Ed\tau+O$. Hence the first integral reduces to

$$2\left[\int_0^\infty Edt\right]^2.$$

The contribution (20) therefore finally provides in $\overline{J_2(\infty)}$ the term

$$\frac{N^2e^2V^2J_0^3}{4m^2E^2}\left[\int_0^\infty\frac{tdt}{R_0^3}\sin 2n_0t\right]^2. \qquad \ldots\ldots(21)$$

Similar reductions occur in all terms in $\overline{J_2(\infty)}$.

Consider now the first term in (18). This contributes

$$-\frac{NeV}{mE}\int_{-\infty}^{+\infty}\frac{tdt}{R_0^3}J^2\sin v\cos\gamma,$$

and we replace $J^2\sin v\cos\gamma_0$ by

$$J_0^2\sin v_0\cos\gamma_0+2J_0\sin v_0\cos\gamma_0 J_1+J_0^2\cos v_0\cos\gamma_0 v_1 \\ -J_0^2\sin v_0\sin\gamma_0\gamma_1.$$

The first term contributes nothing to the average and the reduction of the others is similar to the work above. The only novel feature comes from the first term in v_1. This reduces to the stage of (19)′ in the form

$$-\frac{3n_0N^2e^2V^2J_0^3}{2m^2E^2}\int_{-\infty}^{+\infty}\frac{tdt}{R_0^3}\int_{-\infty}^{t}d\tau\int_{-\infty}^{\tau}\frac{\sigma d\sigma}{R_0^3}\cos v_0(t)\sin v_0(\sigma).$$

Averaging with respect to α replaces $\cos v_0(t)\sin v_0(\sigma)$ by

$$\tfrac{1}{2}\left(\cos n_0t\sin n_0\sigma-\sin n_0t\cos n_0\sigma\right),$$

and the integrand can be reduced to a simpler form by integration by parts. The final result is then

$$-\frac{3n_0N^2e^2V^2J_0^3}{m^2E^2}\int_0^\infty\frac{t^2dt}{R_0^3}\cos n_0t\int_0^\infty\frac{tdt}{R_0^3}\sin n_0t. \qquad \ldots\ldots(22)$$

On carrying through these and all the similar calculations simplifying the integrals and collecting results, we obtain

$$\overline{\{J_2(\infty)\}} = \frac{N^2 \epsilon^2 J_0^{\,3}}{m^2 E^2 p^2 V^2}\left[\tfrac{2}{3}(A'^2 + B'^2) + \tfrac{10}{3}(A^2 + B^2) + 4\frac{pn_0}{V}(A - C)B\right], \qquad \ldots\ldots(23)$$

in which

$$A = \int_0^\infty \frac{\cos\frac{pn_0}{V}t}{(1 + t^2)^{\frac{3}{2}}}\,dt, \quad B = \int_0^\infty \frac{t\sin\frac{pn_0}{V}t}{(1 + t^2)^{\frac{3}{2}}}\,dt, \quad C = \int_0^\infty \frac{t^2\cos\frac{pn_0}{V}t}{(1 + t^2)^{\frac{3}{2}}}\,dt,$$

and the dashes denote that in these terms n_0 is replaced by $2n_0$.

The calculation of $\overline{\{J_1(\infty)\}^2}$ from (10) is very simple—only the result need be recorded, which is

$$\overline{\{J_1(\infty)\}^2} = \frac{4N^2 e^2 J_0^{\,4}}{3m^2 E^2 p^2 V^2}[A^2 + B^2]. \qquad \ldots\ldots(24)$$

Combining (17), (23) and (24) we find

$$\overline{\Delta E_0} = \frac{N^2 e^4}{mp^2 V^2}\left[\tfrac{2}{3}(A'^2 + B'^2) + \tfrac{4}{3}(A^2 + B^2) + 4\frac{pn_0}{V}(A - C)B\right]. \qquad \ldots\ldots(25)$$

In the limit of a very swift collision $B \to 0$, $A \to 1$, so that

$$\overline{\Delta E_0} \to \frac{2N^2 e^4}{mp^2 V^2}.$$

This is in agreement with the elementary calculation of the effect of an impulse of the proper magnitude, and is a check on the accuracy of the calculations. Another check on the form of (25) is that, neglecting the $(A - C)$ term which arises solely from changes of frequency during the encounter—the special terms in v_1—, the form of (25) is practically identical with Bohr's calculation for an isotropic oscillator. The only difference lies in some of the terms here having an argument $2n_0$, a natural result when the motion is no longer simple harmonic. The $(A - C)$ terms due to progressive changes of frequency are a new feature of the orbital, as opposed to the vibrational, problem.

§ 5. *The final reductions.* The essence of Bohr's calculations is that we may pass over continuously under the postulated conditions from the field of validity of (25) to the field of validity of exact calculations in which the electron is treated as free. In such cases

$$\Delta E_0 = \frac{2N^2 e^4}{mV^2(p^2 + \lambda^2)}, \quad \lambda = \frac{Ne^2}{mV^2}. \qquad \ldots\ldots(26)$$

and combining (25) and (26) we may say that in all cases the mean transfer of energy is given by

$$\overline{\Delta E_0} = \frac{2N^2e^4}{mV^2}\frac{1}{(p^2+\lambda^2)}\left[\tfrac{1}{3}(A'^2+B'^2)+\tfrac{2}{3}(A^2+B^2)+2\frac{pn_0}{V}(A-C)B\right] \quad \text{......(27)}$$

under the restrictions which Bohr specifies. If we denote the function in [ ] by $\Psi\left(\frac{pn_0}{V}\right)$, the rate of loss of energy by an α-particle in passing through a gas containing L such atoms per c.c. is given by

$$-\frac{dT}{dx} = \frac{4\pi LN^2e^4}{mV^2}\int_0^\infty \frac{\sigma d\sigma}{\sigma^2+(\lambda n_0/V)^2}\Psi(\sigma). \quad \text{......(28)}$$

Under the conditions of the problem $\lambda n_0/V$ is extremely small; moreover $\Psi(0) = 1$ and $\Psi(\sigma) \to 0$ exponentially as $\sigma \to \infty$ (see later). We can therefore evaluate (28) by writing

$$-\frac{dT}{dx} = \frac{4\pi LN^2e^4}{mV^2}\left\{\int_0^\epsilon \frac{\sigma d\sigma}{\sigma^2+(\lambda n_0/V)^2}+\int_\epsilon^\infty \Psi(\sigma)\frac{d\sigma}{\sigma}\right\} \text{...(29)}$$

$$= \frac{4\pi LN^2e^4}{mV^2}\log\frac{\gamma V}{\lambda n_0},$$

where to the required approximation $\log\gamma$ can be evaluated as

$$\underset{\epsilon\to 0}{Lt}\left\{\log\epsilon+\int_\epsilon^\infty \Psi(\sigma)\frac{d\sigma}{\sigma}\right\}. \quad \text{......(30)}$$

To proceed with (30) we may express $\Psi(\sigma)$ in terms of squares and products of Hankel's functions $H_0^{(1)}(i\sigma)$ and $H_1^{(1)}(i\sigma)$ which vanish exponentially as $\sigma \to +\infty$. [Alternatively with $K_0(\sigma)$ and $K_1(\sigma)$.] The greater part of the reduction is however almost simpler without! We observe directly from the definitions that

$$\frac{dA}{d\sigma} = -B,\quad \frac{dB}{d\sigma} = C = -A+\frac{B}{\sigma},\quad \frac{d}{d\sigma}\left(\frac{B}{\sigma}\right) = -\frac{A}{\sigma}.$$

Also $$\int B^2\frac{d\sigma}{\sigma} = -\int\frac{B}{\sigma}\frac{dA}{d\sigma}d\sigma = -\left[\frac{BA}{\sigma}\right]-\int A^2\frac{d\sigma}{\sigma}.$$

Thus $$\int(A^2+B^2)\frac{d\sigma}{\sigma} = -\left[\frac{BA}{\sigma}\right].$$

No other formulae are required, except the form of B for small σ. This is not simply obtainable except from the expression as a Hankel's function. We have*

$$B(\sigma) = \tfrac{1}{2}\pi i\sigma H_0^{(1)}(i\sigma) = -\sigma(\log\tfrac{1}{2}\sigma+0{\cdot}5772)+O(\sigma^2\log\sigma).$$

---

* Jahnke and Emde, *Funktionentafeln*, p. 95; Watson, *Bessel's Functions*, pp. 73, 60.

Thus, from (27), retaining only leading terms,

$$\int_\epsilon^\infty \Psi(\sigma)\frac{d\sigma}{\sigma} = \left[-\frac{1}{3}\frac{AB}{\sigma}\right]_{2\epsilon}^\infty + \left[-\frac{2}{3}\frac{AB}{\sigma} - A^2 + B^2\right]_\epsilon^\infty,$$

$$= -\tfrac{1}{3}(\log\epsilon + 0{\cdot}5772) - \tfrac{2}{3}(\log\tfrac{1}{2}\epsilon + 0{\cdot}5772) + 1;$$

$$\log\gamma = 1 - 0{\cdot}5772 + \tfrac{2}{3}\log 2 = 0{\cdot}885,$$

$$\gamma = 2{\cdot}42.$$

Finally to collect the results in the usual form (1), we have only to recall that $n_0 = 2\pi\omega$, where ω is the orbital frequency of revolution, and insert the value of λ from (26), obtaining

$$-\frac{dT}{dx} = \frac{4\pi LN^2e^4}{mV^2}\log\frac{\gamma mV^3}{2\pi Ne^2\omega}, \quad (\gamma = 2{\cdot}42). \qquad \ldots\ldots(1)$$

The value of γ for the elastically bound electrons, treated by Bohr, is 1·123.

It is a pleasure to record my thanks to the International Education Board for the opportunity, which they have so materially helped to provide, of working in Prof. Bohr's Institute in Copenhagen.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1923—1924.

ANNUAL GENERAL MEETING.

October 29, 1923.

In the Cavendish Laboratory

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:

Mr C. T. Heycock.

Vice-Presidents:

Prof. Seward.
Dr H. Lamb.
Mr J. Barcroft.

Treasurer:

Mr F. A. Potts.

Secretaries:

Prof. Baker.
Mr F. W. Aston.
Mr J. Gray.

Other Members of Council:

Mr H. Hamshaw Thomas.
Mr R. H. Fowler.
Mr E. Cunningham.
Mr T. C. Nicholas.
Dr E. H. Griffiths.
Prof. J. T. Wilson.
Mr J. M. Wordie.
Mr G. I. Taylor.
Mr H. M^cCombie.
Mr F. P. White.
Mr E. V. Appleton.
Mr J. B. S. Haldane.

The following were elected Associates of the Society:

L. Bastings.
E. H. Boomer.
H. J. J. Braddick, Trinity College.
W. J. Bull, King's College.
D. H. de Burgh, Trinity Hall.
A. P. Cary, Christ's College.
Miss G. E. Coit, Newnham College.
P. A. M. Dirac, St John's College.

J. G. H. Frew, Emmanuel College.
H. S. Hirst, Trinity College.
G. G. Jones, Emmanuel College.
L. H. Martin, Trinity College.
V. Nath, Trinity Hall.
T. H. Osgood, Emmanuel College.
W. C. Quayle, Trinity College.
T. N. Seth, Fitzwilliam Hall.
J. H. Wadia, Emmanuel College.
A. H. Wilson.
W. Wilson, St John's College.

The following Communications were made to the Society:

1. On the motion of spheres, circular and elliptic cylinders through viscous fluid. By W. J. HARRISON, M.A., Clare College.

2. On the derivation of the equations of transfer of radiation and their application to the interior of a star. By E. A. MILNE, M.A., Trinity College.

3. (1) The conics through fives of six points.
(2) Certain nets of plane curves.
By F. P. WHITE, M.A., St John's College.

4. Some refinements of the theory of dissociation equilibria. By C. G. DARWIN, M.A., Christ's College, and R. H. FOWLER, M.A., Trinity College.

5. The fundamental theorem of Denjoy integration. By J. C. BURKILL, B.A., Trinity College.

6. On the Correction for Non-Uniformity of Field in Experiments on the Magnetic Deflection of β-Rays. By D. R. HARTREE, B.A., St John's College.

7. On the solution of certain difference equations. By T. M. CHERRY, B.A., Trinity College.

8. On the formulae of one-dimensional kinematics. By W. BURNSIDE, M.A., Pembroke College.

9. Note on the twelve points of intersection of a quadri-quadric curve with a cubic surface. By W. P. MILNE, M.A., Clare College.

10. The effect of deviations from the inverse square law on the scattering of α-particles. By E. S. BIELER, Ph.D., Gonville and Caius College. (Communicated by Mr R. H. Fowler.)

11. The stability of the periodic states of the triode oscillator. By W. H. M. GREAVES, M.A., St John's College.

12. The structure and life-history of *Lipotropha* n.g., a new type of Schizogregarine, parasitic in the fat body of a dipterous larva (*Systenus*). By D. KEILIN, Ph.D., Magdalene College. (Communicated by Mr J. Gray.)

November 12, 1923.

In the Botany School.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Associates of the Society:

W. A. Hyde, Trinity Hall.
J. F. Lehmann, Trinity College.
C. Presswood, Emmanuel College.

Mr P. Lake delivered a Lecture entitled "Wegener's theory of Continental Drift," which was illustrated by Lantern slides.

November 26, 1923.

In the Comparative Anatomy Lecture Room

MR C. T. HEYCOCK, PRESIDENT IN THE CHAIR.

The following were elected Fellows of the Society:

F. A. Bannister, B.A., Clare College.
D. C. Carroll, B.A., Trinity Hall.
W. R. Dean, B.A., Trinity College.
E. Hindle, M.A., Magdalene College.
P. L. Kapitza, Ph.D., Trinity College.
Hon. Sir Charles A. Parsons, Sc.D., St John's College.
K. A. Pearson, B.A., King's College.
F. C. Phillips, B.A., Corpus Christi College
C. T. Preece, M.A., Trinity College.
T. G. Room, B.A., St John's College.
D. R. Rutnam, B.A., Trinity Hall.
R. Stoneley, M.A., St John's College.
T. H. Turney, B.A., Trinity Hall.
G. P. Wells, B.A., Trinity College.
E. V. Whitfield, B.A., Trinity College.
J. R. Wilton, M.A., Trinity College.

The following were elected Associates of the Society:

Miss M. Taylor, Girton College.
W. L. Webster, Trinity College.

The following Communications were made to the Society:

1. The Haemoglobin of Arenicola. By J. BARCROFT, M.A., King's College, and H. BARCROFT, King's College.

2. Direct measurements of Axial Gradients in Embryonic Tissue. By C. SHEARER, Sc.D., Clare College.

3. A Mathematical Theory of Natural and Artificial Selection. Part I. By J. B. S. **Haldane**, M.A., Trinity College.

4. (1) The spawning of Sea-Urchins.
   (2) The migration of a Red Sea Crab through the Suez Canal.

By H. **Munro Fox**, M.A., Gonville and Caius College.

5. The Parasitism of *Helminthosporium gramineum* Rab. [Leaf-stripe disease of Barley]. By N. J. G. **Smith**, Emmanuel College. (Communicated by Mr F. T. Brooks.)

6. A leaf index and its application to varieties of the Potato. By R. N. **Salaman**, M.D., Trinity Hall.

December 3, 1923.

In the Cavendish Laboratory.

Dr H. Lamb, Vice-President, in the Chair.

The following were elected Fellows of the Society:

D. Keilin, M.A., Magdalene College.
G. C. Steward, M.A., Emmanuel College.

The following were elected Associates of the Society:

Miss L. W. Laverick.
Miss J. MacWillis.

The following Communications were made to the Society:

1. On a simple form of stereoscope and its applications. By C. T. R. **Wilson**, M.A., Sidney Sussex College.

2. (1) On an electrostatic oscillograph.
   (2) On oscillographic study of a Coolidge X-ray tube.

By J. A. **Crowther**, Sc.D., St John's College.

3. On the number of lines, in hyperspace, meeting four regions. By R. **Vaidyanathaswamy**. (Communicated by Mr H. W. Turnbull.)

4. On the generalisation of a theorem of Steiner. By Professor H. F. **Baker**.

5. On an extension of Wallace's pedal property for the circumcircle. By H. W. **Richmond**, M.A., King's College.

6. Angular momentum and electronic impacts. By P. M. S. **Blackett**, B.A., King's College.

7. On errors of observation. By W. **Burnside**, M.A., Pembroke College.

January 21, 1924.

In the Cavendish Laboratory.

PROFESSOR SIR ERNEST RUTHERFORD, IN THE CHAIR.

The following was elected a Fellow of the Society:

H. Godwin, B.A., Clare College.

The following were elected Associates of the Society:

J. M. Luck, Gonville and Caius College.
J. H. Quastel, Trinity College.

The following Communications were made to the Society:

1. An early formulation by Stokes of the theories of the rotatory polarizations of light. By Professor Sir **JOSEPH LARMOR**.

2. Theorems for a cubic curve in space. By Professor **H. F. BAKER**.

3. On the angle of incidence in soaring flight. By **E. H. HANKIN**, Sc.D., St John's College.

4. Canonical forms of the quaternary cubic associated with arbitrary quadrics. By **H. W. TURNBULL**, M.A., Trinity College.

5. Complexes of conics and the Weddle surface. By **C. G. F. JAMES**, B.A., Trinity College.

6. On Graeffe's method for complex roots of algebraic equations. By **S. BRODETSKY**, M.A., Trinity College, and **G. SMEAL**.

7. On the problem of four bodies. By **J. BULL**, M.A., St John's College.

February 4, 1924.

In the University Chemical Laboratory.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:

W. S. Bristowe, B.A., Gonville and Caius College.

The following Communications were made to the Society:

1. The preparation of sulphuryl chloride. By Professor Sir **WILLIAM POPE**.

2. A geometrical pattern having the symmetry of quartz. By Professor **T. M. LOWRY**.

3. The heat of combustion of ethylene. By **C. T. HEYCOCK**, M.A., King's College, and **W. H. MILLS**, M.A., Jesus College.

4. The thallous thallic halides. By **A. J. BERRY**, M.A., Downing College.

5. The stability of gas films adsorbed on tungsten. By W. G. Palmer, M.A., St John's College.

6. The configuration of the ammonium radical. By W. H. Mills, M.A., Jesus College, and E. H. Warren, B.A., Jesus College.

7. A note on protecting and sensitising colloidal sols. By E. K. Rideal, M.A., Trinity Hall.

8. The mechanism of molecular activation. By R. G. W. Norrish, B.A., Emmanuel College.

9. The mechanism of the rusting of iron. By U. R. Evans, M.A., King's College.

February 18, 1924.

In the Botany School.

Professor Seward, Vice-President, in the Chair.

The following were elected Fellows of the Society:

B. W. F. Armitage, M.A., St John's College.
R. W. Gurney, B.A., Trinity Hall.

The following were elected Associates of the Society:

A. G. Nasini, Trinity Hall.
M. H. Salaman, Trinity College

Professor H. F. Newall delivered a Lecture entitled "Sun-spots," which was illustrated by Lantern slides.

March 3, 1924.

In the Comparative Anatomy Lecture Room.

Mr J. Barcroft, Vice-President, in the Chair.

The following was elected a Fellow of the Society:

Hon. W. H. Lever, M.A., Trinity College.

The following Communications were made to the Society:

1. The relation between the affinity for certain gases and the position of the spectral bands in the haemoglobin of Vertebrates. By M. L. Anson, J. Barcroft, M.A., King's College, H. Barcroft, A. E. Mirsky, and C. Stockman. (Communicated by Mr J. Barcroft.)

2. The relation of Flagellates and Ciliates to P_H. By J. T. Saunders, M.A., Christ's College.

3. (1) Note on the penetration of hydroxyl ions into gelatin.
(2) The relation of Cilia to Oxygen.
By J. Gray, M.A., King's College.

4. On remains of extinct Proboscidea in the Museums of Geology and Zoology in the University of Cambridge. By C. **Forster Cooper**, M.A., Trinity College.

5. Poncelet's porism for conics. By W. **Burnside**, M.A., Pembroke College.

6. The equation of state of a gas. By J. E. **Jones**. (Communicated by Mr R. H. Fowler.)

7. The penetration of waves into a variable medium. By J. A. **Wilcken**. (Communicated by Professor Sir J. Larmor.)

8. The differential invariants of the Laguerre group. By **Tadahiko Kubota**. (Communicated by Professor H. F. Baker.)

9. The electromagnetic equations as basis of Einstein's quadratic form. By R. **Hargreaves**, M.A., St John's College.

May 5, 1924.

In the Cavendish Laboratory.

Mr C. T. Heycock, President, in the Chair.

The following were elected Associates of the Society:

M. A. F. Barnett.
A. E. Mirsky.
I. M. Puri.
V. A. Putterill.
Miss M. Stephenson.

The following Communications were made to the Society:

1. Researches in the theory of determinants. By Major P. A. **MacMahon**.

2. On problems of random flight, and the conduction of heat. By W. **Burnside**, M.A., Pembroke College.

3. The Integral Expansions of Arbitrary Functions connected with Integral Equations. By J. **Hyslop**, St John's College. (Communicated by Professor E. W. Hobson.)

4. On the condition that five lines, in fourfold space, should be on a quadric, or be chords of a quartic curve. By F. **Bath**, King's College. (Communicated by Mr H. W. Richmond.)

5. Trajectories of small horizontal velocity in a resisting medium. By J. E. **Littlewood**, M.A., Trinity College.

6. The optical properties of matter. By C. G. **Darwin**, M.A., Christ's College.

7. The applicability and deformation of surfaces. By B. M. **Sen**. (Communicated by Professor H. F. Baker.)

May 19, 1924.

In the Cavendish Laboratory.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:

A. E. Cashmore, Ph.D., Sidney Sussex College.

The following was elected an Associate of the Society:

D. H. Black, Emmanuel College.

The following Communications were made to the Society:

1. The high-energy groups in the *RaC* β-ray spectrum. By C. D. ELLIS, Ph.D., Trinity College.

2. The statistical theory of dissociation and ionisation by collision, with applications to the capture and loss of electrons by α-particles. By R. H. FOWLER, M.A., Trinity College.

3. The spectra of some Lithium-like and Sodium-like atoms. By D. R. HARTREE, B.A., St John's College.

4. The relative absorbing powers of the *L*-levels for radiation of varying wave-length. By H. W. B. SKINNER, B.A., Trinity College.

5. (1) Are the overtones of musical sounds always harmonic?

(2) Further developments in the technique for the measurement of the velocity of very rapid chemical reactions.

By H. HARTRIDGE, M.D., King's College.

6. On the measurement of critical potentials of gases. By E. G. DYMOND, B.A., St John's College.

7. (1) An optical interference method of determining Young's modulus for rods.

(2) A recording gyroscope.

By G. F. C. SEARLE, Sc.D., Peterhouse.

8. Some electrical properties of liquid sulphur. By D. H. BLACK. (Communicated by Dr F. W. Aston.)

9. (1) The integrals of differential equations.

(2) Poincare's theorem on the non-existence of uniform integrals of dynamical equations.

By T. M. CHERRY, B.A., Trinity College.

10. Loci having two systems of generating spaces. By H. W. RICHMOND, M.A., King's College, and F. BATH, King's College.

11. The quadratic form for radial acceleration, in the theory of relativity. By R. HARGREAVES, M.A., St John's College.

12. On the early development of the Echinoderm egg. i
G. S. Carter, B.A., Gonville and Caius College.

13. On the appearance of gas in the tracheae of insects. By D
Ph.D., Magdalene College.

14. A comparison of the molecular weights of the prot
G. S. Adair, M.A., King's College.

15. The parasitism of *Plowrightia ribesia*. By Miss I. A.
(Communicated by Mr F. T. Brooks.)

16. The overwintering of apple mildew, *Podosphaera leuc*
England. By R. C. Woodward. (Communicated by Mr F. T.

17. The number of β-particles from Radium *E*. By K. G.
B.A., St John's College. (Communicated by Professor Sir E. Ru

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July 14, 1924.

In the Cavendish Laboratory:

Mr C. T. Heycock, President, in the Chair.

The following were elected Fellows of the Society:

F. W. Carter, M.A., St John's College.
J. Burtt Davy, Ph.D., Christ's College.
G. G. Morrice, M.D., Trinity College.

The following Communications were made to the Society:

1. Some experiments with an α-particle counter. By E. V. A
M.A., St John's College, K. G. Emeléus, B.A., and M. Barne

2. On the zeros of the Riemann zeta function. By J. E. Lit
M.A., Trinity College.

3. A case of steady flow of a gas, in two dimensions. By
M.A., St John's College.

4. Note on the Doppler principle and Bohr's frequency c
By P. A. M. Dirac, St John's College. (Communicated by
Fowler.)

5. The applicability and deformation of surfaces. By B.
(Communicated by Professor H. F. Baker.)

6. A system of linear equations with an infinity of unkno
E. C. Titchmarsh. (Communicated by Mr G. H. Hardy.)

7. A mathematical theory of natural and artificial selection.
By J. B. S. Haldane, M.A., Trinity College.

8. The vector algebra of eight associated points, of three
surfaces. By H. W. Turnbull, M.A., Trinity College.

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Transactions of the Cambridge Philosophical Society

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CONTENTS.

PAGE

A generalization of Feuerbach's theorem. By Mr J. P. Gabbatt, Peterhouse 601

Characteristics of complexes of conics in space of four dimensions. By Mr C. G. F. James, Trinity College 621

Note on the Lorentz group. By Mr J. Brill, St John's College . . 630

The symmetric functions of which the general determinant is a particular case. By Major P. A. MacMahon, Sc.D., F.R.S., St John's College 633

On simplexes doubly incident with a quadric. By R. Vaidyanathaswamy. (Communicated by Professor H. F. Baker) 655

On the phrase "equally probable." By Mr W. Burnside, Honorary Fellow of Pembroke College 669

A note on wireless signal strength measurements made during the solar eclipse of 24 *January*, 1925. By Mr E. V. Appleton and M. A. F. Barnett. (With One Text-figure) 672

The action of the electrical counter. By K. G. Emeléus, B.A., St John's College. (Communicated by Dr J. Chadwick.) (With Four Text-figures) 676

An extension of Wallace's, Miquel's and Clifford's theorems on circles. By Mr F. P. White, St John's College 684

Triple binary forms; the complete system for a single (1, 1, 1) *form with its geometrical interpretation.* By Mr W. Saddler, St John's College. (Communicated by Mr H. W. Turnbull) 688

A geometrical treatment of the correspondence between lines in threefold space and points of a quadric in fivefold space. By Mr H. W. Turnbull, Trinity College 694

Theory of statistical estimation. By Mr R. A. Fisher, Gonville and Caius College 700

On the idea of frequency. By Mr W. Burnside, Honorary Fellow of Pembroke College 726

The bursting of soap-bubbles in a uniform electric field. By Professor C. T. R. Wilson, and Mr G. I. Taylor, Trinity College. (With Plates VI and VII) 728

The passage of electrons through small apertures. By J. F. Lehmann and T. H. Osgood. (Communicated by Professor Sir E. Rutherford, F.R.S.) (With Four Text-figures) 731

An apparatus for the investigation of the effect of poisonous substances, and mixed vapours on catalytic activity. By F. H. Constable, B.A., Strathcona Research Student, St John's College. (With Three Text-figures) 738

On pedal quadrics in non-euclidean hyperspace. By Mr J. P. Gabbatt, Peterhouse 751

The temperature distribution in a transformer or other laminated core of rectangular cross section in which heat is generated at a uniform rate. By J. D. Cockcroft, B.A., St John's College. (With Five Text-figures) 759

The theory of the influence of magnetic fields on the stopping power of gases for α-particles. By R. de L. Kronig. (Communicated by Mr R. H. Fowler) 773

On Hausdorff's proof of the extended Riesz-Fischer theorem. By Mr S. Pollard, Trinity College 777

On the representation of the modular group of order $\frac{1}{2}p(p^2-1)$ *as a group of linear substitutions on* $\frac{1}{2}(p-1)$ *symbols, when* p *is a prime of the form* $4n+3$. By Mr W. Burnside, Honorary Fellow of Pembroke College 779

On groups of linear substitutions which contain irreducible metacyclical subgroups. By Mr W. Burnside, Honorary Fellow of Pembroke College 784

A theoretical study of the stopping power of hydrogen atoms for α-particles. By Mr R. H. Fowler, Trinity College. (With One Text-figure) . 793

Proceedings at the Meetings held during the Session 1923—1924 . 804

Statement of Accounts of the Cambridge Philosophical Society, from January 1, 1924, to December 31, 1924 814

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PROCEEDINGS

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Cambridge Philosophical Society.

Notes on Optical Constants. By Mr C. G. DARWIN, F.R.S., Tait Professor of Natural Philosophy in the University of Edinburgh.

I. *The Optical Behaviour of certain Atomic Models.*

[*Received* 24 June, *read* 20 July 1925.]

§ 1. In a recent paper[*] the present writer made an attempt to reduce the ordinary optical constants of matter to a more primitive form by introducing "scattering indices." Their use solves half the problem of optics, for general formulae connect these indices with the behaviour of matter in bulk. There remains the deduction of the scattering indices from whatever we are assuming to be the reaction to light of the actual atoms, and even in the classical theory this encounters rather formidable difficulties of convergence. As long as we are content with no great rigour it is a fairly simple matter to deduce the scattering indices for any assumed model without any of those rather difficult arguments about the polarisation of the medium which play a part in the ordinary presentation.

The essential point which simplifies the process is that the wave given out by each atomic model is spherical and contains therefore a factor $e^{ik(ct-r)}/r$. So as long as we consider a point at a distance of only a small fraction of a wave-length away kr is small and the field is practically *statical*. This makes it easy to work out the Hertzian vector associated with the influence of an external electric force, and the factor e^{-ikr} can be re-introduced afterwards. Though there is perhaps a defect in formal rigour in proceeding in this way, there can I think be no doubt of its correctness, and it certainly leads to the right result in all cases where it can be compared with more precise methods.

We shall first illustrate the procedure with the model with which Lorentz obtains the ordinary dispersion formula, and shall then establish a result quoted but not proved in § 20*, which shows how the magnetic gyration of light can be connected with the idea that magnetism is equivalent to a rotation. Lastly, we shall investigate Drude's[†] "hypothesis of molecular currents." This hypothesis has I think been abandoned completely, but in the course of a study of the dispersion of magnetic gyration certain very peculiar experimental results[†] (with oxygen) suggested that it would be worth re-examining. The present method involves rather troublesome algebra but must I think be regarded as more rigorous than Drude's, and leads to quite different results. The two processes are too far apart to make it easy to see where the discrepancy arises, and I have not attempted to do so in detail, partly because the hypothesis has been abandoned and also because Drude works throughout his book on the inferior principle which obtains the Sellmeier and not the Lorentz dispersion formula.

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[*] "The Optical Constants of Matter," *Camb. Phil. Soc. Trans.* vol. XXIII, No. VI (1924). I shall use the notation of the paper without explanation. References to formulae given there will be marked by a *.

[†] Drude, *Theory of Optics* (English Translation), p. 418.

In the course of the present work I have gained a good deal of experience with the tensor notation and of its utility. The verdict must I think be that as long as we are concerned with pure vectors it is inferior to the vector notation in quickness and convenience, but that the slightest degree of complication, as for instance double refraction, puts the vector notation into difficulty and brings out the power of the tensor algebra.

§ 2. Consider the atom as composed of an electron of charge e which can vibrate isotropically about the origin. Under the influence of an electric force $E_a{}^0 e^{ikct}$ its position ξ_a is given by

$$e\xi_a = \rho E_a{}^0 e^{ikct} \qquad \ldots\ldots(2\cdot1).$$

If the restoring force is $mk_0{}^2c^2$, we have

$$\rho = e^2/mc^2\,(k_0{}^2 - k^2) \qquad \ldots\ldots(2\cdot2),$$

but beyond noting that ρ is a function of k we shall not make any use of this relation. The rigorous deduction of the radiation from (2·1) is well known, but we shall give here a shorter non-rigorous process which will illustrate our method.

Consider a point x_a a small fraction of a wave-length distant from the electron. The electrostatic potential there is $\phi = e/r_1$, where $r_1{}^2 = (x_a - \xi_a)^2$. We suppose that ξ_a is much smaller than x_a and so have $\frac{1}{r_1} = \frac{1}{r} - \xi_a \frac{\partial}{\partial x_a}\frac{1}{r}$, where $r^2 = x_a{}^2$. Now if Π_a is the Hertzian vector $\phi = -\partial\Pi_a/\partial x_a$. Hence considering only the variable part we have $\Pi_a = e\xi_a/r = \rho E_a{}^0 e^{ikct}/r$. Π_a satisfies the wave equation, and to fulfil this condition at other distances we therefore introduce a factor e^{-ikr} and have

$$\Pi_a = \rho E_a{}^0 e^{ik(ct-r)}/r.$$

Let there be N such atoms in unit volume. Then the volume dV gives rise to a scattered wave of amount

$$\Pi_a = N dV \rho E_a{}^0 e^{ik(ct-r)}/r.$$

This is equivalent to (4·1)[*]

$$\Pi_a = (\sigma/4\pi)\, dV E_a{}^0 e^{ik(ct-r)}/r.$$

[†] Siertsema, *Arch. Neerl.* vol. II, p. 291 (1899).

Thus $\sigma = 4\pi N\rho$ and we obtain exactly Lorentz's dispersion by introducing (2·2).

§ 3. We can apply the same method to connect the magnetic gyration of light with the Larmor rotation. An atom in a magnetic field $H = \frac{2mc}{e}\omega$ behaves as though there were no field provided it is regarded from the point of view of axes rotating with angular velocity ω in the negative sense. We suppose the atom isotropic and that its vibration is described by the coefficient ρ of § 2, but we must now remember that ρ is a function of k. It is exposed to an electric force $E_1{}^0 e^{ikct}$. If this is referred to axes rotating with velocity $-\omega$ about x_3 we have

$$E_1' = E_1{}^0 e^{ikct} \cos \omega t = \tfrac{1}{2} E_1{}^0 e^{i(kc+\omega)t} + \tfrac{1}{2} E_1{}^0 e^{i(kc-\omega)t},$$

$$E_2' = E_1{}^0 e^{ikct} \sin \omega t = -\tfrac{1}{2} i E_1{}^0 e^{i(kc+\omega)t} + \tfrac{1}{2} i E_1{}^0 e^{i(kc-\omega)t}.$$

Referred to these axes the atom is isotropic; so in its neighbourhood we shall have

$$\Pi_1' = \tfrac{1}{2}\rho\left(k + \frac{\omega}{c}\right) E_1{}^0 e^{i(kc+\omega)t}/r + \tfrac{1}{2}\rho\left(k - \frac{\omega}{c}\right) E_1{}^0 e^{i(kc-\omega)t}/r,$$

$$\Pi_2' = -\tfrac{1}{2} i\rho\left(k + \frac{\omega}{c}\right) E_1{}^0 e^{i(kc+\omega)t}/r + \tfrac{1}{2} i\rho\left(k - \frac{\omega}{c}\right) E_1{}^0 e^{i(kc-\omega)t}/r.$$

Transform these back to fixed axes, and we have

$$\Pi_1 = \tfrac{1}{2}\left\{\rho\left(k + \frac{\omega}{c}\right) + \rho\left(k - \frac{\omega}{c}\right)\right\} E_1{}^0 e^{ikct}/r,$$

$$\Pi_2 = -\frac{i}{2}\left\{\rho\left(k + \frac{\omega}{c}\right) - \rho\left(k - \frac{\omega}{c}\right)\right\} E_1{}^0 e^{ikct}/r.$$

To describe the effects at greater distances we now introduce the factor e^{-ikr}. If the incident light is not very close to a region of resonance we can expand ρ and have

$$\Pi_1 = \rho E_1{}^0 e^{ik(ct-r)}/r,$$

$$\Pi_2 = -i\frac{\omega}{c}\frac{d\rho}{dk} E_1{}^0 e^{ik(ct-r)}/r.$$

Summing over unit volume we obtain as the scattering indices

$$\sigma_{11} = 4\pi N\rho,$$

$$\sigma_{12} = -4\pi N i\frac{\omega}{c}\frac{d\rho}{dk},$$

where the σ's are defined as in (11·2)*, and so we have proved

$$\sigma_{12} = -i\frac{\omega}{c}\frac{d\sigma}{dk}.$$

This result does not in any way depend on the form of σ, and so will be true, for example, for the Kerr effect, provided that it is legitimate there to regard the magnetism as equivalent to a rotation.

§ 4. In Drude's "hypothesis of molecular currents" an electron is supposed to describe a small circle rapidly, and under the influence of the electric field the whole circle shifts to and fro. This has the effect of loading the magnetic flux in the medium. The model is equivalent to supposing that to each electron is attached a magneton. In an external magnetic field the magnetons are orientated, and as they oscillate they will cause an emission. We shall work out the scattering of such a system.

Take a magneton of which the components have moment $e\mu_\theta$. (The factor e, the charge of the attached electron, is introduced for later simplification; μ_θ is of the physical dimensions of a length.) The vector potential of the magneton is

$$A_\beta = [\beta\gamma\theta]\, e\mu_\theta \frac{\partial}{\partial x_\gamma}\left(\frac{1}{r_1}\right) \qquad \ldots\ldots(4{\cdot}1).$$

As in § 2 we replace $\frac{1}{r_1}$ by $\frac{1}{r} - \xi_\delta \frac{\partial}{\partial x_\delta}\frac{1}{r}$, and drop the constant part. We also substitute

$$e\xi_\delta = \rho E_\delta{}^0 e^{ikct} = \rho E_\alpha{}^0 (\alpha\delta)\, e^{ikct},$$

where we have introduced a "spherical tensor" $(\alpha\delta)$ so as to bring the expression into a standard form. We also have the relation

$$A_\beta = \frac{1}{c}\frac{\partial \Pi_\beta}{\partial t} = ik\Pi_\beta.$$

Hence

$$\Pi_\beta = -[\beta\gamma\theta]\,(\alpha\delta)\frac{\rho\mu_\theta}{ik} E_\alpha{}^0 \frac{\partial^2}{\partial x_\gamma \partial x_\delta}\frac{e^{ik(ct-r)}}{r} \qquad \ldots\ldots(4{\cdot}2).$$

This looks as though it were of the standard form (11·2)*, but is not so as it does not conform to the "duplicate rule," § 12*, and the rule of symmetry between γ and δ. We can satisfy this last rule at once by permuting γ, δ and halving the result. Then (with R for $e^{ik(ct-r)}/r$)

$$\Pi_\beta = \frac{i\rho\mu_\theta}{2k}\{[\beta\gamma\theta]\,(\alpha\delta) + [\beta\delta\theta]\,(\alpha\gamma)\} E_\alpha{}^0 \frac{\partial^2}{\partial x_\gamma \partial x_\delta} R \ \ldots(4{\cdot}3).$$

The application of the duplicate rule is more troublesome. We may recall that there are no waves of the second rank associated with $\sigma_{\alpha\beta\beta\delta}$, $\sigma_{\alpha\beta\gamma\beta}$, and that the wave associated with $\sigma_{\alpha\beta\gamma\gamma}$ is of zero and not the second rank. To satisfy the conditions take

$$\rho_{\alpha\beta\gamma\delta} = (i\rho\mu_\theta/2k)\,\{[\beta\gamma\theta]\,(\alpha\delta) + [\beta\delta\theta]\,(\alpha\gamma)\} \\ + \rho'_{\alpha\beta}\,(\gamma\delta) + \rho''_{\alpha\gamma}\,(\beta\delta) + \rho''_{\alpha\delta}\,(\beta\gamma)$$

and choose $\rho'_{\alpha\beta}$, $\rho''_{\alpha\gamma}$ so that $\rho_{\alpha\beta\gamma\delta}$ shall conform to the rule. To do this multiply by $(\beta\gamma)$. Observe that whereas $(\beta\gamma) \times (\gamma\delta)$ means that β is replaced by δ, we have $(\beta\gamma) \times (\beta\gamma) = 3$. Then

$$\rho_{\alpha\beta\beta\delta} = 0 = (i\rho\mu_\theta/2k)\,[\alpha\delta\theta] + \rho'_{\alpha\delta} + \rho''_{\alpha\delta} + 3\rho''_{\alpha\delta}.$$

Again multiplying by $(\gamma\delta)$

$$\rho_{\alpha\beta\gamma\gamma} = 0 = (i\rho\mu_\theta/2k)\,2\,[\beta\alpha\theta] + 3\rho'_{\alpha\beta} + \rho''_{\alpha\beta} + \rho''_{\alpha\beta}.$$

Changing δ into β we can solve and find

$$\begin{aligned} \rho'_{\alpha\beta} &= 2\,(i\rho\mu_\theta/4k)\,[\alpha\beta\theta], \\ \rho''_{\alpha\delta} &= -\,(i\rho\mu_\theta/4k)\,[\alpha\delta\theta] \qquad \text{......(4·4).} \end{aligned}$$

The addition to (4·3) of $\rho''_{\alpha\delta}\,(\beta\gamma)$, $\rho''_{\alpha\gamma}\,(\beta\delta)$ gives no wave, but for $\rho'_{\alpha\beta}\,(\gamma\delta)$ this is not so, and we must allow for a compensating wave of zero rank. Thus

$$\begin{aligned} \Pi_\beta = (i\rho\mu_\theta/4k)\,E_\alpha\{2\,[\beta\gamma\theta]\,(\alpha\delta) + 2\,[\beta\delta\theta]\,(\alpha\gamma) + 2\,[\alpha\beta\theta]\,(\gamma\delta) \\ - [\alpha\delta\theta]\,(\beta\gamma) - [\alpha\gamma\theta]\,(\beta\delta)\}\,\frac{\partial^2}{\partial x_\gamma \partial x_\delta} R \\ - (i\rho\mu_\theta/2k)\,E_\alpha{}^0[\alpha\beta\theta]\,(\gamma\delta)\,\frac{\partial^2}{\partial x_\gamma \partial x_\delta} R. \end{aligned}$$

In the last line we can replace $(\gamma\delta)\dfrac{\partial^2}{\partial x_\gamma \partial x_\delta}$ by $-k^2$, which exhibits the reduction to zero rank.

We now add on the contribution from the electron as in § 2 and sum over the atoms in unit volume. The result is then exactly of the standard form in (11·2)* and we have

$$\begin{aligned} \sigma_{\alpha\beta} &= \sigma\,(\alpha\beta) + \tfrac{1}{2}ik\sigma\mu_\theta\,[\alpha\beta\theta], \\ \sigma_{\alpha\beta\gamma\delta} &= \tfrac{1}{4}ik\sigma\mu_\theta\{2\,[\beta\gamma\theta]\,(\alpha\delta) + 2\,[\beta\delta\theta]\,(\alpha\gamma) + 2\,[\alpha\beta\theta]\,(\gamma\delta) \\ &\qquad - [\alpha\delta\theta]\,(\beta\gamma) - [\alpha\gamma\theta]\,(\beta\delta)\} \ldots\text{(4·5).} \end{aligned}$$

This is the complete formal account of the optics of the medium. We observe that it differs essentially from the model of § 3 in that it contains a wave of the second rank. This suggests at once that the behaviour of the models will be different for magnetic double refraction, if they are adjusted to be the same for gyration.

§ 5. The equation of wave velocities is from (16·1)*

$$\begin{aligned} N_\eta = N_\alpha \Big\{\Big(\frac{1}{3} + \frac{1}{n^2-1}\Big)\sigma_{\alpha\eta} - \frac{n_\eta n_\beta}{n^2-1}\sigma_{\alpha\beta} - \frac{n_\gamma n_\delta}{7}\,(\sigma_{\alpha\eta\gamma\delta} + \sigma_{\alpha\gamma\eta\delta} + \sigma_{\alpha\gamma\delta\eta}) \\ - \frac{n_\gamma n_\delta}{n^2-1}\,(\sigma_{\alpha\eta\gamma\delta} - n_\eta n_\beta \sigma_{\alpha\beta\gamma\delta})\Big\} \ldots\text{(5·1).} \end{aligned}$$

If we substitute from (4·5) only a few of the fourth rank terms survive; in particular the terms multiplied by 1/7 go right out, which shows that the convergence of the integrations is no more critical than in the ordinary type of medium. The equation reduces to

$$\frac{1}{\sigma}N_\eta = N_a\left[\left(\frac{1}{3}+\frac{1}{n^2-1}\right)(\alpha\eta)-\frac{n_\eta n_a}{n^2-1}-ik\mu_\theta\left\{\tfrac{1}{3}[\alpha\eta\theta]\right.\right.$$
$$\left.\left.+\frac{n_a n_\gamma}{n^2-1}[\eta\gamma\theta]\right\}\right] \quad \text{......(5·2).}$$

To find the gyration we take magnetons $\mu_1=\mu_2=0$, $\mu_3=\mu$ and $n_1=n_2=0$, $n_3=n$. The equations become

$$\frac{1}{\sigma}N_1 = N_1\left(\frac{1}{3}+\frac{1}{n^2-1}\right)+N_2\frac{ik\mu}{3} \quad \text{......(5·3),}$$
$$\frac{1}{\sigma}N_2 = N_1\left(-\frac{ik\mu}{3}\right)+N_2\left(\frac{1}{3}+\frac{1}{n^2-1}\right),$$

$N_3=0$. Thus

$$\frac{1}{3}+\frac{1}{n^2-1}=\frac{1}{\sigma}\pm\frac{k\mu}{3} \quad \text{......(5·4),}$$

and

$$N_2=\pm iN_1.$$

The gyration in unit length is given by $\frac{1}{2}k(n'-n'')$, where n' and n'' are the roots of (5·4). We have approximately

$$\tfrac{1}{2}k(n'-n'')=\frac{(n^2-1)^2}{2n}\frac{k^2\mu}{3}.$$

It is more interesting to compare it with the corresponding effect from § 3 where there is only a wave of zero rank. Instead of (5·4) we have in that case (20·3)*,

$$\frac{1}{3}+\frac{1}{n^2-1}=\frac{1}{\sigma\pm\tau},$$

where $\sigma_{12}=-\sigma_{21}=i\tau$. We thus have

$$\frac{\tau}{\sigma^2}=\frac{k\mu}{3} \quad \text{......(5·5).}$$

For values of k rather near an absorption line σ varies as $1/(k_0^2-k^2)$, and τ by § 3 as $k/(k_0^2-k^2)^2$. Thus (5·5) indicates that Drude's model should give exactly the same dispersion as the other. For a wave length very close to the line the approximation fails, but it is easy to see that the gyration will not change sign. We thus conclude, in complete disagreement with Drude, that there will be nothing in optical observation of gyration to discriminate between the hypothesis of molecular currents and the hypothesis of rotation.

§ 6. To study the double refraction we again take $\mu_1 = \mu_2 = 0$, $\mu_3 = \mu$, but now $n_1 = n$, $n_2 = n_3 = 0$. The equations are

$$\frac{1}{\sigma} N_1 = N_1 \left(-\frac{2}{3}\right) + N_2 \frac{ik\mu}{3} \qquad \text{......(6·1)},$$

$$\frac{1}{\sigma} N_2 = N_1\, ik\mu \left(-\frac{1}{3} + \frac{n^2}{n^2 - 1}\right) + N_2 \left(\frac{1}{3} + \frac{1}{n^2 - 1}\right),$$

$$\frac{1}{\sigma} N_3 = N_3 \left(\frac{1}{3} + \frac{1}{n^2 - 1}\right).$$

The x_3 component is normal. For the other we have approximately

$$\frac{1}{3} + \frac{1}{n^2 - 1} = \frac{1}{\sigma} + \frac{1}{3} k^2 \mu^2 \frac{3 + \sigma}{3 + 2\sigma} \qquad \text{......(6·2)}$$

and
$$N_1 = -\frac{ik\mu\sigma}{3 + 2\sigma} N_2.$$

On the rotational hypothesis we have for the same case

$$\frac{1}{3} + \frac{1}{n^2 - 1} = \frac{1}{\sigma} + \frac{\tau^2}{\sigma^2} \frac{2}{3 + 2\sigma}.$$

If we substitute from (5·5) we make the gyrations the same, and the double refraction will then be in the ratio $3(3 + \sigma) : 2\sigma^2$. We thus see that the double refraction will give a discrimination between the hypotheses, and where it has been observed it does of course in fact support the rotational. Drude's work again disagrees with ours, for he found no double refraction at all.

It appears to me that the present calculations start from exactly the same hypotheses as Drude's, and it is a little disquieting that they should give such different results, but the processes are too far apart to trace the discrepancy. Drude's neglect of the factor $\frac{1}{3}$ in calculating refraction leads him to the Sellmeier formula instead of Lorentz's, and it is not unreasonable to suppose that this should make a capital difference in what is in essence a superposed second order effect. As the hypothesis has, I believe, been completely abandoned it does not seem worth while to search more deeply for the cause of the discrepancy.

Notes on Optical Constants. By Mr C. G. Darwin, F.R.S., Tait Professor of Natural Philosophy in the University of Edinburgh.

II. *The Lateral Scattering from a Gas.*

[*Received* 24 June, *read* 20 July 1925.]

1. The tensor calculus introduced in a recent paper* proves a powerful method of investigating the light scattered laterally as exemplified in the blue of the sky. The late Lord Rayleigh† worked out the effect on the polarisation of having non-spherical molecules, and the present note is practically a repetition of his work but with a somewhat greater generality. The method has the advantage that it does not become unmanageable for molecules of a more complicated type of asymmetry, whereas a perusal of his work suggests that it would be a very complicated matter to give the lateral scattering from a set of arbitrary orientated "optically active" molecules. As this case has at present no practical interest, I have merely indicated the procedure so that it could easily be applied when required.

The calculations depend on the fact that laterally the waves from the molecules are all out of phase, and therefore the intensity is simply the sum of the intensities from each of them. The total intensity can be measured, but the most accurate observation is to study the degree of polarisation observed at right angles to the incident beam. This has been done by the present Lord Rayleigh‡, and we shall here give a short discussion of his results, following in general the lines laid down by his father before these experiments had been made.

2. We define the wave scattered by a molecule by the equation

$$\Pi_\beta = \rho_{\alpha\beta} E_\alpha^0 e^{ik(ct-r)}/r.$$

Here E_α^0 is the periodic electric force at the molecule, Π_β is the Hertzian vector of the scattered wave and $\rho_{\alpha\beta}$ is the molecular scattering tensor. If the atoms were not arbitrarily orientated it would be connected with the scattering index of matter in bulk by the relation $\sigma_{\alpha\beta} = 4\pi N \rho_{\alpha\beta}$, where N is the number of molecules in unit volume. We are assuming that the scattering indices form a tensor of the second rank only and are thus excluding natural gyration, but an inherent magnetic gyration is admissible.

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* "The Optical Constants of Matter," *Camb. Phil. Soc. Trans.* vol. XXIII, p. 137 (1924). I use the notation of that work without explanation, marking references with a *.

† Rayleigh, *Phil. Mag.* vol. XXXV, p. 373 (1918).

‡ Rayleigh, *Proc. Roy. Soc.* A, 97, p. 435 (1920).

Take a set of permanent axes x, and a second set x' which are ixed to the molecule and so are arbitrarily orientated with regard o x. Let the connecting direction cosines be given by

$$x_\alpha = l_{\alpha\phi} x_\phi' \qquad \text{......(2·1).}$$

The constants ρ refer to the x' axes. For these axes the incident force has components $E_\phi' = l_{\alpha\phi} E_\alpha^0$ and the scattered wave is thus

$$\Pi_\chi' = \rho_{\phi\chi} l_{\alpha\phi} E_\alpha^0 e^{ik(ct-r)}/r.$$

Transforming back we have

$$\Pi_\beta = \rho_{\phi\chi} l_{\alpha\phi} l_{\beta\chi} E_\alpha^0 e^{ik(ct-r)}/r \qquad \text{......(2·2).}$$

We first find the refractive index. To do this we must average over all orientations and we find as the scattering indices

$$\sigma_{\alpha\beta} = 4\pi N \rho_{\phi\chi} \overline{l_{\alpha\phi} l_{\beta\chi}},$$

$\overline{l_{\alpha\phi} l_{\beta\chi}}$ vanishes unless $\alpha = \beta$ and $\phi = \chi$ and in that case it is $\frac{1}{3}$. We thus put

$$\overline{l_{\alpha\phi} l_{\beta\chi}} = \tfrac{1}{3} (\alpha\beta)(\phi\chi) \qquad \text{......(2·3),}$$

and have
$$\sigma_{\alpha\beta} = \frac{4\pi}{3} N \rho_{\phi\phi} (\alpha\beta).$$

Thus the refractive index is

$$\frac{3(n^2-1)}{n^2+2} = \frac{4\pi}{3} N (\rho_{11} + \rho_{22} + \rho_{33}) \qquad \text{......(2·4).}$$

3. We next examine the wave emitted by a single atom so as to obtain the intensities of its polarised components. We do this for a point at a great distance from the molecule in a direction of which the cosines are m_α. Then in carrying out the differentiations it suffices to write

$$\frac{\partial}{\partial x_\alpha} \frac{e^{ik(ct-r)}}{r} \quad \text{as} \quad -ikm_\alpha \frac{e^{ik(ct-r)}}{r}.$$

We have

$$E_\kappa = k^2 \Pi_\kappa - \frac{\partial^2 \Pi_\beta}{\partial x_\kappa \partial x_\beta} = k^2 \rho_{\phi\chi} l_{\alpha\phi} l_{\beta\chi} E_\alpha^0 \{(\beta\kappa) - m_\beta m_\kappa\} \frac{e^{ik(ct-r)}}{r} \qquad (3·1).$$

The intensity of the wave is proportional to the product of this by its conjugate. We denote the conjugate of $\rho_{\phi\chi}$ as $\tilde{\rho}_{\phi\chi}$ etc. and have

$$I_\kappa = \frac{k^4}{r^2} \rho_{\phi\chi} \tilde{\rho}_{\psi\omega} l_{\alpha\phi} l_{\beta\chi} l_{\gamma\psi} l_{\delta\omega} E_\alpha^0 \tilde{E}_\gamma^0 \{(\beta\kappa) - m_\beta m_\kappa\} \{(\delta\kappa) - m_\delta m_\kappa\} \quad (3·2),$$

where the expression is *not summed for* κ. This expression is to be averaged for all orientations, and this averaging constituted the troublesome part of the problem. It was carried out by Rayleigh by using the Eulerian angles, but the tensor calculus will simplify the procedure a great deal.

4. We require the average value of $l_{\alpha\phi} l_{\beta\chi} l_{\gamma\psi} l_{\delta\omega}$, say $T^{\alpha\beta\gamma\delta}_{\phi\chi\psi\omega}$. This is evidently a tensor of the eighth rank. The only quantities on which it can depend are "spherical tensors" $(\alpha\beta)$, etc. T must therefore be a sum of terms each involving four factors and containing all the eight letters. The x and x' axes are quite independent so that we shall not get factors like $(\alpha\phi)$ at all. A high degree of symmetry will exist, for α is quite indifferent to χ, ψ, ω though it has a special relationship with ϕ. There are two tensors only satisfying these conditions, viz.:

$$T_1 = (\alpha\beta)(\gamma\delta)(\phi\chi)(\psi\omega) + (\alpha\gamma)(\beta\delta)(\phi\psi)(\chi\omega) + (\alpha\delta)(\beta\gamma)(\phi\omega)(\chi\psi) \qquad \ldots\ldots(4\cdot1),$$

$$T_2 = (\alpha\beta\gamma\delta)(\phi\chi\psi\omega) \qquad \ldots\ldots(4\cdot2),$$

where $(\alpha\beta\gamma\delta)$ is short for

$$(\alpha\beta)(\gamma\delta) + (\alpha\gamma)(\beta\delta) + (\alpha\delta)(\beta\gamma) \qquad \ldots\ldots(4\cdot3).$$

It follows that $T^{\alpha\beta\gamma\delta}_{\phi\chi\psi\omega} = AT_1 + BT_2$, where A, B are two numerical constants. To determine them we have to evaluate two special cases. This can be done by the use of the Eulerian angles; but this is unnecessary, for we need only start by evaluating $\overline{l_{11}{}^2} = \frac{1}{3}$, $\overline{l_{11}{}^4} = \frac{1}{5}$. Then $\overline{l_{11}{}^2 (l_{11}{}^2 + l_{12}{}^2 + l_{13}{}^2)} = \frac{1}{3}$ and so $\overline{l_{11}{}^2 l_{12}{}^2} = \frac{1}{15}$. Next $\overline{l_{11}{}^2 (l_{12}{}^2 + l_{22}{}^2 + l_{32}{}^2)} = \frac{1}{3}$, so that $\overline{l_{11}{}^2 l_{22}{}^2} = \frac{2}{15}$. If we take this last and $\overline{l_{11}{}^4} = \frac{1}{5}$ (the intermediate relation does not give a separate equation) and substitute in the tensor formula we find

$$T^{\alpha\beta\gamma\delta}_{\phi\chi\psi\omega} = \tfrac{1}{6} T_1 - \tfrac{1}{30} T_2 \qquad \ldots\ldots(4\cdot4).$$

Of course with only four factors it is quite possible to treat the matter in detail—there is only one other type of non-vanishing average $\overline{l_{11}\, l_{12}\, l_{21}\, l_{22}} = -\frac{1}{30}$. But if we required an average of six factors there are a great many types that do not vanish and it is troublesome to be sure that none have been overlooked. We require to work out three special cases here, and the resulting formula is

$$l_{\alpha\phi} l_{\beta\chi} l_{\gamma\psi} l_{\delta\omega} l_{\zeta\tau} l_{\eta\upsilon} = T^{\alpha\beta\gamma\delta\zeta\eta}_{\phi\chi\psi\omega\tau\upsilon} = \tfrac{1}{6} T_1 - \tfrac{1}{30} T_2 + \tfrac{1}{105} T_3 \qquad \ldots(4\cdot5),$$

where

$$T_1 = (\alpha\beta)(\gamma\delta)(\zeta\eta)(\phi\chi)(\psi\omega)(\tau\upsilon) + \ldots,$$

$$T_2 = (\alpha\beta)(\phi\chi)(\gamma\delta\zeta\eta)(\psi\omega\tau\upsilon) + \ldots,$$

$$T_3 = (\alpha\beta\gamma\delta\zeta\eta)(\phi\chi\psi\omega\tau\upsilon).$$

In this $(\alpha\beta\gamma\delta\zeta\eta)$ means the sum of the fifteen expressions of the type $(\alpha\beta)(\gamma\delta)(\zeta\eta)$. It will readily be believed that without some process of the present kind the matter would become impossible. These tensors come in if it is required to evaluate the scattering om an optically active molecule, and I have in fact worked out e solution quite easily, but the result is not of sufficient interest o give here.

We may also note that similar formulae can be found when here is an odd number of factors, but these involve the curl ensor. For example,

$$\overline{l_{\alpha\phi} l_{\beta\chi} l_{\gamma\psi}} = \tfrac{1}{6}[\alpha\beta\gamma][\phi\chi\psi] \qquad \text{......(4·6),}$$

a result which we shall use later.

5. We shall now apply our result to (3·2) and have (not summing for κ)

$$I_\kappa = \frac{Nk^4}{30r^2}\rho_{\phi\chi}\tilde{\rho}_{\psi\omega}E_\alpha{}^0\tilde{E}_\gamma{}^0\{(\beta\kappa)-m_\beta m_\kappa\}\{(\delta\kappa)-m_\delta m_\kappa\}(5T_1-T_2) \qquad \text{......(5·1).}$$

Take the force of the incident wave along x_1. Then we can put $\alpha = 1$, $\gamma = 1$ and $E_\alpha{}^0\tilde{E}_\gamma{}^0 = I_0$ the incident intensity. First observe from the direction x_2, so that $m_1 = m_3 = 0$, $m_2 = 1$. Take the strong component, which is that polarised along x_1. Then $\kappa = 1$ and the terms in m_κ drop out. It follows that $\beta = \delta = 1$. We now have T_1 as $(\phi\chi)(\psi\omega) + (\phi\psi)(\chi\omega) + (\phi\omega)(\chi\psi) = (\phi\chi\psi\omega)$ since all the other factors in the terms are unity. In T_2 we have $(\alpha\beta\gamma\delta) = 3$. Thus

$$\frac{I_1}{I_0} = \frac{Nk^4}{30r^2}\rho_{\phi\chi}\tilde{\rho}_{\psi\omega}\,.\,2(\phi\chi\psi\omega) = \frac{1}{15}\frac{Nk^4}{r^2}\{\rho_{\phi\phi}\tilde{\rho}_{\psi\psi} + \rho_{\phi\chi}\tilde{\rho}_{\phi\chi} + \rho_{\phi\chi}\tilde{\rho}_{\chi\phi}\} \qquad \text{......(5·2).}$$

Next take the weak component, for which $\kappa = 3$. Again the terms in m_κ drop out, but now we have to put $\beta = \delta = 3$. In this case the only factors that do not vanish are $(\alpha\gamma)$, $(\beta\delta)$, and so we have

$$T_1 = (\phi\psi)(\chi\omega), \qquad T_2 = (\phi\chi\psi\omega),$$

and therefore

$$\frac{I_3}{I_0} = \frac{1}{30}\frac{Nk^4}{r^2}\rho_{\phi\chi}\tilde{\rho}_{\psi\omega}\{5(\phi\psi)(\chi\omega) - [(\phi\chi)(\psi\omega) + (\phi\psi)(\chi\omega) + (\phi\omega)(\chi\psi)]\}$$
$$= \frac{1}{30}\frac{Nk^4}{r^2}\{4\rho_{\phi\chi}\tilde{\rho}_{\phi\chi} - \rho_{\phi\phi}\tilde{\rho}_{\psi\psi} - \rho_{\phi\chi}\tilde{\rho}_{\chi\phi}\} \qquad \text{......(5·3).}$$

Lastly, take the observation along the direction x_1 and consider the component along x_3. Then $m_1 = 1$, $m_2 = m_3 = 0$, and again the terms in m_κ drop out. We have to take $\beta = \delta = 3$ and it is obvious that we shall get the same value as (5·3).

If now our gas is illuminated by unpolarised light going along x_3 and observed along x_2 we shall have as the two components $I_1 + I_3$ along x_1 and $2I_3$ along x_3. Thus the ratio of the intensity of the two components will be

$$P = \frac{2I_3}{I_1 + I_3} = 2\,.\,\frac{4\rho_{\phi\chi}\tilde{\rho}_{\phi\chi} - \rho_{\phi\phi}\tilde{\rho}_{\chi\chi} - \rho_{\phi\chi}\tilde{\rho}_{\chi\phi}}{6\rho_{\phi\chi}\tilde{\rho}_{\phi\chi} + \rho_{\phi\phi}\tilde{\rho}_{\chi\chi} + \rho_{\phi\chi}\tilde{\rho}_{\chi\phi}} \qquad \text{...(5·4).}$$

For a substance composed of transparent non-magnetic molecules we have $\rho_{\phi\chi} = \rho_{\chi\phi}$ and the ρ's are all real. We choose principal axes and write $\rho_{11} = A$, $\rho_{22} = B$, $\rho_{33} = C$. Then

$$P = 2 \cdot \frac{[3(A^2 + B^2 + C^2) - (A + B + C)^2]}{7(A^2 + B^2 + C^2) + (A + B + C)^2}$$

$$= 2 \cdot \frac{A^2 + B^2 + C^2 - BC - CA - AB}{4(A^2 + B^2 + C^2) + BC + CA + AB} \quad \ldots(5\cdot5),$$

which is the formula given by Rayleigh. Since $A + B + C$, or $\rho_{\phi\phi}$, can be calculated from the refractive index, an observation of the polarisation enables us to find

$$(B - C)^2 + (C - A)^2 + (A - B)^2.$$

6. We shall now apply these results to the experiments of Lord Rayleigh on the common gases. Among his results are the following:

	H_2	N_2	O_2	CO_2
$100P$	3·83	4·06	9·4	11·7

The most remarkable feature is the great value for oxygen compared with nitrogen. In their ordinary physical properties, boiling point, viscosity, dielectric constant and refractive index (including the position of ultraviolet absorption bands), they are not very dissimilar, and yet we find that the scattering of nitrogen is much more like that of hydrogen, though there many of these properties are widely different. This suggested that the scattering might be associated with the one property in which oxygen differs from the others, its magnetism; we shall show that this cannot be so in the next section, but it will be convenient first to make a more definite calculation of the asymmetry of the various molecules.

In order to do this we suppose the molecule to be like a dielectric spheroid, which we naturally assume to be prolate for a diatomic substance. In a uniform electric field E_1, E_2, E_3 the potential of an ellipsoid at a distant external point is

$$V = E_1 \frac{abc}{3} \frac{1}{L + 1/(K_0 - 1)} \frac{x_1}{r^3} + E_2 \frac{abc}{3} \frac{1}{M + 1/(K_0 - 1)} \frac{x_2}{r^3}$$
$$+ E_3 \frac{abc}{3} \frac{1}{N + 1/(K_0 - 1)} \frac{x_3}{r^3},$$

where L, M, N are the "demagnetising coefficients" given by

$$L = \frac{abc}{2} \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)\sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}}, \text{ etc.}$$

and K_0 is the dielectric constant. In our present case we take

$a > b = c$. From V we can at once set down the Hertzian vector, which leads to

$$A = \frac{ab^2}{3} \frac{1}{L + 1/(K_0 - 1)},$$

$$B = C = \frac{ab^2}{3} \frac{1}{M + 1/(K_0 - 1)}.$$

Since $L + 2M = 1$ we can calculate the ratio $b : a$, if we are given $B : A$ and if we know the value of K_0.

I have worked out two cases. The least possible asymmetry of shape will be that when the molecule is supposed a perfect conductor, and this we obtain by taking $K_0 = \infty$. But a more natural assumption is to take the molecules as having such a dielectric constant as will give rise to the proper value for the gas in bulk, or more accurately, as will give the proper refractive index for the light concerned. There must be a good deal of doubt as to what is the proper volume to assign to the molecule, but in default of better information we take $\frac{1}{4}b$, where b is Van der Waals' constant, as the actual volume of the molecules in a cubic centimetre. If n is the refractive index at normal density we then have, regarding the molecules as spheres for this part of the calculation,

$$\frac{3(K_0 - 1)}{K_0 + 2} = \frac{n^2 - 1}{b/4}.$$

If they are spheroids the relation is more troublesome, but in view of the doubt attaching to Van der Waals' constant, it is not worth while treating the more complicated case. The table shows the results.

	H_2	N_2	O_2	CO_2
$100P$	3·83	4·06	9·40	11·70
B/A	0·60	0·59	0·44	0·40
b/a Cond.	0·65	0·65	0·52	0·48
K_0	3·40	3·90	4·20	6·10
$b/a\ (K_0)$	0·31	0·35	0·15	0·23

Of course no great importance attaches to the exact values of the numbers, but still there can be no doubt that the experiments show a very curious difference in the shapes of the nitrogen and oxygen molecules.

7. In ordinary substances a magnetic field produces its optical effects by directly changing the scattering indices of the molecules. But a paramagnetic molecule has a magnetic field of its own and it seemed possible that this might exert an intrinsic gyratory effect —of course of quite a different kind from that corresponding to natural "optical activity." When there is no field the intrinsic gyration will not show itself at all in the propagation of the direct wave, but it would be expected to show itself in the lateral scattering. In a magnetic field the orientation of the molecules should

lead to an extra term in the gyration. There is independent support for such a suggestion in that though the gyration of oxygen is of the same general magnitude as that of the other gases, yet the dispersion of gyration is quite abnormal and unexplained. We shall show that an intrinsic gyration sufficient to account for the excess of the lateral scattering would lead to a gyration in a magnetic field many hundred times too large.

Let us suppose that the scattering tensor for the oxygen atom is as follows: $\rho_{11} = A$, $\rho_{22} = \rho_{33} = B$, $\rho_{23} = -\rho_{32} = iG$, $\rho_{12} = \ldots = 0$. For a transparent medium G must be real and we can at once apply (5·4) and have

$$P = 2 \cdot \frac{4\,(A^2 + 2B^2 + 2G^2) - (A + 2B)^2 - (A^2 + 2B^2 - 2G^2)}{6\,(A^2 + 2B^2 + 2G^2) + (A + 2B)^2 + (A^2 + 2B^2 - 2G^2)}$$

$$= 6 \cdot \frac{(A - B)^2 + 5G^2}{5\,(A + 2B)^2 + 7\,(A - B)^2 + 15G^2}.$$

We assume that $B : A$ has the same value as for nitrogen, and so can deduce $G/(A + 2B)$; and then G can be determined from the refractive index by (2·4).

Let us suppose that each molecule has a magnetic moment μ along the axis x_1', and that a field of strength H acts along x_3. The classical theory then implies a distribution law $e^{l_{31}\mu H/kT}$. It is probable* that we ought to use some quantum law and so shall take for a weak field a distribution $1 + ql_{31}\mu H/kT$, where $q = 1$ on the classical theory. The magnetic moment per unit volume, which measures κ the susceptibility, is

$$\kappa H = N\overline{l_{31}\mu\,(1 + ql_{31}\mu H/kT)} = \tfrac{1}{3}qHN\mu^2/kT,$$

and with μ taken as 2 Bohr magnetons this is roughly correct.

The optical constants are given by

$$\sigma_{\alpha\beta} = 4\pi N\rho_{\phi\chi}\overline{l_{\alpha\phi}l_{\beta\chi}\,(1 + ql_{31}\mu H/kT)}$$

from which we derive by (2·3), (4·6)

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma = 4\pi N\,(A + 2B)/3,$$

and
$$\sigma_{12} = -\sigma_{21} = i\tau_1 = 4\pi Nq\,(\mu H/kT)\,[\tfrac{1}{6}\rho_{23} - \tfrac{1}{6}\rho_{32}]$$

$$= 4\pi N\,\frac{\kappa H}{N\mu}\,iG\dagger$$

$$= i\,\frac{\kappa H}{N\mu}\,\sigma \cdot \frac{3G}{A + 2B},$$

and to this we may add a quantity $i\tau_2$ to represent the direct change in the optical constants produced by the field.

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* See Sommerfeld, *Atomic Structure* (Eng. transl.), p. 251.

† If q is a function of the l's, this may not be accurate, but its order of magnitude must be right.

Siertsema* has measured the gyration of oxygen and, reduced to standard conditions, his result for blue light is

$$\tau = 3{\cdot}37 \times 10^{-14} \times H.$$

The scattering from nitrogen gives $(A - B)/(A + 2B) = 0{\cdot}188$, and using this we find

$$3G/(A + 2B) = 0{\cdot}297.$$

We also have
$$\sigma = 2\,(n - 1) = 5{\cdot}46 \times 10^{-4},$$
$$\kappa = 1{\cdot}23 \times 10^{-7}.$$

The magnetic moment of a gramme molecule of Bohr magnetons is 5600 E.M.U., and so if we take $\mu = 2$ B.M. we have $N\mu = 0{\cdot}50$.

Hence
$$\tau_1 = 3{\cdot}99 \times 10^{-11} \times H.$$

This is altogether too great, and we conclude that the abnormal scattering from oxygen cannot be explained in this way.

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* L. H. Siertsema, *Arch. Neerl.* vol. II, p. 291 (1899).

The β-Ray Spectrum of the Natural L-Radiation from Radium B.

By D. H. Black, M.Sc., Emmanuel College. (Communicated by Prof. Sir E. Rutherford.)

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It has been observed by various workers that, in addition to particles ejected from the various atomic levels of the atoms concerned, some are ejected by rays which correspond to the natural K X-radiation of those atoms. It has also been shown that some radioactive substances emit β particles of comparatively low energy. Therefore it was thought advisable to make an attempt to see whether this "soft" radiation has a line spectrum; for one would expect to find traces of the natural L-radiation, just as one finds evidence of the K-radiation, provided that the effect is sufficiently intense.

The problem is not such a straightforward one as is the detection of the ordinary β-ray spectra. One of the chief troubles is the fact that a great deal of "general" radiation from the walls of the apparatus is present at these low energies, and also probably from the "continuous" background. All this radiation tends to mask any line spectrum which might be present.

The apparatus used was the same as for the measurement of the β-ray spectra of mesothorium 2 and of thorium products. This makes use of the now well-known focussing method, causing the groups of particles having definite velocities to fall in lines on a photographic plate. From the dimensions of the apparatus and from the distances of the lines from reference spots on the plate the radius of curvature, ρ, of the paths of the β particles can be measured. Knowing the value of the magnetic field, H, the product $H\rho$ can be found, and from this the energies of the electrons in the various groups can be calculated.

Sources of radium B + C, radium C, thorium B + C + D and mesothorium 2 have been tried, but so far evidence of a definite spectrum has only been obtained with radium B + C sources. The reason why no lines were detected from the thorium B + C + D and the mesothorium 2 sources is probably due to the fact that it is extremely difficult to obtain clean sources of these substances; and owing to the comparatively low energy of the β particles concerned they become diffused if any impurities are present on the sources.

Using sources of radium B (with radium C in equilibrium) on a platinum wire, 0·25 mm. in diameter, several lines have been recorded and these are set out in the following table. Separate sources of radium C were obtained but no lines were observed from this substance although the same equivalent exposures were made in the two cases. Therefore the lines are to be attributed to radium B. This is also borne out from β-ray absorption experiments which show that radium B has a much softer radiation than radium C.

Number	Intensity	$H\rho$	Energy (volts)
1	?	221	4,290
2	v.f.	264	6,140
3	f.	272	6,480
4	m.f.	283	7,050
5	m.	294	7,610
6	m.s.	306	8,210
7	v.f.	315	8,720
8	f.	326	9,320
9	f.	335	9,850
10	m.s.	346	10,470
11	f.	350	10,730
12	f.	357	11,160
13	?	366	11,690
14	v.f.	376	12,370
15	v.f.	381	12,670

As far as can be judged at present it appears that the electrons are ejected by L X-rays from an atom whose atomic number is 83, acting on the M, N and O levels of the same atom. The agreement of the observed with the calculated values will be reserved for a future paper, when it is hoped to have some more data dealing with the subject. The object of this paper is merely to record the fact that these lines have been observed.

In conclusion I wish to thank Sir Ernest Rutherford for suggesting this problem and for his great interest throughout the experiments. I am also indebted to Dr J. Chadwick for preparing the thorium B and mesothorium 2 sources and to Mr G. R. Crowe for the radium B and radium C sources.

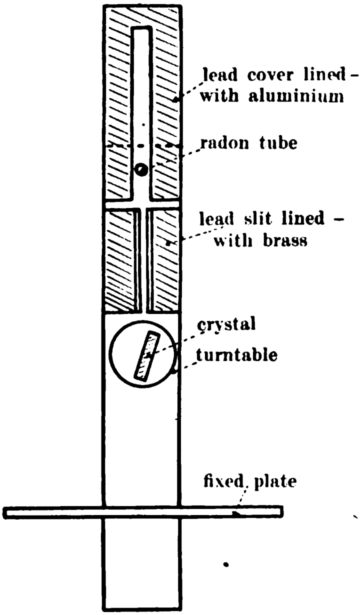
The Natural X-ray Spectrum of Radium B. By Professor Sir ERNEST RUTHERFORD, F.R.S., and W. A. WOOSTER, B.A., Peterhouse.

[*Received* 26 August, *read* 26 October 1925.]

It is now well known that radium B is an isotope of lead of atomic number 82 with a mass 214, and consequently, if the atoms of radium B are bombarded by an external source of electrons, the spectrum excited in it should be identical with that of lead atomic number 82. A very interesting question arises with regard to the *L* radiation emitted by a source of radium B during its spontaneous transformation. At the moment of the expulsion of the disintegration electron from radium B, the internal atomic structure of radium B corresponds to an element of number 82, but an instant later, when the electron has escaped from the nucleus, the charge on the latter is 83 and there must follow a reorganisation of the external electrons. Under these conditions, we cannot be certain whether the *L* spectrum of radium B should correspond to an element of number 82 or 83. Since the excitation of the *L* spectrum is for the most part due to the action of the rays from the nucleus, the spectrum should correspond to number 82 if the emission of the γ-ray *precedes* the escape of the disintegration electron and number 83 if it is *subsequent* to this process.

In 1914, Rutherford and Andrade determined the γ-ray spectrum of radium B and radium C by the photographic method, using the same crystal of rock-salt as was used by Moseley for his original experiments. The soft γ-radiation gave a well-marked spectrum containing two strong lines for reflection angles about 10° and 12°. By extrapolation from Moseley's law which had just been published, it was concluded that this natural *L* spectrum must be accorded to an atomic number 82. This conclusion was strengthened when Siegbahn determined directly the X-ray spectrum of ordinary lead and found values for the strong lines in close accord with those found in the γ-ray spectrum. The results thus appeared to support the view that the natural *L* spectrum of radium B was identical with that of lead. In the interval, there has been an intensive study of the β-ray spectrum of radium B and its interpretation in terms of homogeneous groups of γ-rays emitted by the nucleus, and the question whether the number 82 or 83 governs the spectrum of radium B during its disintegration has assumed considerable importance. With increase of accuracy of the measurements, it is possible to decide from a study of the excited spectra whether the number 82 or 83 is involved, nd from the investigation described below, Ellis and Wooster onclude that the evidence favours the number 83. Even more efinite evidence has been obtained by Black in some work on he low velocity β-ray spectrum accompanying the emission of he *L* radiation from radium B. He found that the energy of the roups of β-rays could be accounted for by assuming a number 83, ut was seriously in error for a number 82.

These observations pointed to the need for a new determination of the natural spectrum of radium B to settle with certainty whether it is to be ascribed to an element 82 or 83. For this purpose, we used a method very similar to that employed by Rutherford and Andrade, using at first a good crystal of rock-salt and a radon tube as a source. The general experimental arrangement can be seen from the diagram. The slit was 3 mm. wide, giving a beam of considerable divergence (1½-3°). This was used with a rotating crystal in the usual manner, the distances of the source and the lines on the plate from the axis of rotation being made the same. In order to prevent the photographic plate from being fogged by β-rays, the source, lead slit and crystal were placed between the poles of a large electromagnet giving a field of 2500 gauss. Two distances of the source from the crystal were used, about 8 and 11 cm. respectively. The plate was kept fixed and the spectrum was obtained on both sides of the main beam, the angle of reflection being calculated from the separation of the two lines without reference to the central beam. The lines measured were the two strongest lines α_1 and β_1* of the *L* spectrum, since a knowledge of these wave-lengths was sufficient for our purpose. With a radon tube containing 100 millicuries it was necessary to expose on each side for about 12 hours. The results of the photographs using rock-salt are shown in Table I.


lead cover lined - with aluminium
radon tube
lead slit lined - with brass
crystal
turntable
fixed plate

Column I shows the distance of the photographic plate from the axis of rotation of the crystal, and the second gives the experimentally determined separation of corresponding lines due to the X-rays from the radon tube. In the third and fourth columns, the theoretical separation, under the same experimental conditions, of the lines due to atoms of atomic numbers 83 and 82 respectively are deduced. For calculating these figures we used the data of D. Coster*. It can be seen that in every case the separation of the two lines agrees well with that calculated for 83. To confirm the accuracy of the measurements and to substantiate that the natural *L* spectrum of radium B is distinct from that of ordinary lead bombarded by electrons, we photographed the latter with exactly the same experimental arrangement, a slit of the same width being substituted for the radon tube. The X-rays were generated at an anticathode coated with solder, and care was taken to insure that they passed centrally down the slit system.

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* By β_1 we denote the close doublet $\beta_1\beta_2$ which was not resolved in our experiments.

Table I.

Col. I	Col. II		Col. III		Col. IV		Col. V	
Distance of crystal from plate	Distances between corresponding lines on the photographic plate							
	Measured for radon tube		Deduced for atomic numbers				Measured for lead X-rays	
			83		82			
	α_1	β_1	α_1	β_1	α_1	β_1	α_1	β_1
cm.	cm.	cm.	cm.	cm.	cm.	cm.	cm.	cm.
8·00	—	5·67	6·92	5·65	7·14	5·84	7·15	5·85
	6·93	—	"	"	"	"	7·12	5·82
	—	5·61	"	"	—	—	—	—
	6·87	5·64	"	"	—	—	—	—
11·00	—	7·75	9·52	7·77	9·81	8·03	9·80	8·01
	—	7·73	"	"	"	"	9·80	8·01
	9·50	—	"	"	"	"	9·80	8·02

The results are shown in column V of the above table, and it will be seen that the results are in close agreement with the calculated values in the fourth column and quite distinct from the values obtained with the radon tubes. These experiments provide strong evidence that the natural *L* spectrum from a radon tube corresponds with that from an atom of atomic number 83.

Although this evidence was quite satisfactory we thought it necessary, in view of the importance of the experiment, to confirm this result by using a calcite crystal although this involved exposures three times as long as in the case of rock-salt. We are indebted to Prof. W. L. Bragg for kindly selecting and testing a calcite crystal. The results are given in Table II, which is arranged in the same way as Table I. The agreement in this case, between the actual separation of the lines and the value computed for an atomic number 83, is seen to be better than in the first case, as would be expected in view of the greater regularity of calcite crystals.

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* *Zeit. f. Phys.* 6, p. 188 (1921).

Table II.

Distance of crystal from plate	Distances between corresponding lines on the photographic plate							
	Measured for radon tube		Deduced for atomic numbers				Measured for lead X-rays	
			83		82			
	α_1	β_1	α_1	β_1	α_1	β_1	α_1	β_1
cm.	cm.	cm.	cm.	cm.	cm.	cm.	cm.	cm.
8·00	6·37	—	6·37	—	6·57	—	6·58	5·40
	—	—	—	—	—	5·39	6·57	5·39
	—	—	—	—	—	—	6·57	5·39
11·00	8·77	—	8·77	—	9·03	7·41	9·02	7·42
	—	—	—	—	—	—	9·03	7·41

A molybdenum photograph was also taken, showing the same good agreement.

It is certain that the original measurements on the natural *L* spectrum were in some way in error, at any rate for these strong lines, but the experiments were carried out so long ago, in the early days of X-ray spectroscopy, that it is difficult to trace the error to its source. The full examination of the spectrum involves a large amount of work and will be undertaken later.

Our result that the atomic number is 83 is in accord with other experimental evidence. The bearing of this work on the general picture of the β-ray disintegration is considered in the following paper by Ellis and Wooster.

We would like to thank Mr B. W. Robinson for great help in the latter part of the experiments.

The Analysis of the β-Ray Spectrum due to the Natural L Radiation of Radium B. By D. H. Black, M.Sc., Emmanuel College. (Communicated by Prof. Sir E. Rutherford.)

[*Received* 26 August, *read* 26 October 1925.]

In a previous paper the values for some lines of the β-ray spectrum of the natural L radiation of radium B have been given. The purpose of the present paper is to show that many of these lines can be accounted for by assuming that rays corresponding to L X-rays, but generated within the atom, act on various atomic levels. Before proceeding to do so, however, it will be advisable to give a fuller account of the experimental methods.

As was mentioned in the previous paper, the apparatus used was essentially the same as had been used to obtain the β-ray spectrum of mesothorium 2* and of some thorium disintegration products†. In the present case special arrangements had to be made to measure the strength of the magnetic field, H, as the lines observed lie well below any of the standard lines from radium B as measured by Ellis and Skinner‡. For this purpose a coil was wound on an ebonite former and attached to a shaft which could be rotated through 180°. This coil was connected in series with a reflecting fluxmeter and the secondary of a mutual inductance. On the rotating shaft there was also fixed a cam which could be adjusted to break the current in the primary circuit of the inductance at any convenient instant during the rotation of the coil. The current in the primary was in such a direction that the current induced in the secondary on breaking the current was opposite in direction to that produced by the rotation of the bobbin in the magnetic field. To measure the value of H the rotating coil was introduced into the magnetic field and the current in the primary of the inductance adjusted until the fluxmeter showed no deflection when the coil was rotated, the value of the current being measured by a carefully calibrated ammeter.

No particular care was taken in winding the search coil, nor was the absolute value of the mutual inductance of any consideration, for the apparatus was calibrated in the following way: An exposure was made with a radium B source with a value of the field which caused several of the β-ray lines as measured by Ellis and Skinner § to fall on the plate. The current which had to be broken in the primary of the inductance in order to balance the throw of the coil was measured. On measuring up the known lines an accurate value of H was obtained and thus the ratio of the current in the primary to the strength of the field was obtained. This ratio was remeasured from time to time in order to keep check on any possible variations in the apparatus. Knowing this ratio, the value of any magnetic field could be calculated from the value of the current broken in the primary in order to obtain a balance.

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* *Roy. Soc. Proc.* A, vol. 106, p. 632 (1924).

† *Ibid.* A, vol. 109, p. 166 (1925).

‡ *Ibid.* A, vol. 105, p. 60 (1924).

§ *Loc. cit.*

Great difficulty was experienced in finding a photographic plate with sufficient contrast for detecting these comparatively slow β-rays. Ordinary X-ray and process plates were useless as the blackening from the general radiation completely masked any lines that were present. Half-tone plates were then tried and several lines were recorded. Following the experience of Dr Aston in his positive ray work some Schumann plates manufactured by Hilgers were tried. These gave by far the best results and have therefore been used for the greater part of this work.

Some of the values of the lines given in the previous paper have been slightly altered and a corrected table is set out below. All the lines except those marked (?) have been found at least twice and the values agree to 1 part in 250. Owing to the variations in the density of the Schumann plates through pressure and developing marks the intensities are only very approximate.

Table I.

No.	Intens.	$H\rho$	V (volts × 10^{-3})
1	?	220·6	4·29
2	v.f.	264·3	6·14
3	f.	272·2	6·48
4	m.f.	283·4	7·06
5	m.	294·5	7·62
6	m.s.	305·6	8·20
7	v.f.	315·4	8·72
8	f.	326·0	9·32
9	f.	335·3	9·85
10	m.s.	345·4	10·44
11	f.	349·7	10·71
12	f.	356·8	11·14
13	?	365·6	11·69
14	v.f.	376·2	12·37
15	v.f.	380·8	12·66

Table II sets out the energies of the absorption levels for atomic numbers 82 and 83. The levels N_{VI}, N_{VII} and O are bracketed together as it would be impossible to distinguish between them. In Table III are given the energies and intensities of the L X-rays of lead (82) and bismuth (83).

Assuming that the β-ray lines are due to electrons being ejected from M, N, and O levels by L X-rays, Table IV gives the comparison of the observed with the calculated values. The energy of the ejected electron will be the energy of the X-ray minus the energy of the level from which the electron is ejected. The numbers in brackets in column 2 are the relative intensities of ejection from the various levels as found by Robinson* when dealing with X-rays.

Table II. *Absorption Energies* (*in volts* × 10^{-3}).

Level	At. no. 82	At. no. 83
M_{I}	3·84	4·00
M_{II}	3·55	3·70
M_{III}	3·06	3·17
M_{IV}	2·58	2·70
M_{V}	2·48	2·59
N_{I}	0·89	0·96
N_{II}	0·75	0·79
N_{III}	0·67	0·68
N_{IV}	0·44	0·48
N_{V}	0·41	0·46
N_{VI} (N_{VII}, O)	0·15 (0·14, 0·14)	0·18 (0·18, 0·15)
M_V (Platinum) 2·12		
N_V (,,) 0·31		

Table III. *Energies of X-rays* (*in volts* × 10^{-3}).

(*Siegbahn's classification*)

Ray	Intens.	At. no. 82	At. no. 83
$L\alpha_1$	10	10·52	10·81
$L\beta_1$	8	12·58	12·79
$L\beta_2$	6	12·58	12·93
$L\gamma_1$	4	14·72	15·21

The difference of wave-length between the $L\beta_1$ and $L\beta_2$ X-rays is so small that it is practically impossible to distinguish between them by this method and therefore $L\beta_2$ is not included in the table. It will be noticed that two lines are included as being due to the $L\alpha_1$ ray acting on the M_V and N_V levels of platinum. Since platinum was the metal on which the sources were deposited this might reasonably be expected.

The table shows very definitely that the agreement for atomic number 83 is far better than for atomic number 82. Taking the atomic number as 82 it is only possible to account for six of the observed lines, none of which is at all strong. If, however, 83 is taken as the correct atomic number, it is possible to account or eleven out of the fifteen lines, including all the stronger ones nd not including either of the two doubtful lines. Moreover the ntensities of the observed lines agree, on the whole, with the ntensities found by Robinson* in his X-ray measurements.

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* *Proc. Roy. Soc.* A, vol. 104, p. 464 (1923).

Table IV. *Energies in volts* × 10^{-3}.

Exciting ray	Level of origin	Atomic number 82		Atomic number 83	
		Energy calc.	Energy obs.	Energy calc.	Energy obs.
$L\alpha_1$	M_{V} (6)	8·04	—	8·22	8·20 (m.s.)
	M_{IV} (6)	7·94	—	8·11	
	M_{III} (5)	7·46	—	7·64	7·62 (m.)
	M_{II} (2)	6·97	7·06 (m.f.)	7·11	7·06 (m.f.)
	M_{I} (1)	6·68	—	6·81	—
	N_{V} (5)	10·11	—	10·35	*10·44 (m.s.)
	N_{IV} (4)	10·08	—	10·33	
	N_{III} (4)	9·85	9·85 (f.)	10·13	—
	N_{II} (3)	9·77	—	10·02	—
	N_{I} (2)	9·63	—	9·85	*9·85 (f.)
	O (3–4)	10·38	—	10·66	10·71 (f.)
	M_{V} (Pt)	8·40	—	8·69	8·72 (v.f.)
	N_{V} (Pt)	10·21	—	10·50	*10·44 (m.s.)
$L\beta_1$	M_{V} (6)	10·10	—	10·40	*10·44 (m.s.)
	M_{IV} (6)	10·00	—	10·29	
	M_{III} (5)	7·52	—	9·82	*9·85 (f.)
	M_{II} (2)	7·03	—	9·29	9·32 (f.)
	M_{I} (1)	8·74	8·72 (v.f.)	8·99	—
	N_{V} (5)	12·17	—	12·53	*12·66 (v.f.)
	N_{IV} (4)	12·14	—	12·51	
	N_{III} (4)	11·91	—	12·31	12·37 (v.f.)
	N_{II} (3)	11·83	—	12·20	—
	N_{I} (2)	11·69	11·69 (?)	12·03	—
	O (3–4)	12·44	12·37 (v.f.)	12·64	*12·66 (v.f.)
$L\gamma_1$	M_{V} (6)	12·24	—	12·62	*12·66 (v.f.)
	M_{IV} (6)	12·14	—	12·51	
	M_{III} (5)	11·66	11·69 (?)	12·04	—
	M_{II} (2)	11·17	11·14 (f.)	11·51	—
	M_{I} (1)	10·88	—	11·21	11·14 (f.)

Lines marked * occur more than once.

It will be noticed that more than one possible origin has been indicated for three of the lines; and that some of the theoretical values are bracketed, the difference being too small to show as two distinct lines on the plate. An examination of the plates in the ordinary way shows that some of the lines are wider than others, and it was thought that they might be combinations of lines as mentioned above. If this were true one would expect the line whose energy is observed to be $10 \cdot 44 \times 10^3$ volts to be made up of at least five lines as shown.

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* *Loc. cit.*

In order to test this theory an attempt was made to analyse this particular line by means of a photo-micrometer. Unfortunately the Schumann plates used were very uneven in density and it was impossible to obtain definite readings with the necessary resolving power. However, three different sets of readings did show evidence of five different lines, but the positions of the peaks denoting these lines were not sufficiently well defined to permit of accurate measurements between them.

As was mentioned in the previous paper, the lines cannot have been due to radium C, as on testing with a source of radium C alone no lines were observed. The atomic number of radium B is 82, but the lines are almost certainly due to L X-rays from atoms whose atomic number is 83 acting on atomic levels of the same atomic number. Therefore this gives very strong support to the theory that the γ-rays of an atom act on the various atomic levels of that atom *after the nuclear change has taken place*; thus bearing out the slight evidence put forward in the two papers on the β-ray spectra of mesothorium 2 and of some thorium disintegration products. This means that the γ-ray cannot precede the nuclear β-ray during the disintegration and would suggest that the γ-ray is due to the ejection of the β-particle.

Confirmation of these results has now been obtained by other workers on the subject using different methods, and the correctness of this conclusion seems assured.

Thorium B is an isotope of radium B and also undergoes a β-ray change. If also in this case the γ-rays are emitted after the disintegration one would expect to obtain a corpuscular L spectrum from this substance exactly the same as from radium B. As mentioned in the previous paper, attempts to obtain this spectrum failed owing to the sources being dirty. Recently however a clean source was obtained by exposing a platinum wire to thorium emanation. Unfortunately it was not possible to make more than one exposure and so the results cannot be relied upon to any great extent. Owing to the fact that the general radiation from thorium B seems to be much greater than that from radium B it was extremely difficult to measure some of the lines. The values are set out in the following table, together with the values from radium B for comparison.

It will be seen that the agreement is good on the whole, although there appears to be one definite exception in the case of one of the stronger lines. Why this should be so is not evident; but lacking further observations one cannot stress this point.

Table V. *Lines from thorium B.*

Thorium B		Radium B	
$H\rho$	Intens.	$H\rho$	Intens.
270·8	m.	271·6	f.
283·1	m.	283·4	m.f.
292·5	?	294·5	m.
303·4	m.s.	305·9	m.s.
310·1	?	315·4	v.f.
334·0	m.	335·2	f.
344·4	m.	345·8	m.s.
375·8	f.	376·2	v.f.
379·5	f.	380·8	v.f.

Seeing that the hypothesis that conversion of the γ-rays occurs after the nuclear change has taken place has now been shown to be probably correct, a list of the γ-rays from such radioactive substances as I have previously worked with is given below.

Table VI. *Energies of γ-rays in volts* $\times 10^{-5}$.

No.	Ms. Th_2	Th. B	Th. C+D	Ra. D
1	0·578	0·747	0·408	0·472
2	1·291	2·41	1·446	—
3	1·838	3·02	2·108	—
4	3·18	—	2·33	—
5	3·38	—	2·53	—
6	4·62	—	2·59	—
7	9·14	—	2·79	—
8	9·69	—	2·92	—
9	—	—	5·18	—
10	—	—	6·58	—
11	—	—	26·49	—

Summary. The β-ray spectrum due to the natural L radiation of radium B has been measured. The results give strong support to the hypothesis that electrons are ejected by γ-rays after the nuclear change in the atom has taken place. The corresponding corpuscular spectrum of thorium B is practically the same as that of its isotope radium B, suggesting that in this case also the γ-rays are emitted after the disintegration.

My thanks are due to Sir Ernest Rutherford for his interest and encouragement throughout this work; to Dr J. Chadwick for the trouble to which he went to obtain a clean thorium B source; and to Mr G. R. Crowe for preparing the radium B sources.

The Atomic Number of a Radioactive Element at the moment of emission of the γ-rays. By Dr C. D. Ellis, Trinity College, and W. A. Wooster, B.A., Peterhouse.

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It is well known that the β-ray type of disintegration is usually accompanied by the emission of γ-rays, and it is a matter of great importance to decide which of these two phenomena occurs first, that is, whether the γ-rays are emitted before the nucleus disintegrates or afterwards. It is not practicable to attempt a direct solution of this problem, but since the emission of the disintegration electron results in a change of the nuclear charge of + 1 it would be sufficient to determine the atomic number of the radioactive body at the moment of emission of the γ-rays. For instance, an atom of the β-ray body radium B has an atomic number 82, but after disintegration when it becomes radium C it has an atomic number 83, so that if the γ-rays were known to be emitted from a body of atomic number 82 it would mean that they were emitted *before* the disintegration, whereas if the atomic number 83 were found it would show that they were emitted *afterwards*.

This information can be obtained from the β-ray spectrum of the body, because it is well known that this corpuscular spectrum is caused by the internal conversion of the γ-rays in the same atom that emits them. The energies of the β-ray groups due to a γ-ray frequency ν will thus be

$$E_1 = h\nu - K_{\text{abs}}, \quad E_2 = h\nu - L_{\text{abs}} \ldots \text{etc.}$$

If we knew the frequency of the γ-ray we could, from measurements of the corpuscular spectrum, determine the K absorption energy. This would give at once the atomic number of the atom in which the γ-ray was absorbed, and this will also be the atomic number at the moment of emission of the γ-rays since the conversion is internal.

This is in effect the measurement we have carried out in the case of radium B and radium C, although the method consisted in the comparison of the already mentioned natural β-ray spectrum with that excited in platinum. The energies of these latter groups will be

$$E_{Pt} = h\nu - K_{Pt} \ldots \text{etc.}$$

so that the difference of corresponding excited and natural groups is

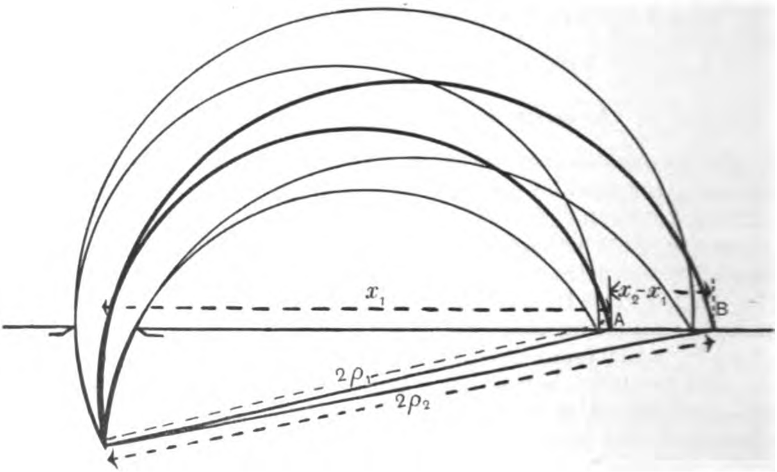
$$E_{Pt} - E_{\beta} = K_{\beta} - K_{Pt}.$$

Everything in this equation is known except K_{β}, the absorption energy of the radioactive atom, and so this quantity can be determined.

The method of measuring corpuscular spectra is well known. The beam of ejected electrons, limited by a slit, is bent round by a uniform magnetic field and the electrons of different velocities are spread out into a spectrum and recorded on a photographic plate. It might be thought that it would be sufficient to measure first the natural spectrum and then the excited spectrum, that is, to determine E_{Pt} and E_{β} separately. Such measurements have frequently been carried out in the past but never with sufficient accuracy. To determine the quantity K_{β} to one per cent., which is certainly required for this purpose, it would be necessary to measure the energies of the groups to at least 1 part in 300. This would be very laborious owing to the difficulty of measuring the magnetic field, but we avoided this by the simple expedient of taking both photographs simultaneously on the same photographic plate. It was an important point that the effective source of the excited and of the natural electrons was identical, and it will be seen that the separation of the two lines on the photographic plate gave a direct measure of the atomic number. The lines would be about three millimetres apart if the γ-ray was emitted before the disintegration and nearly four millimetres if it was emitted afterwards. The exact value of the magnetic field now becomes of small importance: if it is known to one or two per cent. it is sufficient, and the whole accuracy of the experiment depends on the accuracy with which the sources of the two types of electrons can be considered identical.

Experimental Arrangement and Results.

The apparatus was the same as is ordinarily employed in photographing corpuscular spectra, the radius of curvature of the paths varying between 4, 5 and 7·5 cms. The source was about 1·5 cms. below the slit which had a width 0·5 cm. The method of obtaining the excited spectra was also the same as has been used before; a thin-walled glass tube of external diameter 0·4 mm. was filled with radium emanation and arranged to fit closely inside a platinum tube. This platinum tube had an external diameter of 0·97 mm. and was made by boring out a piece of rod. The walls of this tube had a thickness of 0·25 mm. which is sufficient to absorb almost completely the primary β-ray emission from the radium B and C in the emanation tube. The γ-rays from these bodies, however, penetrate the platinum, being partially absorbed and giving rise to the photo electrons which constitute the excited spectrum. These electrons are liberated throughout the body of the metal, but it is only those ejected from the surface layer which contribute to the heads of the lines to which measurements are taken. Those electrons which come from deeper layers are retarded in their passage out and produce only a general shading off of the excited line on the low velocity side. The extremely thin layer which is responsible for the edge of the line may be seen from the following considerations. An arbitrary distance of 0·1 mm. inside the edge of the line corresponds to a retardation of about 1·4 $H\rho$, and a thickness of 0·0013 mm. of platinum is sufficient to produce this change. It was possible on the best photographs to locate the edge of the excited line to less than one-tenth of a millimetre, so it is clear that the measurements of the excited spectra are concerned only with the electrons liberated from the outer skin of platinum of thickness 0·001 mm. Before carrying out an experiment the platinum sheath was exposed to a small quantity of radium emanation and a weak source of radium B and C deposited on its surface. This platinum tube, when slipped over the emanation tube, now becomes the source of two corpuscular spectra, the natural spectrum due to the active material deposited on the outside, and the excited spectrum liberated by the γ-rays. The important point here is that all these electrons can be considered to come from the same source, that is the surface of the platinum, and in comparing different lines no correction is necessary for the size of the source, in fact it is not necessary even to know the position of the source very accurately. This may be seen from the following figure. Suppose



A and *B* are the positions of the edges of the natural and excited lines respectively, then if *H* is the value of the magnetic field the quantities $H\rho_1$ and $H\rho_2$ determine the energies of the two groups of electrons and the difference $H\rho_2 - H\rho_1$ is to be measured and compared with that calculated on the two possible assumptions about the atomic number of the radioactive element when it emits the γ-rays.

Now the calculated value of $H\rho_2 - H\rho_1$ comes to be either about 45 or 55, so it is sufficiently accurate to use a formula accurate to 1 per cent. and write

$$H\rho_1 - H\rho_2 = \frac{H}{2}(x_2 - x_1).$$

The magnetic field was taken from a calibration curve and the only quantity that had to be measured was the distance *AB* of the two lines on the plate.

The results for the three main γ-rays of radium B and the strongest of radium C are shown in the table. The separations of the two lines were measured by a photometer as the visual method was found to be unreliable owing to the different characters of the lines.

Table I

Details of γ-ray tested	$\delta H\rho$ obs.	$\delta H\rho$ calculated if emission	
		precedes disintegration	follows disintegration
γ-ray E 3 of Ra B giving β-ray line $H\rho$ 1410	60 61	49	62
γ-ray E 6 of Ra B giving β-ray line $H\rho$ 1677	56 56 58	44	57
γ-ray E 9 of Ra B giving β-ray line $H\rho$ 1938	52 48	42	53
γ-ray E 15 of Ra C giving β-ray line $H\rho$ 2980	57	46	56

It can be seen that the results indicate very clearly that the γ-ray is emitted after the disintegration, and the definiteness of this conclusion may be emphasised once again by pointing out that a decision between the two atomic numbers depended on finding whether the distance between the two lines was three or four millimetres, and there is no possible ambiguity in the interpretation. It should be noted that there is a small correction which might reasonably be made to take into account the fact that the electrons liberated from the platinum will be ejected in the forward direction. This correction is very small and has not been included in the results, but it would tend to give a still closer agreement with the greater of the two atomic numbers.

We have only one value for a single γ-ray of radium C, but the agreement is good. It would be laborious to extend this work to the higher energy γ-rays of radium C, and the results already obtained seem sufficient to justify the general conclusion that the γ-rays are emitted after the disintegration.

This result agrees with three other lines of evidence, and a full discussion is given in a following paper, but we wish to emphasise here two special features of this measurement. The first is its directness. The atomic number is found by a phenomenon which occurs immediately after the γ-ray has left the nucleus. The second point is that the atomic number is found for each γ-ray separately. In this method we were not forced to deal with the γ-rays collectively as in the experiments of Rutherford and Wooster and again of Black, but were enabled to investigate each γ-ray in turn.

The β-ray Type of Disintegration. By Dr C. D. Ellis, Trinity College, and W. A. Wooster, B.A., Peterhouse.

[*Received* 9 September, *read* 26 October 1925.]

Introduction. In three other papers in this number of the *Proceedings* will be found direct experimental evidence that in a β-ray disintegration the γ-rays are emitted after the electron has been ejected from the nucleus. In this paper we discuss this result in more detail and especially its bearing on the general picture of the β-ray disintegration.

We divide this paper into three sections, of which the first is concerned with the question of the moment of emission of the γ-rays and the second and third with the question of the mechanism of emission of the γ-rays and of the disintegration electron respectively.

It is justifiable to separate the second from the third section because it is now generally accepted that in all disintegrations there are not more than two fundamental phenomena, the emission of the disintegration particle (α or β) and the emission of γ-rays. Although to a certain extent the emission of the disintegration particle may be regarded as the cause of the emission of the γ-rays, yet it will be seen that it is possible to keep separate the particular problems, which are capable of discussion at the present time. This point of view depends largely on the fact that there are several radioactive substances, both α and β type, which disintegrate without the emission of γ-rays. The emission of γ-rays cannot be regarded as a necessary feature of every disintegration, rather it is a possible addition which is usually present in the β-ray type, and rarely present in the α-ray type.

The γ-rays that are emitted are monochromatic and form a nuclear line spectrum, characteristic of the nucleus in the same way that the optical and X-ray spectra are characteristic of the electronic system. Evidence of combinations between the γ-rays has been found, and this has been taken to show the existence of quantum states inside the nucleus, in fact it has been possible to suggest nuclear systems of levels for several β-ray bodies. There has however always been one great difficulty which has prevented the further discussion of these results, which was to know whether the γ-rays were emitted before or after the disintegration, that is, whether the level system referred to the nucleus before the disintegration or to the new product formed afterwards. This question has now been settled by experiment and is discussed in the next section.

The moment of emission of the γ-rays. As early as 1913 Rutherford viewed the γ-rays as being excited by the ejection of the disintegration electron and therefore considered the emission to take place after the disintegration. Meitner* has stressed the importance of this problem and, adopting on general grounds the same point of view, has used it in interpreting her well known experiments on β-ray spectra. She has made this hypothesis more precise than was possible earlier and has suggested that the γ-rays are emitted during the reorganisation to a fresh quantised state after the disturbance caused by the departure of the disintegration particle. Smekal† has published several theoretical papers treating this problem from a general quantum standpoint, and has concluded that the γ-rays might be emitted simultaneously with the disintegration or afterwards but not before.

The conclusions of these authors were based on general principles which might be summed up by saying that it was easier to understand the γ-rays being emitted after than before the disintegration; but it was not possible at that time to justify this decision by experiment. In fact, of the two experiments which bore on this question one suggested emission before and the other emission after the disintegration. The latter conclusion was reached from a study of the natural β-ray spectrum. This consists of several groups of electrons of definite velocities which owe their origin to the γ-rays. The γ-rays emitted from the nucleus are sometimes absorbed in the atom which has emitted them, and corresponding to each frequency there will result several groups of electrons according to the particular level in which the γ-ray is converted. Thus the emission of one frequency ν will give several groups of energies

$$h\nu - K_{\text{abs}}, \quad h\nu - L_{\text{I}_{\text{abs}}}, \quad h\nu - L_{\text{II}_{\text{abs}}}, \ldots \text{ etc.}$$

It has been possible to identify these groups in the β-ray spectra and it may easily be seen how this provides evidence on the problem we are considering. For instance, in the case of radium B (atomic number 82) there is one soft γ-ray whose energy lies midway between the K and L absorption energies. Photographs of the corpuscular spectrum due to this γ-ray show well defined groups due to conversion in the L_{I}, L_{II}, L_{III}, M, N and O levels. The fact that all the three L groups are found removes any ambiguity in deciding the origin of the groups, and we can be sure that the difference of the energies of the softest and hardest group gives the value of $L_{\text{I}} - O$ for the atom at the moment of conversion of the γ-ray. This value agrees well with the X-ray data for a body of atomic number 83, but is distinctly too great for a body of atomic number 82‡. Since radium B has an atomic number

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* Meitner, *Naturwissenschaft*, Bd. III, p. 160, 1924.
† Smekal, *Zeit. f. Physik*, Bd. 28, p. 142, 1924.
‡ Ellis and Skinner, *Proc. Roy. Soc.* A, vol. CV, p. 177.

82 before disintegration and 83 afterwards this result shows clearly that *this particular γ-ray* is emitted after the disintegration. Unfortunately it is not possible to apply the same criterion to decide the moment of emission of the other γ-rays of radium B or of those of radium C. Here one can identify groups due to conversion in the K and L levels, but only one L level can be found and there is no way of deciding which L level should be taken. This renders the evidence ambiguous since the difference $K - L_{\text{III}}$ for atomic number 82 is almost the same as $K - L_{\text{I}}$ or $K - L_{\text{II}}$ for atomic number 83*.

However, if there had been no more evidence available it would have been simplest to extend to *all* the γ-rays the result found for this one soft γ-ray, but there was an important experiment of Rutherford and Andrade which led to the opposite conclusion for the remaining γ-rays. They found that radium B during its disintegration emitted an L X-ray spectrum, which, both in wave-length and intensity, agreed exactly with later measurements of the L X-ray spectrum of lead (atomic number 82). These experiments were admittedly carried out in the very early days of X-ray spectroscopy, but the agreement was such that it was very difficult to believe it to be fortuitous. There was no doubt that this L spectrum was due to the ionisation of the atom by the main γ-rays already mentioned and therefore these experiments showed that they were emitted before the nucleus changed its atomic number from 82 to 83 by the emission of the disintegration electron. There seemed no escape from these experimental facts, and Ellis and Skinner†, in spite of the difficulties involved, accepted the direct interpretation and concluded that while the main γ-rays were emitted before the disintegration the one soft γ-ray arose afterwards.

New measurements by Black‡ of the β-ray spectra of several radioactive bodies showed the need for a fuller investigation of this point, and the problem has been attacked by three different methods which are described in papers in this issue. All three methods agree in showing that the γ-rays are always emitted after the disintegration, and there is no doubt that the original experiments of Rutherford and Andrade were in some way in error. This represents so important an advance that before considering the significance of this result we will examine the evidence in some detail.

We have already indicated that in the one case where β-ray spectra can lead to any result they show emission after the disintegration. The experiments of Rutherford and Wooster, and of Black, may be considered together since by different methods they both showed that radium B (atomic number 82) emits an *L* X-ray spectrum corresponding to atomic number 83. This *L* spectrum arises in a somewhat indirect manner and its origin may be traced back as follows to the emission of the main γ-rays. These γ-rays are sometimes converted in the *K* level, and the result of this ionisation is the emission of the *K* X-ray spectrum. The strongest lines of this spectrum are due to transitions from the L_{III} and L_{II} levels to the vacant *K* level. The reorganisation of the atom now ionised in the *L* levels gives rise to the *L* X-ray spectrum which has been measured by the above authors. There is a certain amount of direct conversion of these γ-rays with the *L* level which will add its effect to the above process. It can be seen that the emission of the *L* spectrum is mainly a tertiary phenomenon and the atomic number is only found after several events have occurred since the original emission of the γ-ray. Of course it is far simpler to consider that the primary event also occurs after the atomic number has changed, but we think it necessary to emphasise the assumption that is made. It is just possible that the γ-ray might be emitted and ionise the *K* ring before the electron left the nucleus but that the subsequent emission of the X-rays occurs after the new electronic system has been established. Any process of this nature seems very improbable on general grounds, but it is very satisfactory that we do not have to depend on general arguments but can produce experimental evidence to show that this does not occur. The experiments of Ellis and Wooster have to do with this same problem and are described in a paper in this issue. The method of finding the atomic number was somewhat different in this case, but the point we would emphasise is that the experiments gave more direct information. The phenomenon which was utilised to determine the atomic number was the *initial* conversion of the γ-rays in the *K* ring. This is the first event which follows the emission from the nucleus, and in this case we again find the same result of emission after the disintegration.

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* Ellis and Skinner, *loc. cit.* p. 173.

† Ellis and Skinner, *Proc. Roy. Soc.* A, vol. cv, p. 191, 1924.

‡ Black, *Proc. Roy. Soc.* A, vol. cv, p. 633, 1924.

We may sum up by saying that four different and independent experiments each lead to the same result, that the γ-rays are emitted after the disintegration.

There is now no longer any doubt as to the correct manner of analysing the β-ray spectra. Black has analysed his measurements on thorium B, thorium D, and mesothorium 2 in terms of the two possible atomic numbers, and it is only necessary to point out that the value of the γ-rays and level systems deduced for the greater of the atomic numbers is to be accepted as correct. With the exception of one γ-ray Ellis and Skinner used atomic number 82 for their analysis of radium B, whereas it is now seen that 83 should have been used. The detailed analysis of the spectrum is unaffected except that where the L_{III} level was used for the higher frequency γ-rays it is now necessary to substitute L_{I} or L_{II}. It is not possible to decide on numerical grounds whether L_{I} or L_{II} should be employed, but as Meitner has pointed out L_{I} is preferable since this choice expresses the fact that once the frequency of the radiation is sufficiently above that of the absorption frequency the main absorption will take place in the most heavily bound sub-group. The energies and wave-lengths of the γ-rays are, of course, changed and we give a table showing the new values.

Table I. *Revised Tables of γ-Rays.*

γ-Rays of Radium B.

(Ellis and Skinner, *Proc. Roy. Soc.* A, vol. cv, p. 174, 1924.)

Name	Energy in volts	Intensity	λ in *XU*
γ C1	$0{\cdot}536 \times 10^5$	—	230·3
γ E3	2·43(3)	25	50·7
γ E4	2·60(0)	6	47·5
γ E6	2·97(0)	30	41·6
γ E9	3·54(0)	40	34·9
Doubtful γ-rays deduced from 1 β-ray line			
γ E1	1·97	2	62·7
γ E2	2·06	1	59·9
γ E5	2·76	3	44·7
γ E16	4·71	1	26·2
γ E17	4·82	1	25·6

The "intensities" represent visual estimates of the relative blackness of the β-ray line due to conversion of the γ-ray in the *K* level.

γ-Rays of Radium C.

(Ellis, *Camb. Phil. Soc. Proc.* vol. xxii, p. 374, 1924.)

Name	Energy in volts	Intensity	λ in *XU*
γ E3	2·75	4	44·9
γ E5	3·33	2	37·1
γ E7	3·89	6	31·7
γ E9	4·29	3	28·8
γ E15	6·12	30	20·2
γ E23	9·41	7	13·1
γ E25	11·30	13	10·92
γ E29	12·48	7	9·90
γ E34	14·26	16	8·66
γ E45	17·78	8	6·94
γ E53	22·19	3	5·56

The level systems for these two bodies are also unchanged in arrangement, but the energies of the levels undergo the same small change. It is sufficient to give a table of the energy values of the new levels and to refer to the original papers for details of the transitions*.

Table II. *Relative Energies of the Levels responsible for the emission of the γ-rays of* Ra B *and* Ra C.

Ra B (volts)	Ra C (volts)
0	0
0·537 × 10^5	0·59 × 10^5
0·625	0·70
2·60	3·10
2·97	3·33
4·077	4·29
	4·47
5·34	9·44
	14·28
	15·59
	22·19

It is natural to seek some connection between the different level systems now that five β-ray bodies have been analysed. It is difficult to come to any definite conclusions although attention may be drawn to a few general features. In the first place it is certainly very striking that a low frequency γ-ray, intermediate between the K and L X-ray frequencies, should be emitted in the case of all these bodies when we consider the large electric fields which must exist in the nucleus. Further, if we restrict our attention to radium B and radium C, thorium B and thorium D, we find no intermediate levels until we reach a group around 300,000 volts. In fact we would suggest very tentatively that in so far as it is possible to describe these level systems in general terms we should, starting from one level at an arbitrary zero, distinguish a set at 50,000 volts, another at 300,000 to 400,000 volts, and then a larger jump to about 900,000 volts and higher energies. A further analogy may be pointed out between these bodies. In both the radium and thorium cases the levels show this general structure, except that whereas radium C is as it were an extended radium B system, thorium D appears to resemble a compressed thorium B system.

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* We regard these level systems as showing the transitions that occur when a normal radium B or C nucleus is ionised. Except for small corrections in the energy they will also show the *structure* of the normal radium B nucleus.

t is suggestive to connect this with the fact that radium C results om radium B by the emission of one negative charge, whereas horium D arises from thorium B by the emission of one α- and ne β-particle, that is one nett positive charge. A calculation with elativistic circular orbits suggests that this is just the behaviour ve should expect if the nuclear fields remain the same in character nd change only in strength, provided that individually they do not ary with the distance more rapidly than does an inverse cube ield. If this result is substantiated it suggests the important onclusion that successive bodies of a family have similar nuclear levels. This point is discussed in the next section where a possible exception to this is indicated at places where α-particle bodies emit γ-rays.

The Emission of the γ-Rays. There is no doubt that following Meitner* we may say that the γ-rays are emitted during the reorganisation of the nucleus after the disturbance caused by the emission of the disintegration particle. We think that, while holding the above opinion as fundamental, it is important to put forward some provisional working hypothesis which supplies more details. In view of the complication of the nucleus and the small amount of evidence available at present it is unlikely that any hypothesis advanced now will ultimately prove correct, but we feel that, provided it is always recognised as provisional, there is a great deal to be gained from a more detailed picture. In many points our hypothesis is easily reconciliable with the more general conclusions drawn recently by Meitner† in a very interesting paper, and, except with regard to the moment of emission, with the view put forward by one of us‡.

It seems best to assume that the γ-rays are emitted by the electronic system of the nucleus, not that this is proved but rather that this being the most obvious assumption it is important to see what consequences follow. In this case we would regard the emission of the electron in a β-ray disintegration as providing an ionisation of the nucleus. Previous to this the electronic system of the nucleus is to be thought of as being in its normal state and there is no possibility of transitions but the disintegration provides an ionisation in some level and transitions are now executed so that the nucleus arrives at its new normal state*. We are indebted to Mr Skinner for pointing out an important advantage of this general point of view in that it leaves open the possibility of accounting for the different extents of the γ-ray spectra emitted by different bodies while still emphasising the similarity of their level systems. For instance, there seems to be a real similarity between the nuclear electronic system of radium B and radium C, yet radium B only emits γ-rays of energy up to 400,000 volts, whereas radium C gives γ-rays as high as 2,000,000 volts. This finds its expression in the diagrams of the radium B and radium C levels that have been given, where the radium B diagram only extends to 400,000 volts, but yet is generally similar in arrangement to the lower portion of the radium C system, the latter extending to over two million volts. We may consider a similar electronic system to be present in radium B as in radium C, that is a system extending to over two million volts, but that the disintegration of radium B only provides an ionisation in the 400,000 volt level, whereas the following radium C disintegration gives a much deeper ionisation. It is clear that a similar explanation may be applied to bodies which emit *no* γ-rays, in which case the ionisation is to be considered to occur in the outermost or highest energy level.

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* The emission of the γ-rays has been discussed several times both by Meitner and Ellis since the first evidence of quantum states in the nucleus was obtained (Ellis, *Proc. Roy. Soc.* A, vol. CI, p. 1, 1922), but whereas Ellis endeavoured to combine this with emission before the disintegration Meitner adopted the view which has now been proved to be correct. The difference between the points of view was otherwise small, but we would emphasise that the credit of first giving a generally acceptable account of the emission of γ-rays is Meitner's and therefore, in respect of priority, our present hypothesis is to be regarded as a possible extension of Meitner's standpoint. It will be noticed that we lay stress on the similarity of the electronic systems in the successive nuclei of the same radioactive family and in general seek to interpret results in terms of level systems. Meitner, keeping to a more general standpoint, has not attempted any detailed treatment, and in fact does not refer to level systems at all. (See Meitner, *infra.*)

† Meitner, *Zeit. f. Physik*, XXVI, p. 169, 1924, and *Naturwissenschaft, loc. cit.*

‡ Ellis, *Camb. Phil. Soc. Proc.* vol. XXII, p. 376, 1924.

It is not necessary to assume that all the nuclei disintegrate in exactly the same way. If it proves useful there is no experimental fact which would prevent us from assuming that the radium B nucleus usually disintegrates by ionising itself in either the 300,000 *or* the 400,000 volt level†. It is a great advantage of our working hypothesis that it suggests in this way an interesting interpretation for both the relative and absolute intensities of the γ-rays.

No discussion of the emission of γ-rays would be complete without considering α-particle bodies. Meitner has discovered the important fact that in certain cases α-ray bodies may also emit γ-rays. This is more the exception than the rule, and in view of the provisional nature of our hypothesis we consider it wiser to concentrate on the normal behaviour of α-particle bodies and to discuss why they do *not* emit γ-rays. Since there is in this case no ionisation of the electronic structure, which according to our hypothesis is mainly responsible for the emission of the γ-rays, there will be no possibility of transitions in the sense already mentioned. It seems necessary to go further and assume that there is no rearrangement of the electrons among the various quantum paths. This is a plausible assumption if the emission of the α-particle only weakens the field without changing radically its "shape," as has been already suggested in another connection. The electrons initially are in their normal states, and keeping always to paths defined by the same quantum numbers they will adjust themselves to the new value of the field and will end up in a normal state. It appears likely that the change of energy will take place continuously at the cost of the energy of the α-particle, because this phenomenon cannot cause an *emission* of energy since in any central field energy will be absorbed in such a change.

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* Meitner has expressed the view that the extent of the γ-ray spectrum depends on the "deepness" of the ionisation, but has refrained from making the further extensions which we propose, and does not believe that the γ-rays give direct information about the stationary states in the nucleus.

† This would probably involve the existence of two maximum velocities for the disintegration electrons, but it can easily be seen that no very marked effect would be produced on the form of the continuous spectrum owing to the peculiar shape of this curve. In the α-ray case where the particles all have the same velocity the effect would be very obvious, and this may be the interpretation of the occasional long range particles found.

The most interesting feature of our hypothesis is that it suggests the existence of a generic connection between the nuclear electronic structures of successive bodies in a family whether the changes are α or β. The level system remains the same, only the field changes, and in the case of a β-change there will be also (after the emission of the γ-rays, if any) one less electron in the outer levels. Such a comparison could not be made without further investigation when the intervening α-ray body emits γ-rays, since the emission in this case may show a fundamental reorganisation of the nucleus.

The Continuous Spectrum of β-Rays. We have already pointed out that the two primary phenomena in the β-ray disintegration are the emission of the disintegration electron and the emission of the γ-rays. We have discussed the γ-rays and the fundamental point of view is substantially in agreement with Meitner, although we have attempted to supply more detail. When we turn to consider the emission of the disintegration electron we find a greater difference of opinion, not only about the interpretation, but even about the experimental facts.

Meitner takes as her starting-point the view that since the quantum dynamics appears to apply to the nucleus the disintegration electron must be emitted with a definite velocity. At the present time this standpoint leads to great difficulties, because all the evidence points to the velocity of emission of the disintegration electron varying between wide limits. Meitner has therefore tried the hypothesis that the continuous spectrum does not consist of the disintegration electrons at all but is due to secondary effects. The objection to this is that if the continuous spectrum is disposed of in this way then there are no electrons left over to fulfil the rôle of disintegration electrons. Further, it is not possible to give a convincing explanation of the continuous spectrum as a secondary effect. For instance, she has suggested that it was formed by the recoil electrons from the Compton effect of the scattering of the γ-rays from the nucleus*. This effect must exist, but it is difficult to estimate its magnitude, and in any case if it did exist to the extent postulated by Meitner it would in the case of radium B produce a continuous spectrum of the wrong energies. The continuous spectrum of this body has a maximum about $H\rho 1550$ and extends beyond $H\rho 2500$. Now the highest frequency γ-ray of radium B is extremely weak and can scarcely be made responsible for the continuous spectrum, but even so it would give an emission only reaching as high as $H\rho 1800$, whereas the Compton effect from the really strong γ-rays would not extend beyond $H\rho 1350$ and would have a maximum far lower. Again, as Meitner has herself recognised, an explanation of this type cannot be applied to radium E which emits no γ-rays.

Even without the argument based on the difficulty of accounting for the continuous spectrum as a secondary effect, we still have plenty of direct evidence that it really consists of the disintegration electrons.

Radium E is very important in this connection, it emits no γ-rays and its entire emission consists of a continuous spectrum of electrons without any definite groups. Now Emeléus† has shown that the number of electrons emitted agrees almost exactly with that to be expected if it were formed by the disintegration electrons, that is, one electron for each atom disintegrating. These results have now been extended to cover both radium B and radium C by Gurney, whose work is in course of publication. It seems unnecessary to labour this point and we may accept the fact that disintegration electrons form a continuous spectrum.

The next point is to consider how this inhomogeneity of velocity has been introduced. We assume that energy is conserved exactly in each disintegration, since if we were to consider the energy to be conserved only statistically there would no longer be any difficulty in the continuous spectrum. But an explanation of this type would only be justified when everything else had failed, and although it may be kept in mind as an ultimate possibility, we think it best to disregard it entirely at present. The most obvious attitude to take with regard to the inhomogeneity of velocity is to consider that it occurs after the electron has left the nucleus. This appears plausible from a quantum standpoint, but there are many difficulties in the way‡.

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* Meitner, *Zeit. f. Physik*, 19, p. 307, 1923.

† Emeléus, *Camb. Phil. Soc. Proc.* vol. XXII, p. 400, 1924.

‡ A great deal of the argument that follows depends on the general form of the radium E continuous spectrum. This we have obtained from some unpublished measurements of Madgwick made in the Cavendish Laboratory, from Kovarik and McKeehan (*Phys. Rev.* vol. VIII, p. 574, 1925) and from Schmidt (*Phys. Zeit.* 8, p. 361, 1907).

One suggestion is that the electron emits continuous γ-rays in ts passage out through the intense electric fields of the atom. Some such effect as this may occur, but there is no question of it being the explanation of the *main* inhomogeneity, because radium E emits no penetrating γ-rays, whereas to account for the observed pectrum a γ-ray emission twice as strong as that of radium B would be necessary.

Meitner has made another proposal to account for the inhomogeneity*. She has suggested that the disintegration electron collides with the planetary electrons in its escape from the atom and in this way loses varying amounts of energy. Now the continuous spectrum seems to be a characteristic feature of all β-ray bodies and any explanation must apply to all the bodies, and we do not think that the collision hypothesis can be correct because of the difficulties met with in the case of radium E. The continuous spectrum of radium E extends to about 1,000,000 volts, showing a maximum at about 300,000 volts, and it is a question of accounting for an average loss of at least 500,000 volts per atom disintegrating if initially every electron is emitted with the maximum velocity. Only 90,000 volts are required to remove a K electron, and more than 19/20 of the disintegration electrons lose more energy than this. It is clear that if this large amount of energy, 500,000 volts, were really to be accounted for in this way then there would be a great many high speed secondary electrons which could not escape detection and also a large amount of K radiation. A small amount of K radiation has been detected, but it seems far too small to agree with this explanation. Again, the work of Emeléus, which has already been quoted, speaks decisively against a large number of secondary electrons. He counted directly the number of electrons emitted with greater energy than 70,000 volts, and he only found one electron per atom disintegrating. His experiments certainly tell us nothing about the number of electrons emitted with less than this energy, but it is sufficient to point out that to account for the 500,000 volts missing energy by means of slow electrons would require eight or nine electrons to be knocked out from each atom. There can be no doubt that a certain number of collisions will occur, but it must be remembered that a collision is not a selective phenomenon, and even without the quantitive arguments we have used it would be difficult to see why nearly every disintegration electron should make such a close collision when electrons can be ejected from the K ring by radiation and not suffer any considerable loss. The degree of inhomogeneity is such that a far larger effect is necessary than can be provided by the collision hypothesis. We feel that collisions produce a subsidiary effect which can be taken into account only when the main cause of the inhomogeneity is understood.

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* *Loc. cit.*

We are thus left with the conclusion that the disintegration electron is actually emitted from the nucleus with a varying velocity. We are not able to advance any hypothesis to account for this but we think it important to examine what this fact implies.

The most obvious assumption is that even if the energy with which the disintegration electron is ejected varies from atom to atom, yet the total energy given out in each disintegration is the same. This would make it possible to consider the nucleus in a completely quantised state both before and after the disintegration, but it brings with it the necessity for finding the missing energy. The amount of this missing energy is large, in radium E it would be more than 500,000 volts per atom, and as we have seen already it is very difficult to find any quantum mechanism to provide it. We will not enter into these difficulties now because there is a direct way of finding out whether the above assumption is true or not. This is to find the heating effect of the β-rays from radium E*. If the energy of every disintegration is the same then the heating effect should be between 0·8 and 1·0 × 10^6 volts per atom and the problem of the continuous spectrum becomes the problem of finding the missing energy.

It is at least equally likely that the heating effect will be nearer 0·3 × 10^6 volts per atom, that is, will be just the mean kinetic energy of the disintegration electrons. This would make it impossible for the nucleus to be rigidly quantised both before and after the disintegration and would denote a lack of definiteness of which there has hitherto been no evidence. It is impossible to estimate the degree of this indefiniteness because the energy scale of the nucleus is not known, but it may be noted that it is of the order of magnitude of the γ-rays.

We have discussed these possibilities solely with the object of emphasising the interest and importance of the continuous spectrum. There seems to be no doubt that it exists and that the explanation of its occurrence is not to be sought in any ordinary secondary effect. Some very interesting phenomenon seems to be involved in the β-ray disintegration, and the first step towards its elucidation is the determination of the heating effect of radium E.

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* We are engaged in making this measurement at present, but it will probably be some time before we obtain definite values as only small sources are available with correspondingly small heating effects.

Assemblies of Imperfect Gases by the method of Partition Functions. By Mr R. H. FOWLER, Trinity College.

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§ 1. *Introduction.* In a recent series of papers* analytical methods have been introduced which allow of a mathematically imple treatment of the theorems of statistical mechanics for the usual assemblies of isolated or effectively isolated systems. By this we mean that the individual component systems may be treated for energy content as if they were never interfered with. It is only then that energy can be assigned to systems rather than to the assembly as a whole, and it is on this partition of energy among the systems that the analysis is based. When this independence, for example between separate atoms, breaks down as in a molecule, and still more in a crystal, we can take the whole complex to be a system. The analysis will still apply, and if we can formulate the dynamical motions of the complex system, we can still make progress. The essential step for any system is to construct its partition function. Examples of such constructions for molecules and crystals will be found in the papers quoted, and are of course otherwise well known.

It will be at least of some formal interest to extend the methods used to assemblies, such as assemblies of imperfect gases, in which the degree of isolation is no longer so high, so as to obtain the usual first approximations of the theory of Van der Waals. This can be done with satisfactory simplicity; in this case, which will be considered first, we may say that the free atoms and molecules continue to act on one another *with forces of short range.* We shall find that it is possible to give a simple account of the dissociative equilibrium of such imperfect gases allowing for the effect of dissociation and recombination in altering the intermolecular fields. [This is worked out of course only to the usual first approximation.] There remain only for consideration the inverse square law forces, gravitational and electrostatic. We shall find that the methods can be naturally extended to deal also with these *forces of long range,* and shall be able to give some sort of analysis of the statistical basis for the very important combined use of Boltzmann's formula with Poisson's equation†.

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* I. Darwin and Fowler, *Phil. Mag.* vol. XLIV, p. 450 (1922); II. *ibid.* p. 823; III. *Proc. Camb. Phil. Soc.* vol. XXI, p. 262 (1922); IV. Fowler, *Phil. Mag.* vol. XLV, p. 1 (1923); V. *ibid.* p. 497; VI. Darwin and Fowler, *Proc. Camb. Phil. Soc.* vol. XXI, p. 392 (1923); VII. *ibid.* p. 730.

† For two examples out of many see Debye and Hückel, "The theory of strong electrolytes," *Phys. Zeit.* vol. XXIV, p. 185 (1923), and Langmuir, *Phys. Rev.* vol. II, p. 450 (1913) and vol. XXI, p. 419 (1923).

In these extensions of equilibrium theory, the part played by the partition functions themselves and the whole machinery of statistical mechanics can be reduced with particular convenience to the calculation of the characteristic function of Planck. When this is specified, the rest is thermodynamics. Moreover, the actual partition functions themselves, especially when forces of long range are considered, naturally more and more take the form of the phase-integrals of Gibbs. A redevelopment of this part of the subject would therefore be hardly necessary, if it were not that there appears still something of importance to be said, particularly with regard to Poisson's equation.

We shall not in this paper be concerned primarily with special applications, but with the general theory of imperfect gases. The theory has however been specially developed with a view to a proper discussion of the equilibrium state of assemblies of ionised atoms at high temperature, to which we shall return in a later paper. The contents of this paper, here revised, formed one chapter of an essay on statistical mechanics with special reference to high temperatures, which was awarded the Adams Prize for 1923–4. No claim is made in this paper for novelty of results except perhaps on minor points. Some such chapter however must form part of any general survey of statistical mechanics, and it is perhaps worth while to publish it here in view of some possible novelty of treatment which may serve to throw a little light on certain points of the theory.

§ 2. *Summary of previous results.* We must recall here the more important formulae of the equilibrium theory as developed in the papers quoted. References will be made under the numbers assigned in the first footnote. To every independent system in the assembly we can assign a partition function $f(z)$, where [I. 469, 475]

$$
\begin{aligned}
f(z) &= \Sigma \varpi_r z^{\epsilon_r} \text{ (quantized)} && \text{......(0·1)} \\
&= \frac{1}{h^s} \int \dots \int z^{\epsilon} \, dp_1 \dots dq_s \text{ (classical)} && \text{......(0·11).}
\end{aligned}
$$

In these forms ϵ is the classical Hamiltonian energy function or ϵ_r the energy of the quantized system in its rth stationary state, ϖ_r is the weight of that state, which is unity for a single state of any non-degenerate system, and h is Planck's constant. In certain cases it is convenient to divide ϖ_r, or multiply h^s, by an integer σ, which is the symmetry number of the system [VII. 731]. The partition functions are equivalent to Planck's Zustandsumme or a generalization of Gibbs' phase integrals. Once they are determined the molecular distribution laws are given at once as average values by the equations [I. 470, 476]

$$
\left.
\begin{aligned}
\bar{a}_r &= N \varpi_r \vartheta^{\epsilon_r} / f(\vartheta) \\
&= N \vartheta^{\epsilon} dp_1 \dots dq_s / h^s f(\vartheta)
\end{aligned}
\right\} (\vartheta = e^{-1/kT}) \quad \text{......(0·2).}
$$

here N is the number of such systems in the assembly. The uations of dissociative equilibrium are, for example [VII. 733],

$$\frac{\bar N_1 \bar N_2}{\bar M} = \frac{f_1(\vartheta) f_2(\vartheta)}{g(\vartheta)} \qquad \text{......(0·3)},$$

molecules 1 and 2, of partition functions $f_1(\vartheta)$ and $f_2(\vartheta)$, combine o form a complex of partition function $g(\vartheta)$, with similar expresions in more complicated cases. When dissociation and recombinaion are occurring N must be replaced by $\bar N$ in (0·2) [VII. 735]. verage energies are in all cases given by [VII. 736]

$$\bar E = \left.\begin{matrix} N \\ \bar N \end{matrix}\right\} \frac{\partial}{\partial\vartheta} \log f(\vartheta) \qquad \text{......(0·4)}.$$

he assemblies obey the laws of thermodynamics [VII. 741] and heir equilibrium properties as thermodynamic systems can be imply deduced from Planck's characteristic function Ψ [V. 506] onstructed by the following summation rule:

Any perfect gas of free molecules of partition function $f(\vartheta)$ and umber N contributes to Ψ

$$kN \left\{ \log \frac{f(\vartheta)}{N} + 1 \right\} \qquad \text{......(0·5)}.$$

Any other single system of partition function $K(\vartheta)$, such as solids n which the individual particles do not exchange places during their otions, contributes

$$k \log K(\vartheta) \qquad \text{......(0·6)}.$$

A similar contribution is made by temperature radiation.

An important partition function is that for free molecules of mass m in a volume V. For these the translatory and internal energies can be considered independently with partition functions $h(\vartheta)$ and $j(\vartheta)$ respectively. Then

$$f(\vartheta) = h(\vartheta) j(\vartheta) = \frac{(2\pi m)^{\frac{3}{2}} V}{h^3 (\log 1/\vartheta)^{\frac{3}{2}}} j(\vartheta) = VF(\vartheta) \quad \text{...(0·7)},$$

say, where $F(\vartheta)$ is a function of ϑ alone.

Any formula for an average value is of general validity, not merely when the average value is numerically large, but in fact only those values large enough to make their fluctuations insignificant will themselves be of significance as the final stage in a physical investigation. It is not however necessary that the average values used should be large at every stage of the investigation.

§ 3. *Gaseous assemblies with molecules surrounded by fields of force of short range.* An immediate extension to assemblies of imperfect gases, in which the free molecules are surrounded by fields of force of short range, can be made by means of the following device*. The device consists in regarding any *pair* of molecules which are within one another's field of force as a *system* to discussed as a whole. Any molecule outside the fields of force of others remains a system as before. The equilibrium state of the assembly can then at once be studied as a problem in dissociation in a gaseous assembly in which every individual *system* is effectively isolated. In a first approximation to the imperfect gas laws it is sufficient to consider interactions in pairs. But multiple interactions and higher approximations could be similarly calculated.

Let us suppose that two simple systems (atoms or molecules) of types α, β have energies E_α, E_β when they are out of each other's fields of force. These energies will then be functions only of the (Hamiltonian) coordinates of the systems α and β respectively. When however they approach each other and enter each other's fields, the energy of the pair will no longer be simply of the form $E_\alpha + E_\beta$, but will contain also a term $E_{\alpha\beta}$ depending on the relative coordinates of the two systems. We shall assume that this term $E_{\alpha\beta}$ depends only on relative position, so that it is of the nature of potential energy. This is equivalent to assuming that the interaction of the two systems on one another is a slow adiabatic variation in Ehrenfest's sense. The energy of the pair is now $E_\alpha + E_\beta + E_{\alpha\beta}$. To give sense to the preceding remarks we must suppose that the interference field is of strictly limited range, but it will be seen later that this assumption is physically unobjectionable, even when, for mathematical convenience, we represent the magnitude of the field by some (sufficiently large) inverse power of the distance. There is then a certain volume $v_{\alpha\beta}$ round α or β in which $E_{\alpha\beta}$ is effectively different from zero. Elsewhere α and β are independent systems, but when the centre of one lies within the $v_{\alpha\beta}$ of the other the pair form a single aggregate, which is itself a system of energy $E_\alpha + E_\beta + E_{\alpha\beta}$. We proceed to construct partition functions.

Let $f_\alpha(\vartheta) = h_\alpha(\vartheta) j_\alpha(\vartheta)$ be the partition function for the free α's, where $h_\alpha(\vartheta)$ is the usual partition function for the translational energy, and $j_\alpha(\vartheta)$ represents the internal quantized motions and rotations (if any). Similarly for the free β's, let $f_\beta(\vartheta) = h_\beta(\vartheta) j_\beta(\vartheta)$. Let $\overline{N}_\alpha, \overline{N}_\beta$ be the average numbers of free α's and β's, N_α, N_β the total number of α's and β's. Let $f_{\alpha\beta}(\vartheta)$ be the partition function for systems which are $\alpha\beta$-aggregates, $\overline{N}_{\alpha\beta}$ in average number. The $f_{\alpha\beta}(\vartheta)$ will contain as factors $j_\alpha(\vartheta)$ and $j_\beta(\vartheta)$, so long as the internal ergies of the α and β are unaltered by the adiabatic establishment of the field of force between the α and β. Besides $j_\alpha(\vartheta)\, j_\beta(\vartheta)$, (ϑ) contains as factor only the partition function for the classical tential and kinetic energy, which splits into factors for the dependent motion of the centre of gravity of the aggregate and e relative motion of the pair in their mutual field of force of otential $E_{\alpha\beta}$.

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* Jeans, *Dynamical theory of gases*, ed. 3, p. 91. The discussion of dissociation and aggregation there given by Jeans on a classical basis by means of the above device is of course inadmissible *in general*. The phenomena are essentially phenomena of the quantum theory. The device however is admirably adapted to the discussion of such classical systems and is so applied here to our classical compound "systems." It is essentially equivalent to Boltzmann's discussion of dissociation, *Vorlesungen über Gastheorie*, II, Abschnitt VI.

It remains to construct this partition function. The element of hase-space for $\alpha\beta$ is

$$(m_\alpha m_\beta)^3\, dx_\alpha \ldots dw_\alpha\, dx_\beta \ldots dw_\beta,$$

here $x_\alpha \ldots z_\alpha$, $x_\beta \ldots z_\beta$ are rectangular cartesian coordinates of osition and $u_\alpha \ldots w_\alpha$, $u_\beta \ldots w_\beta$ the corresponding velocities. This an be transformed to coordinates and velocities of the centre of ravity $x^*, \ldots, w^*$, and coordinates and velocities of α relative to $\xi, \ldots, w$. The Jacobian of the transformation is 1, and we have n element of phase-space

$$(m_\alpha m_\beta)^3\, dx^* \ldots dw^* d\xi \ldots dw.$$

The corresponding prepared weight δ_t is

$$\delta_t = (m_\alpha m_\beta)^3 \frac{dx^* \ldots dw^* d\xi \ldots dw}{\sigma_{\alpha\beta} h^6} \qquad \ldots\ldots(1),$$

where $\sigma_{\alpha\beta}$ is the symmetry number of the aggregate with $\sigma_{\alpha\beta} = 1$ $(\alpha \neq \beta)$, and $\sigma_{\alpha\alpha} = 2$. The corresponding energy is

$$\epsilon_t = \frac{1}{2}(m_\alpha + m_\beta)(\Sigma u^{*2}) + \frac{1}{2}\frac{m_\alpha m_\beta}{m_\alpha + m_\beta}(\Sigma u^2) + E_{\alpha\beta}(\xi, \eta, \zeta) \ldots(2).$$

Thus, in the classical limit, this part of the partition function is

$$\frac{(m_\alpha m_\beta)^3}{\sigma_{\alpha\beta} h^6} \int \overset{(12)}{\ldots} \int e^{-[\epsilon]\log 1/\vartheta}\, dx^* \ldots dw^* d\xi \ldots dw \quad \ldots\ldots(3),$$

where $[\epsilon]$ is given by (2). This reduces at once to

$$h_{\alpha\beta}(\vartheta) . \frac{\left(2\pi \dfrac{m_\alpha m_\beta}{m_\alpha + m_\beta}\right)^{\frac{3}{2}} A(\vartheta)}{h^3 (\log 1/\vartheta)^{\frac{3}{2}}} \qquad \ldots\ldots(4),$$

where $h_{\alpha\beta}(\vartheta)$ is the ordinary partition function for the motion of the aggregate as a whole, and

$$A(\vartheta) = \frac{1}{\sigma_{\alpha\beta}} \int_{v_{\alpha\beta}} e^{-E_{\alpha\beta}(\xi,\eta,\zeta)\log 1/\vartheta}\, d\xi\, d\eta\, d\zeta \quad \ldots\ldots(5).$$

In the expression (4) the form of the factors shows that $A(\vartheta)$

may properly be regarded as the partition function for the potential energy $E_{\alpha\beta}$, and the factor

$$\left(2\pi \frac{m_\alpha m_\beta}{m_\alpha + m_\beta}\right)^{\frac{3}{2}} \Big/ h^3 (\log 1/\vartheta)^{\frac{3}{2}}$$

as the partition function for the kinetic energy of the relative motion.

The gas here considered of "free" molecules and aggregates is one to which the ordinary dissociation theory, in particular equation (0·3), can be at once applied. We find

$$\frac{\overline{N}_{\alpha\beta}}{\overline{N}_\alpha \overline{N}_\beta} = \frac{j_\alpha(\vartheta)\, j_\beta(\vartheta)\, h_{\alpha\beta}(\vartheta) \left(2\pi \dfrac{m_\alpha m_\beta}{m_\alpha + m_\beta}\right)^{\frac{3}{2}} A(\vartheta)}{j_\alpha(\vartheta)\, h_\alpha(\vartheta)\, j_\beta(\vartheta)\, h_\beta(\vartheta) \,.\, h^3 (\log 1/\vartheta)^{\frac{3}{2}}} = \frac{A(\vartheta)}{V} \quad \text{...(6)}.$$

In calculating first order corrections $v_{\alpha\beta}/V$ (and therefore $A(\vartheta)/V$) will be small, so that $\overline{N}_{\alpha\beta}/\overline{N}_\alpha \overline{N}_\beta$ will be small, and $\overline{N}_\alpha$, and $\overline{N}_\beta$ will only differ by first order terms from N_α and N_β. If we ask for $(\bar{a}_{\alpha\beta})_t$, the average number of aggregates with given ranges of relative positional coordinates, we find by the general theory, in particular (0·2),

$$\begin{aligned}(\bar{a}_{\alpha\beta})_t &= \overline{N}_{\alpha\beta}\, e^{-[E_{\alpha\beta}(\xi,\eta,\zeta)]_t \log 1/\vartheta} (d\xi\, d\eta\, d\zeta)_t / \sigma_{\alpha\beta} A(\vartheta) \\ &= \overline{N}_\alpha \overline{N}_\beta\, e^{-(E_{\alpha\beta})_t/kT} (d\xi\, d\eta\, d\zeta)_t / \sigma_{\alpha\beta} V \qquad \text{......(7)}.\end{aligned}$$

The velocity distribution laws can similarly be shown to be unaltered by the field of force. If we recall that $\overline{N}_\alpha$ is effectively N_α, equation (7) reduces to the well-known equilibrium distribution law*, commonly called Boltzmann's Theorem.

In virtue of the properties of $A(\vartheta)$ as a partition function we can at once derive the average potential energy of the assembly due to (α, β) interactions. By (0·4) it is

$$\begin{aligned}\bar{E}_{\alpha\beta} &= \overline{N}_{\alpha\beta}\, \vartheta \frac{\partial}{\partial\vartheta} \log A(\vartheta) \qquad \text{......(8)}, \\ &= \frac{\overline{N}_\alpha \overline{N}_\beta}{V} \vartheta \frac{\partial A(\vartheta)}{\partial\vartheta} \\ &= \frac{\overline{N}_\alpha \overline{N}_\beta}{V} \frac{1}{\sigma_{\alpha\beta}} \int_{v_{\alpha\beta}} E_{\alpha\beta}\, e^{-E_{\alpha\beta}/kT}\, d\xi\, d\eta\, d\zeta \quad (9).\end{aligned}$$

To the first order the $\overline{N}_\alpha$ and $\overline{N}_\beta$ may be replaced by N_α and N_β. We have moreover assumed that $E_{\alpha\beta}$ is due to forces of finite range. It is more usual and convenient in practice to represent molecular forces by forces which fall off like the inverse sth power of the distance. If s is sufficiently large ($s > 4$) the integral in (9) will converge when extended over all space. Any such rapidly convergent integral can be substituted for the finite integral in (9) and elsewhere, without essential modification of the argument, for the integral over all space differs only negligibly from the integral over a small $v_{\alpha\beta}$ of atomic dimensions.

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* Jeans, *loc. cit.* p. 132.

All the general laws established in our former papers must hold for the more general assemblies here considered. They must, for instance, obey the laws of Thermodynamics. In order to study further their equilibrium properties, a possible and sufficient step is to construct the function Ψ, from which all equilibrium properties of the assembly, other than molecular distribution laws, can be obtained by differentiation. This is in general a simpler method than direct calculation for assemblies of imperfect gases. It has been a point of principle in the exposition of the equilibrium theory of statistical mechanics given in former papers that the theory must logically proceed independently of thermodynamical ideas, up to the stage at which it is proved that the assemblies are thermodynamic systems. This however by no means must be taken to imply that there is any logical objection to the use of thermodynamical methods (when convenient) once this stage has been successfully reached.

By (0·5) the function Ψ is given by

$$\Psi/k = \Sigma_\alpha \overline{N}_\alpha \left\{\log \frac{f_\alpha}{\overline{N}_\alpha} + 1\right\} + \Sigma_{\alpha\beta} \overline{N}_{\alpha\beta} \left\{\log \frac{f_{\alpha\beta}}{\overline{N}_{\alpha\beta}} + 1\right\} \quad \text{...(10)},$$

where $\Sigma_{\alpha\beta}$ denotes summation over every pair of molecular types in the assembly, including pairs of like molecules. The partition function f_α must however be carefully interpreted. Its V-factor is no longer exactly V, because if two molecules are closer together than a definite distance, the pair rank as an $\alpha\beta$-aggregate, not as a free α and free β. Thus in f_α we have a V-factor equal approximately to

$$V - \Sigma_\beta N_\beta v_{\alpha\beta} \quad \text{......(11)}.$$

Since $\overline{N}_{\alpha\beta}$ is small the V-factor in $f_{\alpha\beta}$ may be taken to be V, and in all correcting terms $\overline{N}_\alpha$ and N_α need not be distinguished. To this approximation therefore we appear to find that

$$\Psi/k = \Sigma_\alpha \overline{N}_\alpha \left\{\log \frac{VF_\alpha}{\overline{N}_\alpha} + 1\right\} + \Sigma_{\alpha\beta} \overline{N}_{\alpha\beta} \left\{\log \frac{VF_{\alpha\beta}}{\overline{N}_{\alpha\beta}} + 1\right\} - \frac{1}{V} \Sigma_\alpha N_\alpha \Sigma_\beta N_\beta v_{\alpha\beta}.$$

This however is not correct, for the excluded volume relative to each pair of molecules is thus counted twice over in the correcting term to Ψ. After making this correction, we find

$$\Psi/k = \Sigma_\alpha \overline{N}_\alpha \left\{\log \frac{VF_\alpha}{\overline{N}_\alpha} + 1\right\} + \Sigma_{\alpha\beta} \overline{N}_{\alpha\beta} \left\{\log \frac{VF_{\alpha\beta}}{\overline{N}_{\alpha\beta}} + 1\right\} - \frac{1}{V} \Sigma_{\alpha\beta} \frac{N_\alpha N_\beta v_{\alpha\beta}}{\sigma_{\alpha\beta}} \quad \text{......(12)}.$$

We can also reduce (12) to the form of the terms for a perfect gas plus correcting terms (of the first order). We have obviously

$$\overline{N}_\alpha + \Sigma_\beta \sigma_{\alpha\beta} \overline{N}_{\alpha\beta} = N_\alpha \qquad \text{......(13).}$$

and therefore

$$\overline{N}_\alpha = N_\alpha \left(1 - \Sigma_\beta \frac{N_\beta \sigma_{\alpha\beta} A_{\alpha\beta}(\vartheta)}{V}\right) \qquad \text{......(14).}$$

Then, after a simple reduction,

$$\Psi/k = \Sigma_\alpha N_\alpha \left\{\log \frac{VF_\alpha}{N_\alpha} + 1\right\} + \frac{1}{V} \Sigma_{\alpha\beta} N_\alpha N_\beta \left\{A(\vartheta) - \frac{v_{\alpha\beta}}{\sigma_{\alpha\beta}}\right\} \qquad \text{......(15).}$$

$$= \Sigma_\alpha N_\alpha \left\{\log \frac{VF_\alpha}{N_\alpha} + 1\right\} + \frac{1}{V} \Sigma_{\alpha\beta} \frac{N_\alpha N_\beta}{\sigma_{\alpha\beta}} \int \{e^{-E_{\alpha\beta}/kT} - 1\}\, d\xi\, d\eta\, d\zeta \qquad \text{......(16).}$$

Since $p = T\partial\Psi/\partial V$, we find at once

$$p = kT\left[\frac{\Sigma_\alpha N_\alpha}{V} - \frac{1}{V^2} \Sigma_{\alpha\beta} \frac{N_\alpha N_\beta}{\sigma_{\alpha\beta}} \int \{e^{-E_{\alpha\beta}/kT} - 1\}\, d\xi\, d\eta\, d\zeta\right] \text{...(17).}$$

This is the well-known formula of Van der Waals, correct to terms in $1/V^2$ for a mixture of imperfect gases. The integral in (17) is more commonly expressed in polar coordinates, for by ignoring the orientations of our molecules in this connection, we have tacitly assumed that $E_{\alpha\beta}$ is a function of r only (or may be replaced by a mean value for all orientations). Expressed thus we have

$$p = kT\left[\frac{\Sigma_\alpha N_\alpha}{V} - \frac{1}{V^2} \Sigma_{\alpha\beta} \frac{N_\alpha N_\beta}{\sigma_{\alpha\beta}}\, 4\pi \int_0^\infty \{e^{-E_{\alpha\beta}/kT} - 1\}\, r^2 dr\right] \text{...(18).}$$

The method can easily be extended to the case in which orientations are relevant, and the potential is a general function of the relative position, but we shall not take up such extensions here.

An alternative deduction* of Ψ, of some importance for applications, may be of interest here, as it avoids the difficulty of the excluded volumes. Treating the assembly as a perfect gas, we should obtain

$$\Psi/k = \Sigma_\alpha N_\alpha \left\{\log \frac{f_\alpha}{N_\alpha} + 1\right\};$$

we have also the thermodynamic relation

$$\Psi = \int E dT/T^2 = k \int E d\vartheta/\vartheta \qquad \text{......(19).}$$

Provided that we can assign the functional constant of integration we can therefore calculate Ψ from E, but this assignment is impossible on purely thermodynamical grounds as the integration constant is in general an unknown function of the N_α. Here however we already know that part of Ψ which arises from all forms of energy other than intermolecular potential energy together with the integration constant. The extra term in Ψ can be obtained by addition since (19) is linear in Ψ and E. This *extra* term *can* be obtained by integrating (19), for the new integration constant is now fixable. The addition Ψ'/k to Ψ/k must vanish for all temperatures and all N_α when

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* See, for example, Debye and Hückel, *loc. cit.*

$$E_{\alpha\beta} \to 0.$$

We can write, using (9),

$$\begin{aligned}\Psi'/k &= \Sigma_{\alpha\beta} \frac{N_\alpha N_\beta}{\sigma_{\alpha\beta} V} \int \frac{d\vartheta}{\vartheta} \int_{v_{\alpha\beta}} E_{\alpha\beta}\, e^{-E_{\alpha\beta} \log 1/\vartheta}\, d\xi\, d\eta\, d\zeta + const. \\ &= \Sigma_{\alpha\beta} \frac{N_\alpha N_\beta}{\sigma_{\alpha\beta} V} \int_{v_{\alpha\beta}} \{e^{-E_{\alpha\beta}/kT} - 1\}\, d\xi\, d\eta\, d\zeta + const. \quad \ldots(20).\end{aligned}$$

The constant in (20) is some function of the N_α and V independent of T. It could only depend on $E_{\alpha\beta}$, for dimensional reasons, in the form $E_{\alpha\beta}/kT$, and is therefore independent of $E_{\alpha\beta}$. It must therefore be zero, in agreement with (16).

§ 4. *Partition functions for the potential energy of the whole assembly.* A more fundamental method of deriving Ψ'/k is naturally by direct construction of the partition function for the potential energy of the assembly as a whole, and this can be carried out. We shall in this section discuss the general uses of this partition function, without at first restriction to forces of short range. The discussion of § 3 will be to some extent superseded by this section, but it still serves perhaps a useful introductory purpose.

If initially we regard the whole imperfect gas mixture in the assembly as a single system the classical part* of its partition function, $H(\vartheta)$, can be written in the form

$$\begin{aligned}H(\vartheta) &= \frac{m_a{}^{3N_a \ldots}}{h^{3N_a + \ldots}} \int \ldots \int e^{-[\Sigma_a \frac{1}{2} m_a \Sigma_r (u^2+v^2+w^2)_{a,r} + W] \log 1/\vartheta} \\ &\qquad\qquad \times \Pi_a (\Pi_r du_{a,r} \ldots dz_{a,r}) \quad \ldots\ldots(21) \\ &= \Pi_a \left[\frac{(2\pi m_a)^{\frac{3}{2}}}{h^3 (\log 1/\vartheta)^{\frac{3}{2}}}\right]^{N_a} B(\vartheta) \quad \ldots\ldots(22),\end{aligned}$$

where $$B(\vartheta) = \int \ldots \int e^{-W \log 1/\vartheta}\, \Pi_a (\Pi_r dx_{a,r}\, dy_{a,r}\, dz_{a,r}) \quad \ldots(23).$$

In these formulae W is the potential energy of the assembly as a function of all its positional coordinates $(x, y, z)_{a,r}$, and the integrals are to be extended over the whole phase-space for each system. It is therefore assumed that in no case can $W \to -\infty$, but that ultimately there must be repulsive fields between any two particles whose potential $\to +\infty$ when they approach sufficiently closely. No other assumption would be physically consistent with the continued existence of ordinary matter. When $W \equiv 0$, $B(\vartheta) = V^{\Sigma_a N_a}$, and $H(\vartheta)$ reduces to the product of the ordinary individual partition functions. From the form of (21) however it is clear that the classical potential and kinetic energies in a gas can always be handled with separate partition functions. The kinetic energy can always be dealt with in the ordinary way, as if the gases were perfect, by the ordinary partition functions without the V-factor. The potential energy is accounted for by $B(\vartheta)$. It should be observed further that (23) is perfectly general and applies to assemblies in which the imperfections are of any degree and the molecular forces of any range, and whether or no there are external fields of force. If there are no external fields of force W will be a function of the relative coordinates of the molecules only. This will still be true in the limit if there are local boundary fields representing walls of the enclosure.

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* The internal quantized energies of the molecules are accounted for by separate partition functions, which need not be further referred to here.

The partition function $B(\vartheta)$ here introduced is a partition function for what is strictly a single system, the whole gaseous assembly. It therefore appears by itself, not raised to a high power, in the various complex integrals which represent the equilibrium (average) state of the assembly in our theory. This however is no novelty for the same is true of the partition functions for temperature radiation and for single crystals. No essential difference is made thereby either to the analysis or the significance of the results. Average values derived from $B(\vartheta)$ by the usual processes may sometimes be small, and are then better called *frequency ratios*, but they are then without practical significance. But any derived average value which is itself large, in the sense of the ordinary average values of the perfect gas theory, will be equally significant here. The limiting form of $B(\vartheta)$, $V^{\Sigma_a N_a}$, supplies the necessary large factors for the ordinary separate partition functions of the separate molecules.

It is of course necessarily true, and becomes particularly obvious in this section, that the partition functions of our theory are identical with the separable factors in the phase integrals of Gibbs. The further developments will therefore hardly differ at all, whether one starts from the conservative system of many degrees of freedom of Boltzmann or from the canonical ensemble of Gibbs. We shall, for consistency, continue to use the terminology of partition functions. Some of the uses which it is proposed to make of $B(\vartheta)$ do not seem to have been fully investigated before.

It is convenient to use an abbreviated notation. We write

$d\omega_{\alpha,r}$ for the volume element $(dxdydz)_{\alpha,r}$ and shorten $\Pi_\alpha\,(\Pi_r d\omega_{\alpha,r})$ into $\Pi_\alpha (d\omega_\alpha)^{N_\alpha}$ so that

$$B(\vartheta) = \int \dots \int e^{-W \log 1/\vartheta}\,\Pi_\alpha\,(d\omega_\alpha)^{N_\alpha} \qquad \dots\dots(24).$$

We conclude the present section by constructing directly Ψ/k for forces of short range. The partition function $B(\vartheta)$ can at once be seen to contribute to Ψ/k the extra term $\log B(\vartheta)$, just like the contributions of a crystal or temperature radiation after (0·6). The rest of Ψ/k can be calculated as usual from the partition functions $F(\vartheta)$, etc. without V-factors, as for a perfect gas. To derive the formulae of § 3 we have only to evaluate $B(\vartheta)$ approximately. If an α, β-pair lie in a $v_{\alpha\beta}$, $W = E_{\alpha\beta}$ and W is otherwise zero. Multiple occurrences can be ignored. Therefore

$$B(\vartheta) = V^{\Sigma_\alpha N_\alpha} + \Sigma \int_{v_{\alpha\beta}} \frac{d\omega_\alpha}{\sigma_{\alpha\beta}} \int \dots \int \{e^{-E_{\alpha\beta}/kT} - 1\}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha},$$

where Π_α' denotes that integration over a special α-element is omitted. The summation Σ is over every pair of molecules, and the factor $\sigma_{\alpha\beta}$ must be inserted to avoid counting twice over each α, α-pair. Therefore

$$B(\vartheta) = V^{\Sigma_\alpha N_\alpha}\left(1 + \Sigma_{\alpha\beta} \frac{N_\alpha N_\beta}{\sigma_{\alpha\beta} V} \int_{v_{\alpha\beta}} \{e^{-E_{\alpha\beta}/kT} - 1\}\,d\omega_\alpha\right).$$

When therefore we add $\log B(\vartheta)$ to the rest of Ψ/k we restore the V-factors to the perfect gas terms and add the term

$$\Sigma_{\alpha\beta} \frac{N_\alpha N_\beta}{\sigma_{\alpha\beta} V} \int_{v_{\alpha\beta}} \{e^{-E_{\alpha\beta}/kT} - 1\}\,d\omega_\alpha$$

in agreement with (16).

§ 5. *Further applications of* $B(\vartheta)$. In further applications of the partition function $B(z)$ it is sometimes convenient to make use of a general identity between two methods of calculating average values connected with $B(z)$ one of which makes use of $B(z)$ alone. Let $f(z)$ be the combined partition function for the whole of the rest of the assembly. Then in our usual notation, the total number of weighted complexions C is given in a non-dissociating assembly by

$$C = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} B(z) f(z).$$

In the same way we can calculate the number of complexions for which a selected system lies in a selected cell, or in which any specified restriction is imposed on that part or property of the assembly represented by $B(z)$. Subject to this specification it will have a partition function $\delta B(z)$, and we shall find

$$\delta C = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} \{\delta B(z)\} f(z)$$

for the corresponding number of weighted complexions. The regulation average values of the theory for any quantity P are

$$C\bar{P} = \Sigma P\delta C.$$

We therefore find, assuming that summation and integration may be inverted,

$$C\bar{P} = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{E+1}}\{\Sigma P\delta B(z)\}f(z).$$

If now we define a function $P(z)$ by the equation

$$[P(z)]\,B(z) = \Sigma P\delta B(z) \qquad \text{......(25)},$$

then $$C\bar{P} = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{E+1}}[P(z)]\,B(z)f(z) = C[P(\vartheta)].$$

We therefore find that the standard average value of P is $[P(\vartheta)]$ as defined by equation (25). Such mean values may often be more simply calculated by (25). The restriction to non-dissociating assemblies is inessential, for we have proved that in dissociating assemblies all average values are the same as those in the non-dissociating assembly formed by fixing the dissociations at their average values, and this theorem will remain true under the generalizations here introduced.

We can now calculate simply, in a manner exactly analogous to the calculations of distribution in space for perfect gases [I. 455], the average number of α-molecules in a selected $d\omega_\alpha$. We have

$$B(\vartheta) = \Sigma_r\delta_r B(\vartheta), \qquad \bar{a}_\alpha B(\vartheta) = \Sigma_r r\delta_r B(\vartheta),$$

where $\delta_r B(\vartheta)$ is that part of $B(\vartheta)$ which corresponds to the case of r α-molecules in $d\omega_\alpha$. Thus

$$\delta_0 B(\vartheta) = \int \ldots\ldots\ldots\ldots \int_{(V,\,V-d\omega_\alpha)} e^{-W\log 1/\vartheta}\,\Pi_\kappa (d\omega_\kappa)^{N_\kappa},$$

integration being over the volume $V - d\omega_\alpha$ for every α and over V for every other species. Similarly

$$\delta_r B(\vartheta) = \frac{N_\alpha!}{(N_\alpha - r)!\,r!}\int \ldots\ldots \int_{(d\omega_\alpha)} (d\omega_\alpha)^r \int \ldots\ldots\ldots\ldots \int_{(V,\,V-d\omega_\alpha)} e^{-W\log 1/\vartheta}\,\Pi_\kappa^{(r)} (d\omega_\kappa)^{N_\kappa}.$$

The binomial coefficient gives of course the number of different selections of α-molecules, which place r in $d\omega_\alpha$, and in $\Pi_\kappa^{(r)}$ r integrations with respect to $d\omega_\alpha$ are omitted. Thus

$$\bar{a}_\alpha B(\vartheta) = N_\alpha \sum_{r=1}^{N_\alpha} \frac{(N_\alpha - 1)!}{(N_\alpha - r)!\,(r-1)!}\int \ldots\ldots \int_{(d\omega_\alpha)} (d\omega_\alpha)^r$$

$$\times \int \ldots\ldots\ldots\ldots \int_{(V,\,V-d\omega_\alpha)} e^{-W\log 1/\vartheta}\,\Pi_\kappa^{(r)} (d\omega_\kappa)^{N_\kappa} \qquad \text{......(26)}.$$

$d\omega_\alpha$ is "physically" small the integrand for the *last* integration ll be a constant over $d\omega_\alpha$, so that (26) may be written

$$B(\vartheta) = N_\alpha d\omega_\alpha \sum_{r=0}^{N_\alpha-1} \frac{(N_\alpha-1)!}{(N_\alpha-1-r)!\,r!} \int_{(d\omega_\alpha)} \cdots\cdots \int (d\omega_\alpha)^r$$

$$\times \int_{(V,\,V-d\omega_\alpha)} \cdots\cdots\cdots\cdots \int e^{-W\log 1/\vartheta}\,\Pi_\kappa^{(r+1)}\,(d\omega_\kappa)^{N_\kappa}$$

$$= N_\alpha\, d\omega_\alpha \int_{(V)} \cdots \int e^{-W\log 1/\vartheta}\,\Pi_\kappa{}'\,(d\omega_\kappa)^{N_\kappa} \qquad \text{......(27),}$$

which just one α-integration is omitted*. If there are no external elds and no long range forces in W then the coefficient of $N_\alpha d\omega_\alpha$ (27) must be a constant Ω, independent of the coordinates of ω_α, and

$$\int \Omega\, d\omega_\alpha = B(\vartheta) = V\Omega \qquad \text{......(28).}$$

Combining (27) and (28) we find then the familiar result

$$\bar{a}_\alpha = N_\alpha d\omega_\alpha / V \qquad \text{......(29).}$$

This is the systematic derivation of (29) for imperfect gases! Certain subsidiary steps will be useful later. If there are long range forces, these may, as we shall see, build up what are external fields from the point of view of any specified element of the assembly. Equation (29) is then no longer true except when referred to an elementary assembly which is itself small enough for the variation of the external part of W over it to be negligible. We return to this in § 7.

By an exactly similar argument the average number of α, β-pairs simultaneously in selected $d\omega_\alpha$, $d\omega_\beta$ is

$$\frac{N_\alpha N_\beta\, d\omega_\alpha\, d\omega_\beta}{B(\vartheta)} \int \cdots \int e^{-W\log 1/\vartheta}\,\Pi_\kappa{}''\,(d\omega_\kappa)^{N_\kappa} \qquad \text{......(30).}$$

With no effective long range or external forces the integral in (30) must be a function only of the relative coordinates of the selected $d\omega_\alpha$ and $d\omega_\beta$. We may then write it

$$\Omega e^{-W_{\alpha\beta}\log 1/\vartheta},$$

where $W_{\alpha\beta} \to 0$ at infinite separation and Ω is constant. This *defines* $W_{\alpha\beta}$ as a function of the relative coordinates of $d\omega_\alpha$ and $d\omega_\beta$. $W_{\alpha\beta}$ *so defined* may be called *the average potential of* β *in the specified position in the field of* α. $W_{\alpha\beta}$ may depend on the average positions of a large number of other molecules and be itself a function of T. Again

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* Equation (27) is more usually arrived at by saying that the coefficient of N_α divided by $B(\vartheta)$ is the probability or frequency with which a selected α-molecule lies in $d\omega_\alpha$ and that therefore the expectation of all α-molecules in $d\omega_\alpha$, or their average number, is given by (27). The arrangement of the proof in the text follows more closely the normal line of development adopted in these papers.

$$\iint \Omega e^{-W_{\alpha\beta}\log 1/\vartheta} d\omega_\alpha d\omega_\beta = B(\vartheta) = \Omega V^2,$$

provided that $W_{\alpha\beta} \to 0$ rapidly for large separations (which will always be the case in applications). Thus the average number of pairs is

$$N_\alpha N_\beta e^{-W_{\alpha\beta}/kT} d\omega_\alpha d\omega_\beta / V^2 \qquad \text{......(31)}.$$

and, after integration with respect to $d\omega_\alpha$, the average number of β's in a selected region near any α anywhere in the assembly is

$$N_\beta e^{-W_{\alpha\beta}/kT} d\omega_\beta / V \qquad \text{......(32)}.$$

Formulae (31) and (32) are of course familiar forms of Boltzmann's theorem. The present discussion however may perhaps add something to some other expositions in that it puts clearly in evidence the meaning of $W_{\alpha\beta}$ for which the theorem is true*.

§ 6. *Dissociative equilibria for molecules of finite extension.* The formulae of the last sections take a specially simple form when we can think of the constituent systems merely as possessing a definite size (rigid atoms or molecules) without further fields of force. In this limiting case $E_{\alpha\beta} \to +\infty$ inside $v_{\alpha\beta}$ and is zero outside, so that Ψ/k reduces to

$$\Psi/k = \Sigma_\alpha N_\alpha \left\{\log \frac{VF_\alpha}{N_\alpha} + 1\right\} - \frac{1}{V}\Sigma_{\alpha\beta} \frac{N_\alpha N_\beta v_{\alpha\beta}}{\sigma_{\alpha\beta}} \qquad \text{...(33)}.$$

In (33) $v_{\alpha\beta}$ denotes the volume round the centre of mass of an α into which the centre of mass of $\alpha\beta$ cannot penetrate and *vice versa*. This equation can also be obtained directly. It provides a satisfactory starting-point for investigating the equilibrium state of an assembly in which we take account of a sequence of possible stationary states of any atom and associate with each a definite size, which may differ from state to state.

Suppose that there are present in the assembly systems of a certain type in a number of different stationary states, and that these are initially regarded as different systems specified by different α's. Systems other than the states of this system will be specified by other α's. To determine the equilibrium we can either vary Ψ or use directly statistical methods, but the former is the simpler for imperfect gases. As a typical variation suppose N_0 increased

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* Defined in this way $W_{\alpha\beta}$ is the *free energy* of β in the field of an α (Einstein. *Ann. de Phys.* vol. XVII, p. 549 (1905)). This connection between the free energy and the partition functions is expressed in its most general form in the structure of Ψ.

δ **and** N_a diminished by δ, N_0 and N_a being the numbers of the **cial** system in its normal and αth (excited) state. Then

$$= d\Psi/k = \delta \log \frac{VF_0}{N_0} - \delta \log \frac{VF_a}{N_a} - \frac{\delta}{V} \Sigma_\beta N_\beta v_{0\beta} + \frac{\delta}{V} \Sigma_\beta N_\beta v_{a\beta} \qquad \text{......(34),}$$

d **therefore**

$$\frac{N_a}{N_0} = \frac{F_a e^{-\Sigma_\beta N_\beta v_{a\beta}/V}}{F_0 e^{-\Sigma_\beta N_\beta v_{0\beta}/V}} \qquad \text{......(35).}$$

w if we were discussing this assembly without reference to the **es of** its constituent systems, we should of course treat all the **ecial** systems together and construct the normal partition **nction**

$$b(\vartheta) = \Sigma_a \varpi_a \vartheta^{\epsilon_a}$$

take account of the distribution of *internal* energy. Here we mporarily treat each state as a separate system and the partition nctions of the separate states are connected by the equation

$$F_a/F_0 = \varpi_a \vartheta^{\epsilon_a}/\varpi_0 \vartheta^{\epsilon_0} \qquad \text{......(36).}$$

nserting this ratio in (35) and putting N $(= \Sigma_a N_a)$ for the total umber of special systems we have*

$$N_a = N \varpi_a \vartheta^{\epsilon_a} e^{-\Sigma_\beta N_\beta v_{a\beta}/V}/u(\vartheta) \qquad \text{......(37),}$$

here

$$u(\vartheta) = \Sigma_a' \varpi_a \vartheta^{\epsilon_a} e^{-\Sigma_\beta N_\beta v_{a\beta}/V} \qquad \text{......(38).}$$

he Σ_a' summation is of course over the states of the special ystems only.

In all calculations we have only to replace $b(\vartheta)$ and its terms y $u(\vartheta)$ and its terms in order to take full account of changes of xcluded volumes. If we use $u(\vartheta)$ in this way we may now take ll the separate states of the special systems together again as before. The dissociative equilibrium in such an assembly will be given by (0·3) etc. provided that $b(\vartheta)$ is replaced by $u(\vartheta)$ in the partition function for each species. This can be seen at once by reconstructing Ψ/k in terms of $u(\vartheta)$. We have, by (36)–(38),

$$\frac{F_a e^{-\Sigma_\beta N_\beta v_{a\beta}/V}}{N_a} = \frac{u(\vartheta)\, h(\vartheta)}{N} = \frac{F(\vartheta)}{N},$$

where $F(\vartheta)$ is the complete modified partition function. The terms under Σ_a in Ψ/k in (33) which belong to the special system become

$$N\left\{\log \frac{VF(\vartheta)}{N} + 1\right\} + \frac{1}{V} \Sigma_a' N_a (\Sigma_\beta N_\beta v_{a\beta}).$$

When every system has been treated in this way we find

$$\Psi/k = \Sigma_r N_r \left\{\log \frac{VF_r(\vartheta)}{N_r} + 1\right\} + \frac{1}{V} \Sigma_{a\beta} \frac{N_a N_\beta}{\sigma_{a\beta}} v_{a\beta} \qquad \text{...(39),}$$

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* Special cases of equations equivalent to (37) and (38) were first given in connection with high temperature atmospheres by Urey, *Astrophys. J.* vol. LIX, p. 1 (1924), and independently by Fermi, *Zeit. für Phys.* vol. XXVI, p. 54 (1924).

where Σ_r is a summation over the separate systems and $\Sigma_{\alpha\beta}$ remai a summation over every pair of states of the separate systems. I can be verified at once that in the equilibrium state

$$\frac{\partial \Psi}{\partial N_\alpha} = 0 \ (all\ \alpha) \qquad \text{......(40).}$$

Thus variations of N_r can be carried out without explici varying any N_α. The usual equilibrium distribution laws will b at once obtained in terms of $F_r(\vartheta)$, or $(u(\vartheta)\,h(\vartheta))_r$.

It must be borne in mind that all the foregoing formulae ar necessarily only correct to the first power in $1/V$, so that th *exponential* correcting factors are largely illusory. At the same tim the use of the formulae seems to be justifiable for rough quantitati work, right outside the range in which the corrections are smal and in fact for all orders. [It must be remembered that states fo which the correcting terms are large will *ipso facto* not appea effectively in the equilibrium state.] A closer examination of th point will moreover explain away a numerical discrepancy betwee the formulae of this section and those of Urey. Equation (33) ma equally well be written in the form

$$\Psi/k = \Sigma_\alpha N_\alpha \left\{\log \frac{(V - \frac{1}{2}\Sigma_\beta N_\beta v_{\alpha\beta})\,F_\alpha}{N_\alpha} + 1\right\} \qquad \text{......(41).}$$

Equations (33) and (41) have the same theoretical validity, bein both correct to terms in $1/V$. Equation (41) however is the exac form of Ψ/k for a mixture of gases which obeys exactly Van de Waals' equation

$$p = kT\Sigma \frac{N_\alpha}{V - b_\alpha} \qquad \text{......(42),}$$

where

$$b_\alpha = \tfrac{1}{2}\Sigma_\beta N_\beta v_{\alpha\beta} \qquad \text{......(43).}$$

If now we work out the equilibrium state by varying Ψ/k we fin that

$$\frac{(V-b_0)\,F_0}{N_0}\, e^{-\frac{1}{2}\Sigma_\beta N_\beta \frac{v_{0\beta}}{V-b_\beta}} = \frac{(V-b_\alpha)\,F_\alpha}{N_\alpha}\, e^{-\frac{1}{2}\Sigma_\beta N_\beta \frac{v_{\alpha\beta}}{V-b_\beta}} \quad \text{(44).}$$

To the first order in b/V (44) reduces to (35). Urey ignores th difference between $V - b_0$ and $V - b_\alpha$, and so gets a factor $\frac{1}{2}$ in his formula which is wrong for *small* b/V. This however is no great matter. For larger b/V, on identifying all the $V - b_\beta$, (44) reduce to

$$\frac{F_0}{N_0}\, e^{-\frac{1}{2}\frac{p}{kT}v_0^*} = \frac{F_\alpha}{N_\alpha}\, e^{-\frac{1}{2}\frac{p}{kT}v_\alpha^*} \qquad \text{......(45).}$$

where v_0^* and v_α^* are mean excluded volumes for the system in th states 0 and α. This is the form used by Urey. The form suggested e in (35) and (37) which is correct for small b/V differs from (45) en used for all b/V only in the omission of the factor $\frac{1}{2}$. This erence is of no importance as the formula cannot be numerically rect. What is important is that the approximate agreement of) and (45) justifies to some extent the use of (35) etc. for all in rough numerical calculations. For the reason given above e inaccuracies so introduced are not likely ever to be serious.

It may be noted in conclusion that similar formulae can be tained, of exactly the same validity, for the general case in which is given by (16), so that the excluded volumes are now functions the temperature, and may in fact be negative.

§ 7. *Inverse square law forces.* The only important long range rces ($s \leqslant 4$) which act between actual atoms and molecules pear to be gravitational and electrostatic forces, following the verse square law ($s = 2$). When such forces, and (or) external elds are acting, the arguments of § 5 are modified in the following anner. Down to equation (27) included the formulae are unffected. We must suppose of course that W includes the usual harp repulsive fields of small extent round each molecule or other ystem, as well as the gravitational or electrostatic potentials, o that it never happens that $W \to -\infty$. But after (27) we can o longer conclude that the coefficient of $N_a d\omega_a$ is independent of he coordinates of $d\omega_a$. Instead we can define a function w_0 by he equation

$$\Omega e^{-w_0 \log 1/\vartheta} = \int \underset{(V)}{\cdots} \int e^{-W_0 \log 1/\vartheta} \Pi_a{}' (d\omega_a)^{N_a} \qquad \ldots(46),$$

where Ω is a constant adjusted to make w_0 take any convenient assigned value at an assigned point. We then have

$$\bar{a}_a = N_a \Omega\, e^{-w_0 \log 1/\vartheta}\, d\omega_a / B(\vartheta) \qquad \ldots\ldots(47),$$

where
$$\Omega \int_V e^{-w_0 \log 1/\vartheta}\, d\omega_a = B(\vartheta).$$

Equation (47) reduces at once to the usual distribution law in a field of force, namely

$$\bar{\nu}_t = \bar{\nu}_0\, e^{-(w_t - w_0)/kT} \qquad \ldots\ldots(48),$$

where $\bar{\nu}_t$ and $\bar{\nu}_0$ are the average (equilibrium) concentrations of the a-molecules in different volume elements $d\omega_t$ and $d\omega_0$. The average potential energy defined by (46) is the potential for which alone Boltzmann's law (48) will be strictly true.

It is desirable to investigate the average potential energy w_0 and the average potential energy $W_{a\beta}$ of § 5 more closely, for it does not follow without further investigation that they agree with the value of W calculated when the rest of the assembly is in its average state. If we differentiate (46) with respect to x_0, y_0, z_0, th coordinates of $d\omega_\alpha$, we find

$$\frac{\partial w_0}{\partial x_0}\int\underset{(V)}{\cdots}\int e^{-W/kT}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha}=\int\underset{(V)}{\cdots}\int\frac{\partial W}{\partial x_0}\,e^{-W/kT}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha}\quad(49)$$

and two similar equations. In (49) W is of course the total potenti energy of the gas phase in any configuration, and therefo $-\partial W/\partial x_0$, $-\partial W/\partial y_0$, $-\partial W/\partial z_0$ are the force components actin on the selected α in $d\omega_\alpha$. The function w_0 is therefore so defin that its partial derivatives are the average values of the parti derivatives of W. In (49) however it is convenient to distingui between the parts of $\partial W/\partial x_0$ which arise from forces of long an short range. If these are distinguished by suffixes l and s so th $W = W_l + W_s$, it may happen that

$$\int\underset{(V)}{\cdots}\int\frac{\partial W_s}{\partial x_0}\,e^{-W/kT}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha}=0\qquad\ldots\ldots(50).$$

This will always happen when $W_l = 0$, and will continue to happe when $W_l \neq 0$, so long as no alterations in $\bar{\nu}_t$ are introduced by W sufficient to cause a failure in the perfect gas laws. When such a failure begins to occur W_s must contribute, and contribute in such a way as to account for the difference between (48) and the laws o hydrostatic equilibrium, which can always be written in the form

$$\frac{\partial p}{\partial s}=-\rho\,\frac{\partial w_0'}{\partial s}\qquad\ldots\ldots(51),$$

where w_0' is the potential energy per unit mass due to the lon range forces*.

Returning to the case in which (50) holds, we find that

$$\frac{\partial w_0}{\partial x_0}\int\underset{(V)}{\cdots}\int e^{-W/kT}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha}=\int\underset{(V)}{\cdots}\int\frac{\partial W_l}{\partial x_0}\,e^{-W/kT}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha}\qquad\ldots\ldots(52).$$

If we differentiate (52) we find

$$\left[\frac{\partial^2 w_0}{\partial x_0{}^2}-\frac{1}{kT}\left(\frac{\partial w_0}{\partial x_0}\right)^2\right]\int\underset{(V)}{\cdots}\int e^{-W/kT}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha}$$
$$=\int\underset{(V)}{\cdots}\int\left\{\frac{\partial^2 W_l}{\partial x_0{}^2}-\frac{1}{kT}\left(\frac{\partial W_l}{\partial x_0}\right)^2\right\}e^{-W/kT}\,\Pi_\alpha'\,(d\omega_\alpha)^{N_\alpha}.$$

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* It is interesting and easy to verify this equivalence by direct calculation as for terms of the first order in the deviations from the perfect gas laws. We have to identify (48) in the form

$$\frac{1}{\rho}\frac{\partial\rho}{\partial s}=-\frac{1}{kT}\frac{\partial w_0}{\partial s}$$

with (51), calculating directly p and the potential of a molecule inside the gas due to the intermolecular forces. [For the latter see, e.g., Fowler, *Phil. Mag.* vol. XLIII. p. 785 (1922), §§ 6, 7.]

ad two similar equations. By addition

$$\left.\nabla^2 w_0 - \frac{1}{kT}\Sigma\left(\frac{\partial w_0}{\partial x_0}\right)^2\right]\int\underset{(V)}{\cdots}\int e^{-W/kT}\,\Pi_a{}'\,(d\omega_a)^{N_a}$$

$$= \int\underset{(V)}{\cdots}\int\left[\nabla^2 W_l - \frac{1}{kT}\Sigma\left(\frac{\partial W_l}{\partial x_0}\right)^2\right] e^{-W/kT}\,\Pi_a{}'\,(d\omega_a)^{N_a} \quad \ldots(53).$$

ow in (53) W_l is in general a potential energy due to inverse square aw forces. It may therefore, after the usual local smoothing in ω_0, be taken to obey Poisson's equation

$$\nabla^2 W_l = \mu\rho,$$

vhere ρ is the smoothed local density of matter in $d\omega_0$ in any coniguration. The element $d\omega_0$ must now be supposed to be only "physically" small in the usual sense. The value of μ will depend on the precise mixture of electrostatic and gravitational potentials oncerned. We therefore find that w_0 satisfies the equation

$$\nabla^2 w_0 - \mu\bar{\rho} = -\frac{1}{kT}\Sigma\left\{\overline{\left(\frac{\partial W_l}{\partial x_0}\right)^2} - \left(\frac{\partial w_0}{\partial x_0}\right)^2\right\} \quad \ldots\ldots(54).$$

The terms on the right are the *fluctuation* in the resultant force on a molecule in $d\omega_0$ divided by $(-kT)$.

If this fluctuation can be ignored then the equation for w_0 reduces to Poisson's

$$\nabla^2 w_0 - \mu\bar{\rho} = 0 \quad \ldots\ldots(55).$$

Under these conditions the equilibrium state of the assembly may be calculated by the combined use of Boltzmann's and Poisson's equations (48) and (55)—a fertile procedure recently in frequent use. It seems however to be necessary to consider, with more care than is usually allowed, the conditions for the validity of this joint use of (48) and (55). I have not succeeded in obtaining any general conditions for the effective validity of (55), so that at present it seems to be only possible to proceed by the method of verification *a posteriori*. It does not seem to be possible to calculate this fluctuation explicitly but it may be legitimate to assume that the fluctuation is at least never much greater than $\Sigma\,(\partial w_0/\partial x_0)^2$, and that when only large scale forces are involved (not local intermolecular effects) the fluctuation will in general be very small compared to $\Sigma\,(\partial w_0/\partial x_0)^2$*. The fluctuation term arising from any rigidly fixed field of force, of origin external to the gaseous phase of the assembly, will of course be strictly zero, and can be absorbed at once into w_0 without further investigation.

As an example consider the equilibrium state of an isolated isothermal gravitating gas of which each small element is effectively perfect*. This problem has been treated by Milne† on the basis of equations (48) and (55). There is no explicit solution for ρ or w_0, but it is found that if r is the distance from the centre of the gravitating mass

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* At exceptional points $\Sigma\,(\partial w_0/\partial x_0)^2=0$ and the fluctuation of course does not vanish.

$$\rho \sim \frac{2}{r^2}\left(\frac{kT}{m}\right)\frac{1}{4\pi G};$$

G is the gravitational constant. It follows that, to the same approximation,

$$w_0 = 2kT \log r + const.$$

If then we compare the ignored term $(\partial w_0/\partial r)^2/kT$ with a term retained such as $\partial^2 w_0/dr^2$ we find that their numerical ratio is 2. If therefore we may assert that in general the fluctuation must be small compared to $(\partial w_0/\partial r)^2$, we can at once justify the neglect of the fluctuation in the customary analysis of the problem.

A similar example is the electron atmosphere in equilibrium with a metal at high temperature. If the form of the atmosphere is effectively that of the gap between the parallel plates of an infinite condenser the problem admits of the exact solution ‡

$$e^{w_0/2kT} = A \cos \frac{Bx}{\sqrt{(2kT)}},$$

where A and B are constants of which B depends on the electron density at a standard potential (zero). Here again the numerical ratio of the ignored term $(\partial w_0/\partial x_0)^2/kT$ and the retained term $\partial^2 w_0/\partial x_0^2$ is less than 2, and the same conclusions can be drawn if the same hypothesis is admitted.

A similar investigation can be made for the local field $W_{\alpha\beta}$ of equation (31), if it depends effectively on inverse square law forces. In this case the minuteness of gravitational forces from the molecular standpoint is such that their local effect in clustering the molecules can be ignored. Electrical forces however are large from the molecular standpoint, and, even when their long-range effects are zero owing to zero or negligible average space charges, may exercise a decisive control over local conditions by the establishment of a state of polarization near the selected ion or near the surface of a conducting solid. As before we may conclude that the $W_{\alpha\beta}$ of (31) will satisfy Poisson's equation—if not for the neighbourhood of a single ion at any rate on the average for a number of such neighbourhoods—when we have carried out the usual local smoothing and ignored the fluctuation. The volume elements near any one ion to which this smoothing must now be applied are small on the molecular scale, but the smoothing process can probably be taken to apply correctly to the sum of a large number of such neighbourhoods similarly situated with respect to a large number of similar ions. It does not however here seem possible to assert that the fluctuation in any one of these small regions will be small compared to the average electric intensity, and it therefore appears to me only safe to conclude that the fluctuation term is negligible in the analogue of (54) if $\frac{1}{kT}\Sigma\left(\frac{\partial W_{\alpha\beta}}{\partial x_0}\right)^2$ is itself small compared to $\partial^2 W_{\alpha\beta}/\partial x_0^2$, etc.

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* In order to discuss such an assembly completely it is necessary to suppose the problem somewhat idealized so that the mass of gas is contained in a reflecting enclosure so large that molecular impacts on the walls do not effectively alter the position of the centre of gravity of the mass of gas, or consequently its momentum, which must be fixed by the conditions of the problem. [This is not strictly realizable. Such conditions can be formally accounted for by additional selector variables both for momenta and positional coordinates.

† Milne, *Trans. Camb. Phil. Soc.* vol. XXII, p. 483 (1923).

‡ See, for example, Davison, *Phil. Mag.* vol. XLVII, p. 544 (1924).

Consider for example the polarization field round an ion in solution according to the recent successful theory of strong electrolytes initiated by Debye and Hückel*. Using Boltzmann's and Poisson's equations on the assumption that the square and higher powers of $W_{\alpha\beta}/kT$ can be neglected, they determine $W_{\alpha\beta}$, and hence the mean electrostatic potential V at the centre of any ion of charge ϵ. The extra term in E is determined at once as $\frac{1}{2}\Sigma\epsilon V$ and the extra term in Ψ by (19). They find (to come to details) that ($W_{\alpha\beta}$ is here a potential energy not an electrostatic potential)

$$W_{\alpha\beta}=\frac{z_\alpha z_\beta \epsilon^2}{D}\frac{1}{r}e^{-\kappa r},$$

in which $z_\alpha\epsilon$, $z_\beta\epsilon$ are the charges on the two ions in multiples of the charge on the electron, D the dielectric constant and κ a positive function of the temperature and concentrations given by

$$\kappa^2=\frac{4\pi\epsilon^2}{DkT}\Sigma_\alpha \frac{N_\alpha}{V}z_\alpha^2.$$

Now part of $W_{\alpha\beta}$, namely $z_\alpha z_\beta\epsilon^2/Dr$, is due to the charge of the α-ion itself and is non-fluctuating. The fluctuating part of the local field is therefore

$$W'=\frac{z_\alpha z_\beta \epsilon^2}{D}\frac{1}{r}(1-e^{-\kappa r}).$$

When we compare $\frac{1}{kT}\left(\frac{\partial W'}{\partial r}\right)^2$ with the retained term $\frac{\partial^2 W'}{\partial r^2}$ we find, after a little reduction, that the ratio is always less than W'/kT. This will necessarily be small when $W_{\alpha\beta}/kT$ is small. In so far therefore as the assumption of Debye and Hückel is adequate, the use of Poisson's equation appears to be legitimate. It seems to me however that it is no use including higher powers of $W_{\alpha\beta}/kT$

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* Debye and Hückel, *Phys. Zeit.* vol. XXIV, pp. 185, 305 (1923), and later papers.

in Poisson's equation in the discussion of purely *local* effects, for in so far as these further terms are relevant, it seems to me as if Poisson's equation itself may fail*. The fact that, owing to the field of the selected α-ion, $W_{\alpha\beta} \to \infty$ as $r \to 0$, is trivial and does not invalidate the argument, because such regions do not contribute effectively to the electrostatic energy, and must in any case be really excluded by the finite extension of the ions.

In view of the foregoing discussion, the region of validity of the Debye-Hückel theory may be taken to be the region in which

$$\frac{\kappa z_\alpha z_\beta \epsilon^2}{DkT} \quad (any\ \alpha,\ \beta)$$

is small. This however can often be somewhat simplified, for $W_{\alpha\beta}$ is only used directly to calculate U_e, the electrostatic energy of the assembly, and contributes thereto the term $\frac{1}{2}\kappa z_\alpha^2\epsilon^2/D$. A sufficient criterion of validity for general purposes will therefore be that

$$\frac{\frac{1}{2}\kappa\epsilon^2\Sigma_\alpha N_\alpha z_\alpha^2}{(\Sigma_\alpha N_\alpha)\, DkT} \text{ or } \frac{\sqrt{\pi}\epsilon^3 (\Sigma_\alpha N_\alpha z_\alpha^2)^{\frac{3}{2}}}{(\Sigma_\alpha N_\alpha)(DkT)^{\frac{3}{2}}\, V^{\frac{1}{2}}} \qquad \text{......(56)}$$

should be small. When (56) becomes comparable with unity the theory can only be used tentatively with great caution.

It is evident that the calculations apply at once with the same region of validity to assemblies of ionized gases. In fact in this case the application is the more certain because the calculations envisage every system in the assembly and do not relegate any effects to an arbitrarily specified dielectric constant D.

When the finite size of the ions is explicitly relevant it appears to me that the discussion of Debye and Hückel is hardly adequate. The sizes must be explicitly taken account of by short range forces, which may play an effective part in the distribution laws very near any selected ion, especially if the various ions are of different sizes. A rediscussion of this part of the theory, based explicitly on the partition function $B(\vartheta)$, might be of value, but cannot be undertaken here.

Apart however from these extensions there is one point in Debye's theory for point-ions which seems to invite further analysis—that is the use of (19) for deducing the contribution to Ψ from the electrostatic energy. This involves an integration with respect to T, in which D is regarded as a constant. This assumed constancy has been criticized, but must really be theoretically correct. This is perhaps easier to understand when viewed from the completely molecular standpoint. The factor D takes account the polarizability of the molecules of the solvent. Now the olarizability of a single molecule must be a molecular property ntirely independent of the temperature, and the temperature oefficient of D can only be a secondary effect due to changes in he numbers of polymerized molecules or changes in the relative umbers of molecules in different stationary states. Such changes re primarily changes in the numbers of the different species in he assembly and as such must always be held constant in the ntegration in question, and it is therefore theoretically correct to eep D also constant. It is however perhaps comforting to confirm his by direct calculation. Debye has confirmed the derived result or the osmotic pressure*. In the final section of this paper we onfirm it more generally by a direct alternative calculation of the ree energy itself.

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* Rosseland, *M.N.R.A.S.* vol. LXXXIV, p. 720 (1924), independently arrived at the same conclusion (p. 728). It must be necessary ultimately to fall back on direct calculations from $B(\vartheta)$, in the manner attempted by Milner, *Phil. Mag.* vol. XXIII, p. 551 (1912); vol. XXV, p. 742 (1913). It should be placed to Milner's credit that he gave the first correct explanation of the anomaly of strong electrolytes.

§ 8. *A direct calculation of the contributions of intermolecular orces to the free energy.* Fields of force, external or intermolecular, ontribute to the energy U and the free energy $F\ (= -T\Psi)$ of an ssembly. A familiar method of calculating such contributions to the *energy* is to suppose such fields of force to be gradually built up from zero, the various systems of the assembly being fixed in their average *final* positions. This is the origin of the familiar formula $\frac{1}{2}\int \rho W dv$ in the theory of attractions. A similar method of calculating contributions to the *free energy* is to suppose such fields of force to be gradually built up from zero, the various systems of the assembly being in their *equilibrium distributions at every stage.* Such a method is of general validity. It is clear that such a calculation gives the energy set free or taken up when the force centres are destroyed in a reversible process at constant temperature and volume and this is the change of free energy. It is not difficult to confirm this result using the general form (24) for $B(T)$. We shall be content here with simple special cases, which sufficiently illustrate the general argument.

(*a*) The partition function for the potential energy of a single molecule in an external field of force is

$$\int_V e^{-W/kT}\, dv,$$

and the extra contribution to Ψ/k for a perfect gas of N molecules is

$$\Psi_w/k = N \log\left\{\frac{1}{V}\int_V e^{-W/kT}\, dv\right\}.$$

The potential energy of the gas is (by the usual rule (0·4))

$$U_w = N\int_V W e^{-W/kT}\, dv \Big/ \int_V e^{-W/kT}\, dv,$$

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* Debye, *Recueil des Trav. chim. des Pays Bas*, Sér. IV, 4, p. 597 (1923).

which is obviously equal to

$$N\int_0^1 d\sigma \int_V We^{-W/kT}\,dv \Big/ \int_V e^{-W/kT}\,dv.$$

This is the expression for the work done when the force centres in each molecule are brought by fractions $d\sigma$ into the external field with the fixed *final* distribution of molecules. With equilibrium distributions at each stage, the work done however is

$$N\int_0^1 d\sigma \left\{\frac{\int_V We^{-\sigma W/kT}\,dv}{\int_V e^{-\sigma W/kT}\,dv}\right\},$$

$$= -kTN\int_0^1 d\sigma \frac{d}{d\sigma}\left\{\log \int_V e^{-\sigma W/kT}\,dv\right\},$$

$$= -T\Psi_w = F_w,$$

which is the result stated.

(*b*) The electrostatic potential, in Debye's theory, at the centre of an α-ion is $\kappa z_\alpha \epsilon/D$. If all charges are reduced to the fraction σ this potential is reduced by σ^2 since $\kappa \propto \epsilon$. In estimating energies with *fixed* distribution laws κ must be kept fixed, but in calculating the free energy κ must be allowed to vary. The electrostatic energy therefore contains the factor $\frac{1}{2}$ arising from $\int_0^1 \sigma\,d\sigma$; in the free energy this must be replaced by the factor $\int_0^1 \sigma^2\,d\sigma$, or $\frac{1}{3}$. It follows that the free energy is

$$-\frac{1}{3}\frac{\kappa\epsilon^2}{D}\Sigma_\alpha N_\alpha z_\alpha^{\,2},$$

and
$$\Psi_e/k = \frac{\epsilon^2\kappa}{3DkT}\Sigma_\alpha N_\alpha z_\alpha^{\,2},$$

which is the standard result. In this deduction however there is no possible question as to the temperature variation of D, which must be irrelevant to the formation of Ψ_e.

Note on the virial theorem. In connection with the foregoing discussion of electrostatic effects, this seems a suitable place to clear away an apparently mistaken criticism by Cavanagh[*] of Milner's[†] use of the virial in calculating the osmotic pressure of a strong electrolyte. Cavanagh assumes a certain incorrect but not impossible form for the virial term and shows that its thermodynamic consequences are improbable. This does not appear to me to be any argument whatever against the general validity of the use of the virial, or even against its use in the particular (unnatural) case he discusses. And in fact one can see by quite general arguments that the use of the virial to calculate the osmotic pressure is thermodynamically correct in a wide range of cases including the electrostatic virial here in question.

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[*] Cavanagh, *Phil. Mag.* vol. XLIII, p. 606 (1922), esp. p. 626.

[†] Milner, *loc. cit.* 2nd paper, p. 748.

Suppose that all the forces acting between the ions obey an inverse sth power law. By the virial theorem the extra term in the pressure is

$$\frac{1}{3V}\Sigma\Sigma r\,\phi(r)=\frac{s-1}{3V}U_s,$$

where U_s is the energy of the inter-ionic forces. This form of the pressure term can be confirmed by a purely thermodynamic argument combined with a similarity theorem. If all distances are multiplied by ρ, then all the potential energies are multiplied by $1/\rho^{s-1}$. If therefore at the same time the temperature is multiplied by $1/\rho^{s-1}$ the ratios of the kinetic and potential energies which determine the whole distribution laws will be unaltered and we can conclude that U_s/T is unaltered*. Hence

$$U_s/T=F(T^3V^{s-1}) \qquad \text{......(57).}$$

Now the extra term Ψ_s in the characteristic function is given by

$$\Psi_s=\int^T\frac{U_s}{T^2}\,dT=\int^T F(T^3V^{s-1})\frac{dT}{T}=\frac{1}{3}\int^{T^3V^{s-1}}F(x)\frac{dx}{x}.$$

Therefore

$$p_s=T\frac{\partial\Psi_s}{\partial V}=\frac{s-1}{3}\,\frac{T}{V}F(T^3V^{s-1})=\frac{s-1}{3V}U_s,$$

which is the virial.

Summary. The paper contains a general development of the theory of imperfect gases by means of partition functions. All the usual results of the theory of mixtures of imperfect gases can be simply deduced.

A discussion is given of the dissociative equilibria of imperfect gases, in which the results of Urey and Fermi on the stationary states of the hydrogen atom are considerably generalized.

The effects of long range forces (inverse square law) are discussed in detail, and the range of validity of the combined use of Boltzmann's and Poisson's equations carefully examined, with particular reference to the theory of strong electrolytes advanced by Debye and Hückel. It is concluded that the validity of this theory cannot be regarded as established except for "small" values of the ionic concentrations.

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* This similarity argument is well known for $s=2$, when it plays a great part in the theory of radiative equilibrium of gaseous stars (Eddington). It can be verified by direct calculation in any case for which calculations can be carried through.

An extended form of Kronecker's theorem with an application which shows that Burgers' theorem on adiabatic invariants is statistically true for an assembly.

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Summary.

The following paper is in two parts.

In Part I it is shown that Kronecker's theorem can be extended in the form:

If $| l + l_1\nu_1 + l_2\nu_2 + \dots + l_s\nu_s |$ *is greater than* η *for all sets of integers* $l_1, l_2 \dots l_s$ *less in absolute value than* K/ϵ *and not all zero,* l *also being an integer, then, for any* $x_1, x_2 \dots x_s$, *an integer* q *less than* $L/(\eta\epsilon^s)$ *can be found such that* $q\nu_1 - x_1, q\nu_2 - x_2 \dots q\nu_s - x_s$ *all differ from integers by less than* ϵ. K, L *depend only on* s.

It is an immediate corollary that if

$$| l_1\nu_1 + l_2\nu_2 + \dots + l_s\nu_s + l\nu |$$

is greater than ζ for all sets of integers $l_1, l_2 \dots l_s, l$ less in absolute value than K/ϵ and not all zero while $F(v_1, v_2 \dots v_s, v)$ is periodic, period 1, in $v_1, v_2 \dots v_s, v$, then T can be found between 0 and $L/(\zeta\epsilon^s)$ such that

$$\left| \int_0^1 \int_0^1 \dots \int_0^1 F(v_1 \dots v_s, v)\, dv_1 \dots dv_s dv - \frac{1}{T}\int_t^{t+T} F(v_{10} + \nu_1 t, \dots v_{s0} + \nu_s t, v_0 + \nu t)\, dt \right| < \epsilon N,$$

where K, L depend on s, and N on s and the bounds of

$$\partial F/\partial v_1 \dots \partial F/\partial v.$$

In Part II Burgers' theorem on adiabatic invariants is discussed with the help of this corollary.

It is concluded that if a particular system is taken, it may not be possible to fix a limit such that if the rate of change of the controllable parameter is less than this limit, the whole changes in the invariants will be less than some quantity given in advance.

It is shown, however, that the average change in the invariants for an assembly of systems sufficiently evenly distributed in their phase-space can be made less than a quantity given in advance.

RT I. *An extended form of Kronecker's theorem, with a corollary applying to certain dynamical problems.*

The extension is similar in its object to that given by Landau[*], t a more powerful result is simply arrived at.

Kronecker's theorem asserts that if $\nu_1, \nu_2 \dots \nu_s$ are any s real mbers, then there is a positive integer n not greater than ϵ^{-s} ch that $n\nu_1, n\nu_2 \dots n\nu_s$ all differ from integers by less than ϵ, and ere are arbitrarily large positive integers satisfying the same ndition; and that if further there is no relation with integer efficients $l_1, \dots l_s, l$, not all zero, of the form

$$l_1\nu_1 + l_2\nu_2 + \dots + l_s\nu_s + l = 0 \qquad \text{......(A)},$$

en for any s real numbers $x_1 \dots x_s$ there is an arbitrarily large ositive integer n such that $n\nu_1 - x_1, \dots n\nu_s - x_s$ all differ from ntegers by less than $\epsilon < 1$, while as $m \rightarrow \infty$ the number N of such ntegers n less than m satisfies

$$N \sim m\epsilon^s.$$

This may be expressed geometrically as follows. In s-dimensional space consider the points

$$(\nu_1 - {}_1n_1, \dots \nu_s - {}_1n_s), \quad (2\nu_1 - {}_2n_1, \dots 2\nu_s - \ n_s), \dots$$

where ${}_tn_r$ are integers so chosen that the points lie inside the unit hyper-cube, i.e. $0 \leqslant \nu_1 - {}_1n_1 < 1$, etc.

Then (1) at least one of these points after the first m is arbitrarily near (0, ... 0);

(2) provided there is no relation of the form (A), which would make all the points lie on one of the planes

$$l_1x_1 + l_2x_2 + \dots + l_sx_s + \lambda\mu = 0 \qquad \text{......(B)},$$

where λ is the H.C.F. of $l_1 \dots l_s, l$, and μ any integer, there are points after the first m arbitrarily near any point in the cube, and the points tend to fill the cube uniformly.

Questions arise as to the conditions that the points will ever fill the cube uniformly to a given degree of fine-grainedness and as to a limit to the number of points that must be taken for this to be so.

The answer to the first question is easy. Kronecker's theorem shows that the points fill the cube uniformly unless there is a condition of the form (A); if there is, the points fill the portions of the planes (B) within the cube uniformly unless there is another condition, and so on. There cannot be more than s independent conditions. The fine-grainedness depends on the distance apart of the planes, $\lambda/(l_1^2 + \dots + l_s^2)^{\frac{1}{2}}$.

The second question is more difficult and sufficient conditions only are obtained in this paper. If there are to be points of the series within ϵ of any point, no condition (A) must hold for which $\lambda/(l_1^2 + \dots + l_s^2)^{\frac{1}{2}}$ is greater than 2ϵ; further if such a condition (A) were nearly true a very large number of points of the series would have to be taken before one within ϵ of a point midway between the corresponding planes (B) would be found.

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[*] E. Landau, Scripta Universitatis atque Bibliothecae Hierosolymitanarum; 1923, *Über Diophantische Approximationen.*

This suggests a condition of some such form as

$$| l + l_1\nu_1 + \dots + l_s\nu_s | > \eta \qquad \dots\dots(C),$$

for all integers l, $l_1 \dots l_s$, not all zero, such that

$$| l_1 |, \dots | l_s | < K/\epsilon,$$

where η is small, and K depends only on s.

It will be shown that there is then an integer q less than $L/\eta\epsilon^s$, where L depends only on s, such that to an accuracy of ϵ the first q of these points form the vertices within the cube of equal s-dimensional parallelepipeda completely filling the cube and of largest diameter less than ϵ.

It may be that there is such an integer q less than L/η but this is neither proved nor disproved in this paper.

This is shown in the following way:

By the ordinary Kronecker's theorem integers p less than ϵ^{-s} and $p_1 \dots p_s$ can be found such that $| p\nu_r - p_r | < \epsilon$ $(r = 1 \dots s)$.

It follows easily that integers p less than $2/\eta\epsilon^{-s}$ and $p_1 \dots p_s$ can be found such that $| p\nu_r - p_r | < \epsilon$ and

$$| \nu_r - p_r/p | < \epsilon\eta \qquad (r = 1 \dots s).$$

From the last, and (C), $l_1p_1 + l_2p_2 + \dots + lp \neq 0$ for values of l considered in (C). ......(D).

It is easy to show that the points

$$(m\,(p_1/p) + m_1, \quad m\,(p_2/p) + m_2, \dots),$$

for $m = 0 \dots p - 1$ and $m_1 \dots m_s$ any integers, form the vertices of parallelepipeda filling all space, and (D) ensures that these parallelepipeda have dimensions less than a constant multiple of ϵ.

The required result then follows at once.

The extended form of Kronecker's theorem.

If $\nu_1, \nu_2 \dots \nu_s$ are any s real numbers such that

$$| l + l_1\nu_1 + l_2\nu_2 + \dots + l_s\nu_s | > \eta$$

when l, $l_1 \dots l_s$ are any integers, not all zero, such that

$$| l_1 |, | l_2 |, \dots | l_s | < \sqrt{s}\,(\tfrac{4}{3})^{\frac{1}{2}(s-1)}/\epsilon,$$

then integers q, $q_1 \dots q_s$, and s sets of s numbers $(\xi_r, \dots \xi_{rs})$, $(r = 1 \dots s)$ can be found such that:

1. $0 < q < 2^s s^{\frac{1}{2}s+1} \left(\dfrac{4}{3}\right)^{\frac{1}{2}(s-1)} \dfrac{1}{\eta\epsilon^s}$,

2. $| q\nu_r - q_r | < \frac{1}{2}\epsilon/\sqrt{s} \qquad (r = 1 \dots s)$,

3. $\xi_{r1}^2 + \xi_{r2}^2 + \dots + \xi_{rs}^2 < \epsilon^2/s$,

4. *For every set of s numbers $x_1 \dots x_s$, one and only one set of:*
   s integers $m_1 \dots m_s$;
   an integer m such that $0 \leqslant m < q$;
   s numbers $y_1 \dots y_s$ such that $-\frac{1}{2} < y_r \leqslant \frac{1}{2}$ $(r = 1 \dots s)$,
   can be found for which

$$x_r = m_r + m\frac{q_r}{q} + \xi_{1r}y_1 + \xi_{2r}y_2 + \dots + \xi_{sr}y_s \qquad (r = 1 \dots s)$$

and a fortiori

$$\sum_{r=1}^{s} (x_r - m_r - m\nu_r)^2 < \epsilon^2.$$

Let $\{a\} = a - n$ where n is an integer and $-\frac{1}{2} < \{a\} \leqslant \frac{1}{2}$.

Lemma 1. *If $\nu_1, \nu_2 \dots \nu_s$ are any s real numbers, and ϵ any positive number less than* 1, *then there is a positive integer p such that integers $p_1 \dots p_s$ can be found for which $| p\nu_r - p_r | < \epsilon$ $(r = 1 \dots s)$ and $0 < p \leqslant \epsilon^{-s}$.*

Proof. If N is the integer next greater than ϵ^{-s}, there must be two, m, n, of the integers 0, 1, … $N - 1$, such that real numbers $x_1 \dots x_s$, $y_1 \dots y_s$ and integers $p_1 \dots p_s$ exist satisfying

$$\left.\begin{array}{c} m\nu_r + \epsilon > x_r > m\nu_r - \epsilon \\ m\nu_r + \epsilon > y_r > n\nu_r - \epsilon \\ x_r - y_r = p_r \end{array}\right\} \quad (r = 1 \dots s),$$

since otherwise $m\nu_1 + \epsilon > x_r > m\nu_r - \epsilon$, $r = 1 \dots s$, for $m = 0$, $1 \dots N - 1$, define N non-overlapping intervals

$$\tfrac{1}{2} > \{x_r\} > -\tfrac{1}{2} \qquad (r = 1 \dots s),$$

each of size ϵ^s, which contradicts $N > \epsilon^{-s}$.

Clearly $p = | m - n |$ satisfies the conditions imposed.

Lemma 2. *If $\nu_1, \nu_2 \dots \nu_s$ are any s real numbers, and ϵ, η any positive numbers less than* 1, *then there is a positive integer p such that integers $p_1 \dots p_s$ can be found for which*

$$| p\nu_r - p_r | < \epsilon \qquad (r = 1 \dots s) \qquad \dots\dots(2{\cdot}1),$$

$$| p\nu_r - p_r | < p\epsilon\eta \qquad (r = 1 \dots s) \qquad \dots\dots(2{\cdot}2),$$

$$0 < p < 2/(\eta\epsilon^s) \qquad \dots\dots(2{\cdot}3).$$

Proof. If the value of p given by Lemma 1, which satisfies (2.1) and (2.3), also satisfies (2.2), then Lemma 2 is true.

If not, let q be the largest positive integer for which integers $q_1 \dots q_s$ can be found so that

$$|q\nu_r - q_r| < \epsilon \qquad (r = 1 \dots s),$$

and for some r $$|q\nu_r - q_r| > q\epsilon\eta,$$

involving $q < 1/\eta$.

Let $$\lambda_r = 2(q\nu_r - q_r) \qquad (r = 1 \dots s),$$ $$p = 2lq,$$ $$p_r = 2lq_r + l_r \qquad (r = 1 \dots s),$$

where $l, l_1 \dots l_s$ are integers found by Lemma 1 such that

$$|l\lambda_r - l_r| < \epsilon \qquad (r = 1 \dots s),$$ $$1 \leqslant l < \epsilon^{-s}.$$

Then $$|p\nu_r - p_r| = |l\lambda_r - l_r| < \epsilon \qquad (r = 1 \dots s).$$

Since $p > q$ it follows that $|p\nu_r - p_r| < p\epsilon\eta$ $(r = 1 \dots s)$, while as required $p = 2lq < 2/(\eta\epsilon^s)$.

Lemma 3. *If* $\nu_1, \nu_2 \dots \nu_s$ *are* s *real numbers such that*

$$\left|\sum_{r=1}^{s} l_r\nu_r + l\right| > k\eta,$$

when $l, l_1 \dots l_s$ *are integers, not all zero, such that*

$$|l_r| < k/(s\epsilon) \qquad (r = 1 \dots s),$$

then integers $q, q_1 \dots q_s$ *can be found, with no common factor, such that*

$$0 < q < 2/\eta\epsilon^s \qquad \text{......(3·1)},$$

$$|q\nu_r - q_r| < \epsilon \qquad (r = 1 \dots s) \quad (3·2),$$

$$\left(\sum_{r=1}^{s} l_r q_r\right) + lq \neq 0 \qquad \text{......(3·3)},$$

when $l, l_1 \dots l_s$ *are integers, not all zero, such that*

$$|l_r| < k/(s\epsilon) \qquad (r = 1 \dots s).$$

Proof. It is sufficient to take $q, q_1 \dots q_s$ to be $p, p_1 \dots p_s$ given by Lemma 2, divided out by any common factor, noticing that

$$\left|\sum_{r=1}^{s} l_r(\nu_r - (q_r/q))\right| < k/(s\epsilon) \,.\, s \,.\, (q\epsilon\eta)/q = k\eta.$$

Lemma 4. *If* $q, q_1 \dots q_s$ *are integers such that*

$$\left(\sum_{r=1}^{s} l_r q_r\right) + lq \neq 0,$$

when $l, l_1 \dots l_s$ *are integers, not all zero, satisfying* $|l_r| < k/(s\epsilon)$ *(Hyp.)*, n, *whatever integers, not all zero,* $k_1 \dots k_s$ *may be, integers* $m, m_1 \dots m_s$ *be found so that*

$$0<\frac{\sum\limits_{r=1}^{s}\left(m\frac{q_r}{q}+m_r\right)k_r}{\left(\sum\limits_{r=1}^{s}k_r^2\right)^{\frac{1}{2}}}<\frac{s}{k}\epsilon. \qquad (Concl.)$$

Proof. If $\sum\limits_{r=1}^{s} k_r q_r = 0$, and h is the highest common factor of $\dots k_s$, then $m_1 \dots m_s$ can be chosen so that

$$\sum_{r=1}^{s}\left(m\frac{q_r}{q}+m_r\right)k_r=\sum_{r=1}^{s}m_r k_r=h,$$

hile from (*Hyp.*), since $\sum\limits_{r=1}^{s}(k_r/h)\,q_r=0$, one at least of $k_1 \dots k_s$ is reater than $hk/(s\epsilon)$, whence (*Concl.*) follows.

If $\sum\limits_{r=1}^{s} k_r q_r \neq 0$, and h is the highest common factor of $\sum\limits_{r=1}^{s} k_r q_r$ nd $qk_1, \dots qk_s$, then $m, m_1, \dots m_s$ can be chosen so that

$$\sum_{r=1}^{s}\left(m\frac{q_r}{q}+m_r\right)k_r=\frac{h}{q},$$

vhile from (*Hyp.*), since $\sum\limits_{r=1}^{s}(qk_r/h)\,q_r-\left(\sum\limits_{r=1}^{s}k_r q_r\right)\Big/h\,.\,q\doteq 0$, one at east of $(q/h)\,k_1 \dots (q/h)\,k_s$ is greater than $k/(s\epsilon)$, whence (*Concl.*) follows.

Lemma 5. *If* $q, q_1 \dots q_s$ *are integers without a common factor, and* S *denotes the set of s-dimensional points*

$$(p\,(q_1/q)+p_1,\ p\,(q_2/q)+p_2 \cdots p\,(q_s/q)+p_s),$$

where p *is* $0, 1 \dots q-1$ *and* $p_1 \dots p_s$ *are any integers, then* s *points of* S, $P_r \equiv (\xi_{r1}, \dots \xi_{rs})$ $(r = 1 \dots s)$ *can be chosen so that*

$$OP_r \leqslant \left(\tfrac{4}{3}\right)^{\frac{1}{4}(s-1)}d \qquad (r=1 \dots s) \qquad \dots\dots(5{\cdot}1),$$

where $OP_r^2=(\xi_{r1}^2+\dots+\xi_{rs}^2)\neq 0$ (*O being the point* $(0, \dots 0)$), *and* d *is the perpendicular distance of* P_s *from the flat* $(s-1)$*-fold* $OP_1 \dots P_{s-1}$; *and further the set* Σ *of points*

$$m_1P_1+m_2P_2+\dots+m_sP_s,\ \text{i.e.}\ (\Sigma m_r\xi_{r1}, \dots \Sigma m_r\xi_{rs}),$$

where $m_1 \dots m_s$ *are any integers, is the same as the set* S.

Proof. The nearest point not O of S to O must be further from O than $1/q$.

Let P_1 be as near as any other of these points to O.

Project all the points of S on the flat $(s-1)$-fold through O perpendicular to OP_1.

Let ${}_2P_2$, the projection of P_2', be as near to O as any other these points, not O.

Then $P_2' = {}_2P_2 + {}_2m_1P_1 + {}_2\alpha_1P_1$, where ${}_2m_1$ is an integer a $-\frac{1}{2} < {}_2\alpha_1 \leqslant \frac{1}{2}$.

Let $P_2 = {}_2P_2 + {}_2\alpha_1P_1$; clearly a point of S.

Project all points of S on the flat $(s-2)$-fold through O perpendicular to OP_1P_2.

Let ${}_3P_3$ be as near as any of these points, not O, to O.

A point of S, of which ${}_3P_3$ is the projection,

$$P_3 = {}_3P_3 + {}_3\alpha_2\,{}_2P_2 + {}_3\alpha_1P_1,$$

where $-\frac{1}{2} < {}_3\alpha_2 \leqslant \frac{1}{2}$, $-\frac{1}{2} < {}_3\alpha_1 \leqslant \frac{1}{2}$ can be found.

In this way $P_4 \dots P_s$ can be defined in succession;

$$P_r = {}_rP_r + {}_r\alpha_{r-1}\,{}_{r-1}P_{r-1} + \dots + {}_r\alpha_1P_1 \qquad (r = 2 \dots s),$$

so that O_sP_s, being the perpendicular distance of P_s from $OP_1 \dots P_{s-1}$, is d.

From these definitions

$$O_rP_r^2 \leqslant O_{r+t}P_{r+t}^2 + {}_{r+t}\alpha_{r+t-1}^2 O_{r+t-1}P_{r+t-1}^2 + \dots + {}_{r+t}\alpha_r^2 O_rP_r^2,$$

so that $$O_{s-1}P_{s-1}^2 \leqslant O_sP_s^2 + {}_s\alpha_{s-1}^2 O_{s-1}P_{s-1}^2,$$

whence $$O_{s-1}P_{s-1}^2 \leqslant \tfrac{4}{3}O_sP_s^2 = \tfrac{4}{3}d^2,$$

and $$OP_{s-1}^2 = O_sP_s^2 + {}_s\alpha_{s-1}^2 O_{s-1}P_{s-1}^2 \leqslant \tfrac{4}{3}O_sP_s^2 = \tfrac{4}{3}d^2,$$

and so on, obtaining in succession

$$OP_{s-2}^2 \leqslant (\tfrac{4}{3})^2 d^2, \dots, OP_1^2 \leqslant (\tfrac{4}{3})^{s-1} d^2.$$

This proves (5·1).

Any point P is of the form $\sum_{r=1}^{s} (m_r + \alpha_r) P_r$ where m_r is an integer, $-\frac{1}{2} < \alpha_r \leqslant \frac{1}{2}$. Thus if P is a point of S, so is $\sum_{r=1}^{s} \alpha_rP_r$, P' say; $\alpha_1 \dots \alpha_s$ must then all be zero, for if not, let α_r be the last not zero. Projecting perpendicular to $OP_1 \dots P_{r-1}$ the projection of P' is nearer to O than ${}_rP_r$, contradicting the definition of ${}_rP_r$.

Thus any point of S is a point of Σ.

Clearly any point of Σ is a point of S.

Proof of the main theorem.

To obtain the extended form of Kronecker's theorem it is only necessary to combine Lemmas 3, 4 and 5, noticing that since $q, q_1 \dots q_s$ have no common factor, all the points of S are different, so that any point $(x_1 \dots x_s) = P$ can be put in the form

$$P = \sum_{r=1}^{s} (n_r + y_r) P_r,$$

where $n_1 \dots n_s$ are integers and $-\frac{1}{2} < y_r \leqslant \frac{1}{2}$ $(r = 1 \dots s)$, in one d only one way, while $\sum_{r=1}^{s} n_r P_r$, a point of Σ, is a unique point $(q_1/q) + m_1, \dots m(q_s/q) + m_s)$ of S, m being 0, 1 … or $q-1$ and $\dots m_s$ integers.

Noticing also that Lemma 4 requires $d < (s/k)\,\epsilon$, when in the mbination ϵ is replaced by $\frac{1}{2}\epsilon/\sqrt{s}$, k by $\frac{1}{2}s\,(\frac{4}{3})^{\frac{1}{2}(s-1)}$ and η by $k = (2\eta/s)\,(\frac{3}{4})^{\frac{1}{2}(s-1)}$, the theorem as stated is obtained.

In problems dealing with multiply-periodic dynamical systems me-averages are often approximated to by replacing them by verages over the variables in which the system is periodic.

A corollary of the extended form of Kronecker's theorem gives recise conditions under which an assigned degree of approximaion can be obtained.

Corollary to the extended form of Kronecker's theorem.

If $F(v_1 \dots v_s, v)$ *is periodic of period* 1 *in* $v_1 \dots v_s, v$, *if for all* $\dots v_s, v$,

$$\left|\frac{\partial F}{\partial v_1}\right|, \dots \left|\frac{\partial F}{\partial v_s}\right|, \left|\frac{\partial F}{\partial v}\right| < M,$$

nd if $\nu_1 \dots \nu_s, \nu$ *are any* $s+1$ *real numbers such that*

$$|\, l_1\nu_1 + \dots + l_s\nu_s + l\nu \,| > \zeta,$$

when $l_1, \dots l_s, l$ *are any integers, not all zero, such that*

$$|\, l_1 \,|, \dots |\, l_s \,|, |\, l \,| < \sqrt{s}\,(\tfrac{4}{3})^{\frac{1}{2}(s-1)}/\epsilon,$$

then a number T *can be found such that*

$$0 < T < 2^s s^{\frac{1}{2}s+1}\,(\tfrac{4}{3})^{\frac{1}{2}(s-1)}\,\frac{1}{\zeta\epsilon^s},$$

and

$$\left|\int_0^1\int_0^1 \dots \int_0^1 F(v_1 \dots v_s v)\,dv_1 \dots dv_s dv \right.$$
$$\left. - \frac{1}{T}\int_t^{t+T} F(v_{10} + \nu_1 t, \dots v_{s0} + \nu_s t, v_0 + \nu t)\,dt \right| < \epsilon M s^{\frac{3}{2}}.$$

Proof. Suppose ν is the greatest of $\nu_1 \dots \nu_s, \nu$.

In the extended form of Kronecker's theorem replace $\nu_1 \dots \nu_s$ by $\nu_1/\nu \dots \nu_s/\nu$ and η by $\zeta/|\,\nu\,|$; its hypothesis then follows from the above hypothesis.

Take T to be $q/|\,\nu\,|$.

Then
$$0 < T < 2^s s^{\frac{1}{2}s+1}\,(\tfrac{4}{3})^{\frac{1}{2}(s-1)}\,\frac{1}{\zeta\epsilon^s},$$

and

$$\int_0^1\int_0^1 \dots \int_0^1 F(v_1 \dots v_s v)\,dv_1 \dots dv_s dv = \int_v^{v+q}\left\{\iint_D F\,dv_1 \dots dv_s\right\} dv,$$

where D is given by $-\frac{1}{2} < y_r \leqslant \frac{1}{2}$ $(r = 1 \dots s)$, where

$$v_r = v\,(q_r/q) + \xi_{1r}y_1 + \dots + \xi_{sr}y_s + v_{r0},$$

since the ranges correspond, corresponding points having the same value for F. Thus

$$\int_0^1 \int_0^1 \dots \int_0^1 F(v_1 \dots v_s v)\, dv_1 \dots dv_s dv$$
$$= \frac{1}{q} \int_v^{v+q} F\left(v_{10} + v\frac{q_1}{q}, \dots, v\right) dv + R,$$

where
$$|R| < \tfrac{1}{2} M \sum_{p,q} |\xi_{pq}| < \tfrac{1}{2} M \epsilon s^{\frac{3}{2}}.$$

Thus

$$\int_0^1 \int_0^1 \dots \int_0^1 F(v_1 \dots v_s v)\, dv_1 \dots dv_s dv$$
$$= \frac{1}{T} \int_t^{t+T} F(v_{10} + \nu_1 t, \dots, v_0 + \nu t)\, dt + R + S,$$

where, since
$$|q(\nu_r/\nu) - q_r| < \tfrac{1}{2}\epsilon/\sqrt{s}, \qquad (r = 1 \dots s)$$
$$|S| < \tfrac{1}{2} M \epsilon s^{\frac{3}{2}},$$

which proves the corollary.

PART II. *On Burgers' theorem on adiabatic invariants of Hamiltonian systems.*

One of the difficulties in Burgers'[*] treatment of adiabatic invariants is that in general near any state of a multiply-periodic system there are states in which the frequencies are connected by a linear relation with integer coefficients, and it appears that near such states the variation must be carried out arbitrarily slowly for the rates of change of the invariants to average out to any definite approximation; in fact it remains a question whether the changes in the invariants for a given change in the controllable parameter can be made infinitesimal by making the rate of change of the parameter infinitesimal.

Dirac[†] has obtained conditions under which this can be done; in general these conditions will not hold everywhere; Laue[‡] has also considered the question. In this paper the problem will be discussed with the aid of the form of Kronecker's theorem proved in Part I, which simplifies it. Burgers' theorem will be proved to hold statistically under conditions that would hold for systems given by any ordinary functions and not becoming degenerate for a finite range of the parameter. This is really all that is required in the application to quantum theory. Assemblies of quantized systems have to obey the second law of thermodynamics and to imitate classical assemblies in the limit of high quantum numbers. Thus the quantum conditions must be such that they are, statistic-ly, adiabatic invariants, but need not necessarily be so for single stems.

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[*] Burgers, *Proc. Amsterdam Roy. Acad. of Sciences*, 20, p. 163.

[†] P. A. M. Dirac, *Proc. Roy. Soc.* A, 107, p. 725.

[‡] M. Laue, *Ann. der Physik*, 4te Folge, 76, p. 619.

If the total change in the controllable parameter "a" is first xed, the total change in an invariant I_r can be made of order ϵ r those parts of the trajectory of the system in "(I, a)-space" r which $|\Sigma l_r \nu_r| > \eta$, where "$\nu_r$" are the frequencies and l_r are ny integers, not all zero, less than a constant multiple of $1/\epsilon$, by king $\dot{a}$ constant and less than some constant multiple of $\eta . \epsilon^s$—here being s frequencies.

Fixing ϵ, it is only necessary to make the total change in I_r, for arts of the trajectory for which one or other of the finite number f inequalities $|\Sigma l_r \nu_r| < \eta$ holds, of order ϵ. The question is whether his can be done by making η, and so $\dot{a}$, small enough.

Only one of the inequalities need be considered, and can be aken, by a transformation of variables, to be $|\nu_1| > \lambda\eta$, where λ s some constant. The argument can easily be extended to the case where several of the inequalities fail together. It is clear that only changes in I are now in question and we may average over all the angle-variables other than v_1 and get equations, dropping the suffix, of the form

$$\left.\begin{aligned} \dot{I} &= -\dot{a}\frac{\partial}{\partial v}F(a, I, v) \\ \dot{v} &= \nu(a, I) + \dot{a}\frac{\partial}{\partial I}F(a, I, v) \end{aligned}\right\} \qquad \text{......(A)},$$

where $$F(a, I, v+1) = F(a, I, v).$$

As $\dot{a} \rightarrow 0$ the length of the trajectory within $|\nu| < \eta$ will tend to zero unless either

1. $\nu(a, I) = 0$ for some fixed I and a range of a—clearly this will not make I change;
2. the equations

$$\left.\begin{aligned} \nu(a, I) &= 0 \\ \frac{\partial \nu}{\partial a} &= \frac{\partial \nu}{\partial I}\frac{\partial F}{\partial v} \end{aligned}\right\}$$

both hold for some values of (a, I, v) in the range considered. This is equivalent to Dirac's condition for this case. It is the condition that the solutions of

$$\left.\begin{aligned} \frac{dI}{da} &= -\frac{\partial F}{\partial v} \\ \frac{dv}{da} &= \frac{\partial F}{\partial I} \end{aligned}\right\} \qquad \text{......(B)}$$

do not definitely cross $\nu(a, I) = 0$; if they do, what is required can be proved, for it is easy to show that within $|\nu| < \eta$ the difference between solutions of (A) and (B) starting at the same place is less than a constant multiple of the product of $\delta a/\dot{a}$ and the total variation of ν, i.e. of $\eta^2/\dot{a}$, δa being the change in a, and this can be made small compared with η.

An example in which 2. holds and in which I varies can easily be constructed. Suppose $F(a, I, v) = \cos v + \lambda I$, $\nu(a, I) = I + a$, $\dot{a} = \epsilon$, so that the equations become

$$\left.\begin{aligned} \frac{dI}{da} &= \sin v \\ \frac{dv}{da} &= \frac{1}{\epsilon}(I + a) + \lambda \end{aligned}\right\},$$

and
$$a = \int \frac{dv}{\{2/\epsilon\,(v - \cos v + c)\}^{\frac{1}{2}}} + a_0,$$
$$I = \{2\epsilon\,(v - \cos v + c)\}^{\frac{1}{2}} - a - \lambda\epsilon,$$

where a_0, c are constants of integration.

Suppose a starts at some value for which $I + a$ is negative and increases. The negative sign of the radical must be taken, and v decreases till $v - \cos v + c = 0$, when $I + a = -\lambda\epsilon$, dv/da changes sign, and thereafter v increases. It is easy to see that the values of I for large negative and positive values of a are the same and nearly constant. If however $v - \cos v + c = 0$ for the value of v satisfying

$$\frac{d}{dv}(v - \cos v) = 0,$$

then, as a increases, v tends to, but never reaches this value, and

$$I \sim -a - \lambda\epsilon.$$

Thus if initially I, a, v are such that $c = \frac{\pi}{2} + n\pi$, where n is an integer, I will depart widely from its initial value. c depends on ϵ as well as on I, a, and v, but whatever ϵ, I and a, v can be found so that $c = \frac{\pi}{2} + n\pi$.

The state of affairs is similar when two or more conditions $|\Sigma l_r \nu_r| < \eta$ hold at once. If a particular system is taken, and if initially I_r, v_r, a are fixed, then it may be that however small η is, ϵ can be found less than η and such that the final values of I_r differ from their initial values by more than some ΔI_r given in advance.

Nevertheless it is clear that the average changes in I_r can be made as small as we like in an assembly of similar systems sufficient in number for the state of the assembly to be represented sufficiently closely by a continuous density in its phase-space, by taking $\dot{a}$, the same for each system, constant and sufficiently small. A formal proof of this follows, but a simple case will be first considered.

Take the equations for a system with one degree of freedom in the form

$$\left.\begin{aligned} \dot{I} &= -\dot{a}\frac{\partial}{\partial v}F(I, a, v) \\ \dot{v} &= \nu(I, a) + \dot{a}\frac{\partial}{\partial I}F(I, a, v) \end{aligned}\right\},$$

where $$F(I, a, v+1) = F(I, a, v).$$

Suppose $\dot{a} = \epsilon$ independent of a, and

$$|\nu(I, a)| > \eta \qquad \text{......(0·1)},$$

and consider the motion from $I = I_0$, $v = v_0$, $a = a_0$ to $a = a_0 + \delta a$.

Then $$|I - I_0| = O(\delta a),$$

which means $|I - I_0| = \lambda\delta a$, where $|\lambda|$ is less than some constant, so that $|v - v_0 - \nu(I_0, a_0).(a - a_0)/\epsilon| = O(\delta a) + O((\delta a)^2/\epsilon)$.

Thus $$I_{a_0+\delta a} - I_0 = -\int_{a_0}^{a_0+\delta a}\frac{\partial}{\partial v}F(I, a, v)\,da,$$

so that

$$\begin{aligned} |I_{a_0+\delta a} - I_0| &< O\{(\delta a)^2\} + O\{(\delta a)^3/\epsilon\} \\ &\quad + \left|\int_{a_0}^{a_0+\delta a}\frac{\partial}{\partial v}F\left(I_0, a_0, v_0 + \frac{a - a_0}{\epsilon}\nu(I_0, a_0)\right)da\right| \qquad \text{...(0·2)} \\ &= O\{(\delta a)^2\} + O\{(\delta a)^3/\epsilon\} + O\{\epsilon/\eta\} \qquad \text{......(0·3)}. \end{aligned}$$

Now suppose δa is much smaller than Δa, and divide Δa into many intervals like δa, then clearly

$$|I_{a_0+\Delta a} - I_0| = O\{\delta a.\Delta a\} + O\{\Delta a\,(\delta a)^2/\epsilon\} + O\{\Delta a.\epsilon/(\eta\delta a)\},$$

i.e. $$\left|\frac{\Delta I}{\Delta a}\right| = O(\delta a) + O\{(\delta a)^2/\epsilon\} + O\{\epsilon/(\eta\delta a)\}.$$

Any Δa and η having been fixed, δa and ϵ can be chosen so that $|\Delta I/\Delta a|$ is arbitrarily small, i.e. Δa and η being fixed, ϵ can be chosen so that $\Delta I/\Delta a$ is arbitrarily small for $|\nu| > \eta$.

Now consider an assembly with density of representative points $M(v, I, a)$ in (v, I)-space. Along a trajectory M does not vary. By taking η small enough we may suppose that $|\nu| < \eta$ includes only an infinitesimal proportion of (I, a)-space, and that even there $|\Delta I/\Delta a|$ is less than some constant.

Thus the average change in I for the systems,

$$\frac{\iint M\,\Delta I\,dI\,dv}{\iint M\,dI\,dv},$$

can be made infinitesimal by taking ϵ small enough.

The chief points in the extension of this proof are the followin

$\dot{a}$ need not be independent of a if $\ddot{a}/\dot{a}^2$ is bounded;

a transformation is made such that M is no longer consta along a trajectory, but log M is bounded;

as the systems are multiply-periodic, the extended form Kronecker's theorem is used in the transition from (0·2) to (0·3), an it is essential that, after fixing a limit to $\Delta I/\Delta a$ for which a definit number of conditions like (0·1) are required, Δa can still be take as small as required;

lastly, before all the above argument is applied, the range a is cut up into small pieces in order to limit the proportion systems that can get out of the ranges of I for which the hypothe are supposed to hold.

A proof of Burgers' theorem considered as a statistical theorem applying to an assembly.

1. Suppose the equations of motion of a system depending a controllable parameter a are

$$\left.\begin{aligned} \dot{p}_r &= -\frac{\partial}{\partial q_r} H(p, q, a) \\ \dot{q}_r &= \frac{\partial}{\partial p_r} H(p, q, a) \end{aligned}\right\} \quad (r = 1 \ldots s) \qquad \ldots\ldots(1\cdot1),$$

and that the contact transformation given by

$$\left.\begin{aligned} p_r &= \frac{\partial}{\partial q_r} W(q, I', \alpha', a) \quad (r = 1 \ldots s) \\ v_r &= \frac{\partial}{\partial I_r'} W(q, I', \alpha', a) \quad (r = 1 \ldots u) \\ \beta_r' &= \frac{\partial}{\partial \alpha_r'} W(q, I', \alpha', a) \quad (r = 1 \ldots s-u) \end{aligned}\right\} \qquad \ldots\ldots(1\cdot2)$$

makes (p, q) periodic functions with period 1 of $v_1 \ldots v_u$, and $H(p, q, a) = H(I', a)$, so that equations (1·1) transform to

$$\left.\begin{aligned} &\left.\begin{aligned} \dot{I}_r' &= -\dot{a}\frac{\partial}{\partial v_r}\left(\frac{\partial W}{\partial a}\right) \\ \dot{v}_r &= \frac{\partial H}{\partial I_r'} + \dot{a}\frac{\partial}{\partial I_r'}\left(\frac{\partial W}{\partial a}\right) \end{aligned}\right\} \quad (r = 1 \ldots u) \\ &\left.\begin{aligned} \dot{\alpha}_r' &= -\dot{a}\frac{\partial}{\partial \beta_r'}\left(\frac{\partial W}{\partial a}\right) \\ \dot{\beta}_r' &= \dot{a}\frac{\partial}{\partial \alpha_r'}\left(\frac{\partial W}{\partial a}\right) \end{aligned}\right\} \quad (r = 1 \ldots s-u) \end{aligned}\right\} \quad \ldots(1\cdot3),$$

where $$\left(\frac{\partial W}{\partial a}\right) = \frac{\partial}{\partial a} W(q, I', \alpha', a),$$

but is expressed in terms of I_r', α_r', v_r, β_r'.

Clearly $\dot{I}_r'$, $\dot{v}_r$, $\dot{\alpha}_r'$, $\dot{\beta}_r'$ must depend only on p_r, q_r, a and so be periodic functions with period 1 of $v_1 \dots v_u$, i.e.

$$\frac{\partial}{\partial \alpha_r'}, \ \frac{\partial}{\partial \beta_r'}, \ \frac{\partial}{\partial I_r'}, \ \frac{\partial}{\partial v_r} \left(\frac{\partial W}{\partial a}\right)$$

are also periodic, so that

$$\left(\frac{\partial W}{\partial a}\right) = F + \sum_1^u f_r(a)\, v_r,$$

where F is also periodic.

If now $I_r = I_r' + \int^a f_r(a)\, da$ $(r = 1 \dots u)$, the equations become

$$\left.\begin{aligned} \dot{I}_r &= -\dot{a}\frac{\partial F}{\partial v_r} \\ \dot{v}_r &= \frac{\partial H}{\partial I_r} + \dot{a}\frac{\partial F}{\partial I_r} \end{aligned}\right\} r = 1 \dots u, \qquad \left.\begin{aligned} \dot{\alpha}_r' &= -\dot{a}\frac{\partial F}{\partial \beta_r'} \\ \dot{\beta}_r' &= \dot{a}\frac{\partial F}{\partial \alpha_r'} \end{aligned}\right\} (r = 1 \dots s - u)$$

$$\dots\dots(1\cdot4).$$

Let $\bar{F} = \int_0^1 \dots \int_0^1 F\, dv_1 \dots dv_u$ and suppose the equations

$$\left.\begin{aligned} \frac{d\alpha_r'}{da} &= -\frac{\partial \bar{F}}{\partial \beta_r'} \\ \frac{d\beta_r'}{da} &= \frac{\partial \bar{F}}{\partial \alpha_r'} \end{aligned}\right\} (r = 1 \dots s - u) \qquad \dots\dots(1\cdot5)$$

are solved by the contact transformation given by

$$\left.\begin{aligned} \alpha_r' &= \frac{\partial}{\partial \beta_r'} V(I, a, \beta', a) \\ \beta_r &= \frac{\partial}{\partial \alpha_r} V(I, a, \beta', a) \end{aligned}\right\} \qquad \dots\dots(1\cdot6),$$

where $\partial V / \partial a = -\bar{F}$, so that, if

$$G_r = \frac{\partial}{\partial I_r} V(I, a, \beta', a),$$

the equations (1·4) become

$$\left.\begin{aligned}
&\left.\begin{aligned}
\dot I_r &= -\dot a \frac{\partial F}{\partial v_r}\\
\dot v_r &= \frac{\partial H}{\partial I_r} + \dot a \frac{\partial F}{\partial I_r}
\end{aligned}\right\} (r = 1 \dots u)\\
&\left.\begin{aligned}
\dot\alpha_r &= -\dot a \frac{\partial}{\partial \beta_r}(F - \bar F) - \sum_1^u \dot I_t \frac{\partial G_t}{\partial \beta_r}\\
\dot\beta_r &= \quad \dot a \frac{\partial}{\partial \alpha_r}(F - \bar F) + \sum_1^u \dot I_t \frac{\partial G_t}{\partial \alpha_r}
\end{aligned}\right\} (r = 1 \dots s - u)
\end{aligned}\right\} \quad \dots(1\cdot 7).$$

The equations will be taken in this form.

Suppose further that for $a_0 \leqslant a \leqslant a_1$, $I_r' < I_r \leqslant I_r''$, $\alpha_r' \leqslant \alpha_r < \alpha_r''$, $\beta_r' < \beta_r < \beta_r''$, the second differential coefficients of H, F, $\bar F$, G, exist and are bounded, and that for any particular integers $l_1 \dots l_u$ not all zero, the volume of "(I, a)-space" contained in $a_0 \leqslant a \leqslant a_1$, $I_r' \leqslant I_r \leqslant I_r''$ and satisfying

$$\left| l_1 \frac{\partial H}{\partial I_1} + \dots + l_u \frac{\partial H}{\partial I_u} \right| < \zeta$$

can be made arbitrarily small by taking ζ small enough. (1·8)

Suppose the distribution of representative points of an assembly of these systems can, for $a = a_0$, be represented with sufficient accuracy as a differentiable density in "(v, I, β, α)-space" within $I_r' \leqslant I_r \leqslant I_r''$, $\alpha_r' \leqslant \alpha_r \leqslant \alpha_r''$, $\beta_r' \leqslant \beta_r < \beta_r''$, and periodic in $v_1 \dots v_u$.

Then it will be shown that, corresponding to a change of a from a_0 to a_1, if $\dot a$ be taken small enough, $|\ddot a / \dot a^2|$ being less than some constant, the proportion of systems for which the change of I, α, β is greater than given numbers can be made less than a given number.

If none of the systems can, for change of a from a_0 to a_1, have changes in I, α, β greater than some constant, it follows that the average change in I, α, β can be made arbitrarily small.

Note. It is necessary for the proof that all the small quantities considered should remain large compared with the scale of distribution of representative points, which is considered as a continuous density.

The proof follows, divided into stages numbered 2, 3, 4, 5.

2. *Along trajectories starting within* $I_r' \leqslant I_r \leqslant I_r''$, $\alpha_r' \leqslant \alpha_r < \alpha_r''$, $\beta_r' \leqslant \beta_r \leqslant \beta_r''$ *for* $a = a_0$ *and remaining within those limits, as long as* $a < a_1$, *the density of representative points* $M(v, I, \beta, \alpha, a)$ *remains bounded.*

Proof. Along trajectories in "(v, I, β, α)-space"

$$\frac{d}{da}(\log M) = \Sigma \frac{\partial}{\partial I_r}\left(\frac{dI_r}{da}\right) + \Sigma \frac{\partial}{\partial v_r}\left(\frac{dv_r}{da}\right) + \Sigma \frac{\partial}{\partial \alpha_r}\left(\frac{d\alpha_r}{da}\right) + \Sigma \frac{\partial}{\partial \beta_r}\left(\frac{d\beta_r}{da}\right)$$

$$= \Sigma\Sigma \frac{\partial(I_t, G_t)}{\partial(\beta_r, \alpha_r)},$$

so that

$$\left|\frac{d}{da}(\log M)\right| < A,$$

where A is some constant—so long as the trajectory remains within $I_r' \leqslant I_r \leqslant I_r''$, etc.

3. *Given ϵ and ζ, for a region $a_0 \leqslant a \leqslant a_1$, $I_r' \leqslant I_r \leqslant I_r''$, $\alpha_r' \leqslant \alpha_r \leqslant \alpha_r''$, $\beta_r' \leqslant \beta_r \leqslant \beta_r''$, where*

$$\left| l_1 \frac{\partial H}{\partial I_1} + \dots + l_u \frac{\partial H}{\partial I_u} \right| > \zeta \qquad \dots\dots(3\cdot 1),$$

when $l_1 \dots l_u$ are any integers, not all zero, such that

$$|l_1|, \dots |l_u| < D/\epsilon,$$

then

$$\left|\frac{\Delta I_r}{\Delta a}\right|, \left|\frac{\Delta \alpha_r}{\Delta a}\right|, \left|\frac{\Delta \beta_r}{\Delta a}\right|$$

can be made less than $E\epsilon$ for $\Delta a > F\zeta\epsilon^{u-1}$ by taking $\dot a < G\zeta^2\epsilon^{2u-1}$, if $|\ddot a|/\dot a^2$ is less than some constant, where D, E, F, G are certain constants.

Proof. Consider the motion of the system from $t = t_0$ to $t = t_0 + T$, where T is that T given by the corollary to Part I when $\nu_1 \dots \nu_0, \nu$ are replaced by $(\partial H/\partial I_1)_0 \dots (\partial H/\partial I_u)_0$, so that

$$T = O(1/\zeta\epsilon^{u-1}),$$
$$I_r - I_{r0} = O(\dot a T),$$
$$\alpha_r - \alpha_{r0} = O(\dot a T),$$
$$\beta_r - \beta_{r0} = O(\dot a T).$$

Thus

$$\dot v_r - \left(\frac{\partial H}{\partial I_r}\right)_0 = O(\dot a) + O(\dot a T),$$

$$v_r - v_{r0} - \left(\frac{\partial H}{\partial I_r}\right)_0 (t - t_0) = O(\dot a T^2) + O(\dot a T)$$
$$= O(\epsilon),$$

so that

$$\dot I_r - \dot a \frac{\partial F}{\partial v_r}\left(I_{10} \dots, v_{10} + \left(\frac{\partial H}{\partial I_r}\right)_0 (t - t_0) \dots, \alpha_{10} \dots, \beta_{10} \dots\right) = O(\epsilon \dot a)$$

and

$$I_{r_{t_0+T}} - I_{r_{t_0}} = O(\dot a T \epsilon)$$

by the corollary to Part I, since

$$\int_0^1 \dots \int_0^1 \frac{\partial F}{\partial v_r}(I_{10} \dots, v_1 \dots, \alpha_{10} \dots, \beta_{10} \dots)\, dv_1 \dots dv_u = 0.$$

Likewise
$$\alpha_{r_{t_0+T}} - \alpha_{r_{t_0}} = O(\dot a T \epsilon),$$
$$\beta_{r_{t_0+T}} - \beta_{r_{t_0}} = O(\dot a T \epsilon),$$
since
$$\int_0^1 \dots \int_0^1 \left[\frac{\partial}{\partial \beta_r}(F - \bar F) + \sum_1^u \frac{\partial F}{\partial v_t}\frac{\partial G_t}{\partial \beta_r}\right] dv_1 \dots dv_v = 0, \text{ etc.}$$
Now
$$\ddot a / \dot a^2 = O(1),$$
i.e.
$$\frac{d}{da}\log \dot a = O(1),$$
so that
$$\dot a T < B(a_{t_0+T} - a_{t_0})$$
$$> C(a_{t_0+T} - a_{t_0}),$$
where B, C are certain constants, so that for
$$\Delta a > F \,.\, \frac{1}{\epsilon}\zeta^2 \epsilon^{2u-1} 1/\zeta \epsilon^{u-1}$$
$$> F \zeta \epsilon^{u-1},$$
where F is some constant,
$$I_{r_{a_0+\Delta a}} - I_{r_{a_0}} = O(\epsilon \Delta a)$$
by dividing the interval Δa into more than some multiple of $1/\epsilon$ successive periods like t_0 to $t_0 + T$ so that the odd portion is less than some multiple of $\epsilon \Delta a$ and can only contribute of that order to $I_{a_0+\Delta a} - I_{a_0} = \Delta I$.

Likewise for $\Delta \alpha_r$ and $\Delta \beta_r$.

4. *For change of a from a_2 to $a_2 + \delta a$, the average change in I_r, α_r, β_r for systems with*
$$I_r' < \iota_r' \leqslant I_r \leqslant \iota_r'' < I_r'',$$
$$\alpha_r' < A_r' \leqslant \alpha_r \leqslant A_r'' < \alpha_r'',$$
$$\beta_r' < B_r' \leqslant \beta_r \leqslant B_r'' < \beta_r'',$$
for $a = a_2$, is less than some multiple of $\epsilon \,.\, \delta a$, provided that δa is less than some multiple of $\iota_r' - I_r'$, etc. and M is bounded for $a = a_1$, if $\dot a$ is sufficiently small.

Proof. $\Delta I_r = O(\Delta a)$, etc. hold always, so the systems to be considered remain within $I_r' \leqslant I_r \leqslant I_r''$, etc., as long as
$$\delta a = O(\iota_r' - I_r'), \text{ etc.} \qquad \text{......(4·1).}$$

Moreover M remains bounded for them (by 2) ......(4·2).

ϵ having been fixed, by the hypothesis about H (1·8), by taking ζ small enough, the volume of (I, a)-space, and so of (I, v, β, α, a)-space satisfying
$$a_2 \leqslant a \leqslant a_2 + \delta a, \quad I_r' \leqslant I_r \leqslant I_r'', \text{ etc.} \qquad \text{......(4·21),}$$

and which is excluded by the hypothesis of (3·1), may be made less than some multiple of $\epsilon . \delta a$, while at the same time Δa of 3, which has only to be greater than $\zeta \epsilon^{1-u}$, is as small as may be required. ......(4·3)

Also, except for regions excluded by (3·1),

$$\Delta I = O\,(\epsilon \Delta a), \text{ etc.} \qquad \text{......(4·4).}$$

Hence, combining (4·1), (4·2), (4·3), (4·4), by taking $\dot{a}$ small enough, the average change in I_r of the systems

$$\sim \frac{\int_{a_2}^{a_2+\delta a} \int_R \dots \int M \frac{\Delta I_r}{\Delta a} dI_1 \dots dr_1 \dots d\alpha_1 \dots d\beta_1 \dots \Delta a}{\int_{\iota_1'}^{\iota_1''} \dots \int_0^1 \dots \int_{A_1'}^{A_1''} \dots \int_{B_1'}^{B_1''} \dots M d\beta_1 \dots d\alpha_1 \dots dr_1 \dots dI_1 \dots},$$

where R is a region included in (4·31),

$$= O\,(\epsilon \delta a).$$

Likewise for changes in α_r, β_r.

5. The theorem as stated in 1 follows from 2 and 4 by dividing (a_0, a_1) into a sufficient number of ranges δa, since at each stage only an infinitesimal proportion, $O\,(\sqrt{\epsilon})$, of systems can have changes in (I, α, β) that are not infinitesimal, $O\,(\delta a \sqrt{\epsilon})$, while only an infinitesimal proportion, $O\,(\epsilon / \delta a)$, can emerge from

$$\iota_r' \leqslant I_r \leqslant \iota_r'', \text{ etc.},$$

and in the first stage that domain includes all but an infinitesimal proportion, $O\,(\delta a)$, and finally ϵ may be arbitrarily small compared with δa.

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Doublet and Triplet Separations in Optical Spectra as Evidence whether Orbits penetrate into the Core. By Mr D. R. HARTREE, St John's College.

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§ 1. *Introduction.*

Though the mechanism of the "relativity" doublets of X-ray spectra on the one hand and of doublets and triplets of optical spectra on the other is not yet at all clear, it seems possible to find a relation between them and so, taking the formula for the X-ray doublets as verified experimentally (whatever its interpretation), to derive a formula for the separations of a doublet or triplet term of an optical spectrum. This has been done by Landé*.

It is assumed that the splitting of an n_k term of a central field into a doublet, triplet or higher multiplet term is due to some influence which acts chiefly on the innermost part of the orbit, and in order that Landé's relation should be applicable it is essential that the orbit of the series electron corresponding to the doublet (or triplet) term should penetrate into the region of the core occupied by the orbits of the core electrons corresponding to the related X-ray doublet terms.

If these ideas are valid, it seems possible that comparison of the doublet (or triplet) separations of a term of an optical spectrum with Landé's formula may furnish evidence on the question whether the corresponding orbits penetrate or not, and the main object of the present paper is to consider the experimental data from this point of view.

The conclusions arrived at by consideration of this evidence are sometimes at variance with the assignment of quantum numbers suggested by Bohr† and Born‡.

Reference will be made in places to other evidence on the question whether orbits penetrate or not; some aspects of this other evidence will be investigated in a separate paper.

§ 2. *The Landé Formula and its Expression in Terms of Quantum Defect.*

It will be convenient, for the sake of reference, to outline Landé's deduction of his formula for the doublet separations of optical spectra.

It is assumed that an n_k core orbit gives a doublet term with an X-ray "relativity" doublet separation $\Delta\nu$, given by the simple relativity difference of energy between an n_k and an n_{k-1} orbit in a Coulomb field*. Writing R for the Rydberg constant and $\alpha = \pi e^2/ch$ as usual, the "relativity" doublet separation for the n_k orbit in the field of a point charge Z is

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* A. Landé, *Zeit. f. Phys.* vol. xxv, p. 46 (1924).

† N. Bohr, *Ann. der Phys.* vol. LXXI, p. 253 (1923), particularly § 6.

‡ M. Born, *Vorlesungen über Atommechanik* (Springer, 1924), ch. III, § 31.

$$\Delta\nu_i = \gamma R\alpha^2 Z^4/n^3 k\,(k-1) \qquad \text{......(1).}$$

Here γ is the ratio of the complete expression for the relativity doublet separation to its first term, and can be calculated.

As already mentioned, it is assumed that the cause of the doublet separation acts chiefly on the innermost part of the orbit; let Z_i be a mean value of the effective nuclear charge over this part of the orbit; Landé shows that the doublet separation in optical spectra, $\Delta\nu$, is then given approximately by multiplying the X-ray doublet separation, given by (1) with $Z = Z_i$, by the fraction of a radial period which an electron spends in this part of its orbit. Let C be the core charge of the atom, and q the quantum defect of an n_k orbit of the series electron which penetrates the core and which at aphelion is in a field not appreciably different from that of a point charge C; then the formula for the doublet separation thus obtained is

$$\Delta\nu = \gamma R\alpha^2 C^2 Z_i^{\,2}/(n-q)^3\, k\,(k-1) \qquad \text{......(2).}$$

For all orbits of any given value of k which penetrate the core, the difference between Z_i, the effective nuclear charge over the part of the orbit which is chiefly responsible for the doublet separation, and the atomic number N is likely to be very nearly the same, so we can put $Z = N - s$. It must be expected that s will be a function of k, since the orbits of the series electron with different values of k penetrate to different extents. If the orbit of the series electron does not penetrate into the core, it might be expected that the simple relativity formula (1) would hold with Z not much greater than C.

The doublet separation can be expressed in terms of the difference of quantum defect q between the two members of the doublet term. By definition of q,

$$\nu = C^2 R/(n-q)^2,$$

so to the first order

$$\Delta\nu = [2C^2 R/(n-q)^3]\,\Delta q \qquad \text{......(3).}$$

Combining with (2) and putting $Z_i = N - s$, we have

$$\Delta q = \gamma\alpha^2\,(N-s)^2/2k\,(k-1) \qquad \text{......(4),}$$

which is appreciably simpler than (2); the core charge C and the principal quantum number of the series electron have disappeared, so that Δq is—according to this formula—a function of the atomic number N and the azimuthal quantum number k only; i.e. of the element and the sequence to which a term belongs, but not of the degree of ionisation or the position of the term in the sequence of a given spectrum*; and it is actually found from observed spectra that Δq is nearly the same for all terms of a sequence to which the Landé formula is applicable.

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* The part of Landé's paper relevant to the present paper is that dealing with the connection between the doublet separations of optical spectra and the "relativity" doublets of X-ray spectra. The formula for these "relativity" doublets may be taken as known from experimental data on X-ray spectra. The question of the mechanism of the X-ray "relativity" doublet is at present irrelevant.

On this account formula (4) may reasonably be looked on as the fundamental one for the doublet separations in optical spectra, rather than formula (2), and it may be pointed out that it is also physically significant. The quantum defect q is $1/h$ times the difference between the principal action integral $I = \Sigma_s \oint p_s dq_s$ for the actual orbit and for the orbit in the field of a point charge C which has the same energy, and therefore same outer loop if the actual orbit is at aphelion in a field not appreciably different from that of a point charge C. For penetrating orbits the chief part of this difference comes from the part of the actual orbit which lies inside the core, and the innermost part, which is assumed to be chiefly responsible for the doublet separation, is very nearly the same for all penetrating orbits of a given atom with the same k, independently of the state of ionisation or the principal quantum number of the orbit. It may therefore be expected that the difference in I, and so in q, between the orbits corresponding to the two terms of the doublet should be independent of core charge and total quantum number of the orbit concerned†.

§ 3. *Comparison with Observation-Doublet p terms.*

In comparing the theoretical formula with observed doublet and triplet separations, Landé takes formula (2), calculates Z_i for various p terms and plots against N. Here a somewhat different method will be adopted, and account will also be taken of d terms.

From formula (3) we have

$$(\Delta q/\gamma)^{\frac{1}{2}} = [\alpha^2/2k\,(k-1)]^{\frac{1}{2}}\,(N - s) \qquad \ldots\ldots(5),$$

so that $(\Delta q/\gamma)^{\frac{1}{2}}$ would be expected to be linear in the atomic number N, the gradient of the linear relation being $\alpha/[2k\,(k-1)]^{\frac{1}{2}}$.

In deriving formula (2) for doublet separations in optical spectra from formula (1) for "relativity" doublet separations in X-ray spectra it is assumed that at aphelion the orbit of the series electron is in a field not appreciably different from that of a point charge C. It sometimes happens that the first term of a sequence corresponds to

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[*] A formula equivalent to (4) is given by Landé, but no special emphasis is laid on it.

[†] The writer believes that this significance of the constancy of Δq within a sequence of terms of a single spectrum has already been pointed out by Bohr, but has been unable to find the reference.

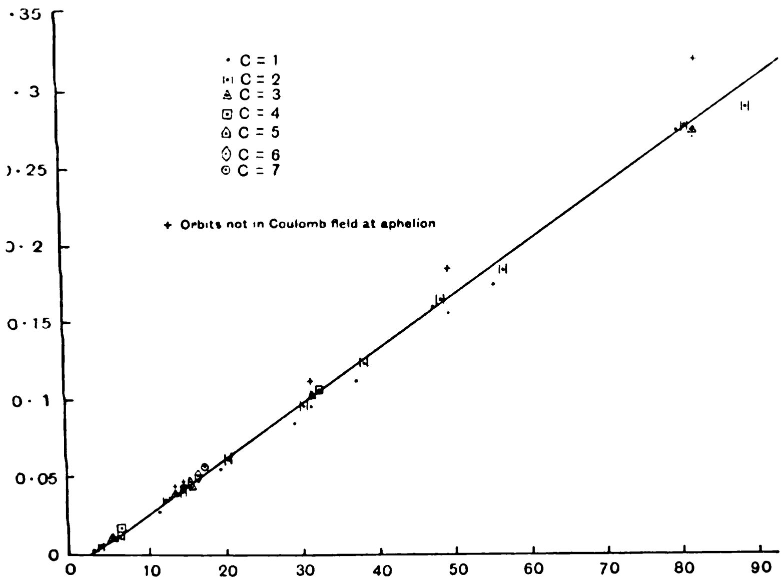

Fig. 1. Separations of p terms of doublet spectra. $(\Delta q/\gamma)^{\frac{1}{2}}$ plotted against atomic number. Straight line is drawn with theoretical gradient $a/2 = 0{\cdot}00365$.

an orbit for which this is not the case[*], and formula (5) is not applicable to it. For the second term of a sequence the added field at aphelion is probably never so great that formula (5) is not applicable.

The values of $(\Delta q/\gamma)^{\frac{1}{2}}$ for the first p terms of all spectra available are plotted in Fig. 1[†]. When it is clear from the term value or otherwise that the field at aphelion is appreciably different from that of a point charge C, the point is marked with a cross, and the value of $(\Delta q/\gamma)^{\frac{1}{2}}$ for the second or third p term is also plotted[‡].

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[*] For example, in an atom whose core orbit consists of a complete group plus a pair of electrons in n_1 orbits (atoms of Al sub-group types, e.g. Al, Si^+, P^{++}, Ga, In, Tl), the aphelion of the first p orbit, which has the same value of n as the two outermost orbits with $k = 1$ already present in the core, is probably not far outside the aphelia of these orbits, so that the field there differs considerably from that of a point charge C. This is shown in the corresponding spectra (at any rate for the less highly ionised atoms of these types), by the fact that the first p term is greater than $C^2R/4$, i.e. the effective quantum number $n - q$ is less than the azimuthal quantum number k. This can only be the case if the field at aphelion of the orbit of the series electron is appreciably different from that of a point charge C.

[†] I am indebted to Mr J. A. Carroll for permission to use some yet unpublished data for the spectra Ga III, Ge IV, Hg II, and Tl III.

[‡] The accuracy with which Δq can be determined from the observed spectra decreases rather rapidly from one term of a sequence to the next, so it is best to determine it from the earliest term available. The value plotted has been found from the second p term except for Ga I and In I for which no lines involving the second p term have been observed.

The values of Δq have been calculated by evaluating q separately for the two members of the doublet and subtracting, and not from the first order formula (3).

The points for the p terms all fit closely to a linear relation of the type to be expected. The best straight line through the observed points (determined by least squares) has a gradient of 0·00347 ± [illegible] while the theoretical linear relation has a gradient $\alpha/2 = 0{\cdot}00$[illegible] which differs from it by several times the probable error, but the observed spectra do not form a random sample of all possible doublet spectra. A straight line with the theoretical gradient can be drawn to fit observed points nearly as well as the best straight line; such a line is shown in the figure, the value of the constant s (which can be thought of as a screening constant) being about 2·2. It might be expected to be about 2. Not too much importance should be given to choosing the linear relation to get the best possible fit for the heavy elements, as the higher order terms in the formula for the X-ray relativity doublet[*] are then appreciable ($\gamma = 1{\cdot}342$ for Ra), and it is by no means certain that the use of the same factor γ in the formula for optical doublet separations as for the X-ray relativity doublet separation is justifiable. It is clear, however, that for the optical spectra also there is some "second-order" effect coming in, of the same order of magnitude as that expressed by the factor γ calculated from relativity considerations.

From the point of view of the main object of this paper, the general fit to the Landé formula is important, but more important is the fact that the values of Δq for p terms of all doublet spectra—including, in particular, the spectra Cu I, Ag I and Au I—are consistent. This would not be expected unless all the orbits penetrated[†]; in particular, if for Cu I and Ag I the p terms correspond to non-penetrating orbits as suggested by Bohr and Born[‡], the values of $(\Delta q/\gamma)^{\frac{1}{4}}$ would be expected to lie much below the curve for other spectra.

Unless the development of ideas of the mechanism of the doublet separations involves a serious departure from Landé's relation between the X-ray and optical doublets, it seems that the observed doublet separations demand that, except for lithium-like atoms, all p orbits

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[*] The series for the relativity doublet separation between the 2_1 and 2_2 terms had to be taken to 2 terms further than those given by Sommerfeld, in order to obtain 4 significant figures in γ for the heavy elements. The formula for γ is

$$\gamma = 1 + x + \frac{53}{50}x^2 + \frac{303}{200}x^3 + \frac{3839}{2000}x^4 + O(x^5),$$

$$x = \frac{5}{8}\alpha^2 Z_i^2 = Z_i^2/30070.$$

In calculating γ, the value of Z_i has been taken as $N - 2$.

[†] Certainly the values of Δq for the p terms of lithium-like atoms agree closely with the linear relation for the p terms of other atoms, and the orbits do not penetrate, but they form a special case, since for them the value of $N - s$ (with the value of s deduced from penetrating orbits) is not far different from the core charge.

[‡] *Loc. cit.* It may perhaps be mentioned that the relation between the first p terms of Cu I, Zn II, Ga III and Ga IV appears to confirm that the corresponding orbits penetrate, and the same is true for the spectra Ag I, Cd II and In III.

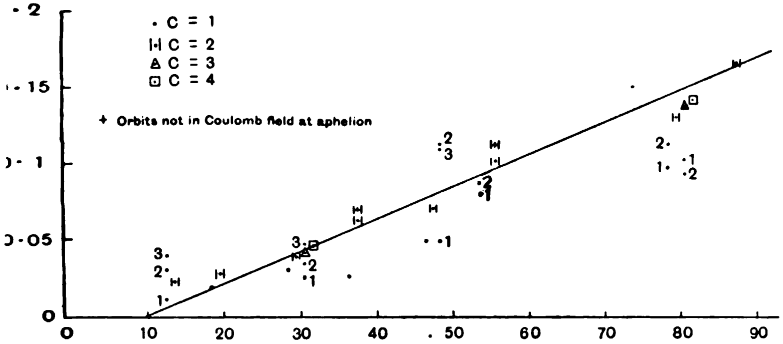

Fig. 2. Separations of *d* terms of doublet spectra. $(\Delta q/\gamma)^{\frac{1}{2}}$ plotted against atomic number. Straight line is drawn with theoretical gradient $a/2\sqrt{3} = 0{\cdot}00210$.

should penetrate. The same appears to be the case for triplet spectra see § 5).

In all cases for which the aphelion of the orbit is not far enough from the core for the added field to be negligible (points marked with a cross on the diagram), the value of Δq is greater than for other terms of the sequence.

§ 4. *Doublet d Terms.*

In the case of the doublet spectra so far known, it is usually fairly certain whether the *p* and *f* orbits penetrate or not (usually the *p* orbits penetrate and the *f* orbits do not), but the *d* orbits are often, as it were, on the border-line between penetrating and non-penetrating orbits, and sometimes it is not easy to be certain of their nature, and any evidence on the subject is interesting and may be of some importance.

The spectra so far observed, for which the orbits corresponding to *d* terms can be taken with certainty to be penetrating, are not many, but they are enough to define approximately a standard relation between $(\Delta q/\gamma)^{\frac{1}{2}}$ and N with which the values for other spectra can be compared. For the spectra Ca II, Sr II, Ba II and Ra II the first *d* term corresponds to an orbit for which the added field at aphelion is far from negligible*; in these cases the corresponding points are marked with a cross and the point for the second *d* term is plotted as in the case of the similar *p* terms; the other spectra at present known, whose *d* terms certainly correspond to penetrating orbits, are Zn II, Ga III, Ge IV, Cd II, Hg II, Tl III; for spectra other than those already mentioned the points for the successive *p* terms are numbered 1, 2, 3 ....

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* Cf. footnote *, p. 907. The first *d* term of Ba II is doubtful. Fowler and Paschen-Götze in their tables give different groups of lines for the doublet and satellite $3d - 2p$. H. N. Russell and F. A. Saunders (*Ap. J.* vol. LXI, p. 60 (1925)) agree with Paschen-Götze, and the $3d$ pair of terms determined from these lines seems more probable than Fowler's, both from its q and Δq. The point plotted has been determined from it.

The first *d* term of Ra II is not known.

From formula (5) it would be expected that for penetrating *d* orbits for which the field at aphelion is negligible, the value of $(\Delta q)^{\frac{1}{2}}$ plotted against *N* would lie on or about a straight line of gradient $a/2\sqrt{3} = 0{\cdot}00210$. The points for those spectra already specified do lie fairly close to a straight line of the theoretical gradient, confirming Landé's formula as far as the variation of doublet separation with *k* is concerned; and, as for the *p* terms, the value of Δq for a term corresponding to an orbit for which the added field at aphelion is not negligible is larger than the value for the rest of the sequence.

It will be seen that the point for Cs I lies no further from the straight line drawn than those from some other penetrating orbits, which suggests that the *d* terms of Cs I correspond to penetrating orbits. This is interesting, as the other evidence on the nature of the *d* orbits of Cs is inconclusive, though on the whole it seems to suggest that the orbits do not penetrate. If the doublet separations in wave number $\Delta\nu$ are compared, their evidence for penetrating *d* orbits for Cs seems all the more striking. For the first two *d* terms of K I, Rb I and Cs I the values of $\Delta\nu$ are*:

	K I	Rb I	Cs I
First *d* term, $\Delta\nu$	$-2{\cdot}7$	—	97·6
Second *d* term, $\Delta\nu$	—	3·0	42·8

It is impossible to carry out an accurate extrapolation, but if for Cs I the orbits were of the same kind as those of K I and Rb I, the doublet separations would hardly be expected to be greater than about 20 and 10, and actually they are about four times as large. On the other hand, the value of the term indicates that the added field at aphelion is not negligible, so that if the orbits penetrate the value of Δq would be expected to be greater for the first term of the sequence than for the others, and actually it is less. Thus in this case the evidence of the doublet separations is inconclusive, though this is in itself interesting as it means that there are difficulties in accounting for the observed terms whatever the nature of the orbit.

The doublet separations of the *d* terms of other spectra fall into two fairly well-marked groups;

(1) Those which show a large increase of Δq within a sequence, the values for the later terms being greater even than they would be for penetrating orbits;

(2) Those which show little variation within a sequence, and are less than the value for penetrating orbits.

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* For the significance of the negative sign Δq for K I, see § 7. The separation of the first *d* term of Rb I has not been observed.

The doublet separations of the *d* terms of Al I, Ga I, and In I l into the first group; the increase of Δq within the sequence is so at that even $\Delta\nu$, the separation in wave number, increases at first. e reason of the increase seems clear; the cores of these atoms con- t of a closed group with two further electrons in rather large and sely bound orbits; the first *d* orbit (circular) lies outside, but in appreciable added field, since the two outer orbits form a group low symmetry and also are probably very easily polarised; the rihelia of the later *d* orbits probably lie in the region of the aphelia the two outer core orbits, so that the mutual interactions of the re and series electron would be stronger than for the circular orbit. his rough picture thus explains the relative values of Δq for the fferent *d* terms, and it also explains the rapid increase of q within e *d* sequence, but it does not explain why the values of Δq should so large, larger indeed than the values to be expected for penetrat- g orbits.

The triplet *f* terms of Al II show a similar behaviour, and may erhaps be explained in a similar manner.

The doublet separations for the *d* terms of K I, Cu I, Rb I, Ag I, u I and Tl I belong to the second group. Other evidence is in favour f non-penetrating *d* orbits for K, Cu, Rb and probably Ag and Au; he approximate constancy of Δq within the sequences suggests enetrating orbits, but the actual values of Δq seem too small for enetrating orbits, though they are large for non-penetrating orbits, specially as the term values indicate a very small added field. It eems at present as if there is no explanation of large doublet separa- ions for terms corresponding to non-penetrating orbits, on ideas imilar to those underlying Landé's formula. A modification of the heory of multiplets has recently been given by Heisenberg and may provide an explanation; it will be considered shortly in § 8.

For Tl I somewhat different considerations apply; the two outer core orbits have just about the same maximum radius (estimated from the 6_1 term of Tl III) as those of the core of neutral Al, Ga and In, and if the *d* orbits do not penetrate, there seems no reason why values both of the quantum defect q and of Δq should not behave in the same way for Tl I as for Al I, Ga I and In I. But they do not; for Tl I, q and Δq remain nearly constant throughout the *d* sequence, while for the other three spectra mentioned they both increase considerably for the first few terms. This suggests that for Tl the *d* orbits penetrate, but on the other hand the value of $(\Delta q/\gamma)^{\frac{1}{2}}$ is rather small.

§ 5. *Triplet Spectra.*

Landé suggests that the values of Δq for extreme members of a triplet term in triplet spectra should fit his formula if the orbits penetrate and the field at aphelion is negligible. In Fig. 3, values of $(\Delta q/\gamma)^{\frac{1}{2}}$ for the extreme terms of the first and (where observed)

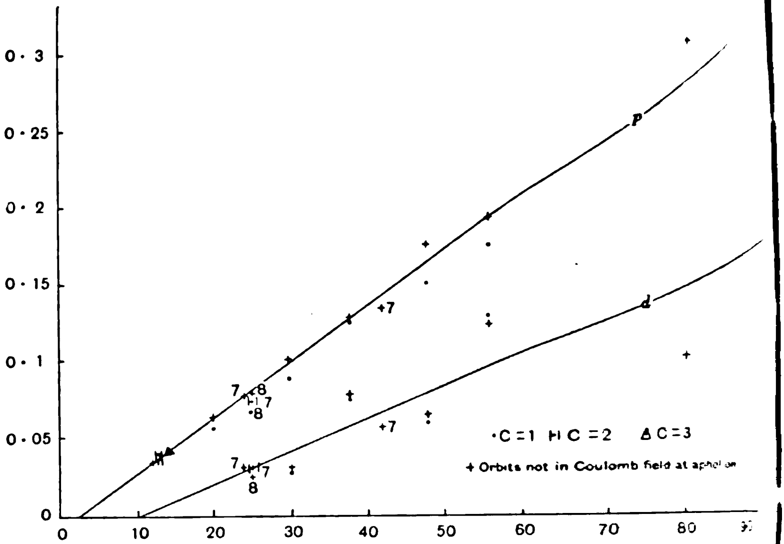

Fig. 3. Separations of *p* and *d* terms of spectra of multiplicity greater than 2 $(\Delta q/\gamma)^{\frac{1}{4}}$ plotted against atomic number. Straight lines are those drawn on Figs. 1 and 2 for *p* and *d* terms of doublet spectra. Numerals 7 and 8 beside points indicate multiplicity of spectrum; points not numbered refer to spectra of multiplicity 3.

second *p* and *d* triplets are plotted; the first *p* triplet usually corresponds to orbits for which the added field at aphelion is not negligible, so that its triplet separations would be expected to be somewhat larger than those given by the Landé formula; the points for such terms are marked with crosses. The straight lines are those determined from doublet spectra.

The points for the *p* terms are mostly below the straight line, but the general fit to it is not bad, and suggests that all the orbits penetrate; this seems to be confirmed by the balance of other evidence at present available, but it disagrees with Bohr's and Born's conclusion (*loc. cit.*) that for Zn I, Cd I, and Hg I the first *p* orbit has quantum numbers 2_2 and does not penetrate.

It might just be mentioned that the ratio $\Delta p_{12}/\Delta p_{23}$ for triplet spectra (Fowler's suffixes) shows a general tendency to depart further and further from the value 2 given by Landé's "interval rule" as the atomic number increases. It is slightly greater than 2 even for light atoms, and is as large as 2·9 for the first *p* term of Hg I and about $10\frac{1}{2}$ and $17\frac{1}{2}$ for the second and third *p* terms.

For *d* orbits the points for triplet spectra do not lie close enough to the line for doublet spectra to enable any very certain conclusions to be drawn; the points for the alkaline earth metals lie high and those for the Zn sub-group are low. For the alkaline earth metals e *d* orbits certainly penetrate partly, but whether they penetrate ght into the core or only into the region of the one outermost core rbit is uncertain; the former is probable, and agrees with Bohr's ssignment of quantum numbers. For the Zn sub-group the small alues of q for the *d* terms suggest that the orbits do not penetrate, he triplet separations are considerably smaller than those to be xpected for penetrating orbits, but they are large compared to the relativity" separations to be expected for non-penetrating orbits.

§ 6. *Spectra of Multiplicity higher than* 3 (*Complex Spectra*).

In all spectra of multiplicity higher than 3 which have so far een analysed, the majority of terms do not belong to series of the ydberg type, and it is convenient to refer to such spectra as a whole s complex spectra. It is interesting to see if Landé's formula can e extended to apply to the extreme members of any term of such pectra; it cannot be expected to apply to all, and a criterion is needed in order to choose the terms which may most reasonably be expected to fit it. A suitable criterion seemed to be this, that the term should belong to a sequence of the Rydberg type. Certainly the Landé formula could hardly be expected to apply if this were not the case, for unless a term belongs to such a sequence it is unlikely that its value is a measure of the energy of a single orbit, and in deriving the Landé formula by multiplying the X-ray relativity doublet by the fraction of a period which the series electron spends in the inner loop of its orbit, the expression for this fraction involving the value (or effective quantum number) of the term assumes that the term value is a measure of the energy of that orbit alone. Whether the Landé formula can be expected to apply in *all* cases when the terms fit a Rydberg formula is not so clear.

There is only one complex spectrum yet analysed for which a Rydberg sequence of *p* terms is definitely known, that is the octet system of Mn I*, and this is also the only complex spectrum for which enough terms of a *d* sequence are known to make sure it is of the Rydberg type. But in Cr I and some other complex spectra there is a *p* term and a *d* term which look rather like the first terms of a Rydberg *p* and *d* sequence respectively; these terms† and those of Mn I already mentioned are given in Table I and the corresponding points plotted in Fig. 3. All points except that for the third *p* term of Mn I refer to orbits for which the term values show that the added field at aphelion is not negligible (though except for Mn I it need

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* M. A. Catalan, *Phil. Trans. Roy. Soc.* vol. CCXXIII, p. 127. Catalan's "wide triplet" series are part of the octet system, and the "narrow triplet" series and the multiplets part of the sextet system.

† The values of the terms for Mn II have been found from the first principal and first diffuse triplet (see Catalan, *loc. cit.* p. 162) assuming $1s = 119{,}000$, which value has been estimated by the writer without reference to the magnitude of the separations of the terms. (*Nature*, vol. CXVI, p. 356 (1925).)

Table I. *Complex Spectra. p and d terms which probably belong to sequences of the Rydberg type.*

Spectrum	$2R$	n_k	$\Delta\nu$ Extreme members of term	q		Reference
Cr I	7	4_2	31043·24 31237·09	2·11985 2·12569	$4^{\bar{7}}_{2j}$	H. Geiseler and W. Grotrian. *Zeit. f. Phys.* vol. XXII, p. 228 (1924)
		4_3 (? 3_3)	12281·33 12289·12	1·01081 1·01176	$4^{\bar{7}}_{3j}$	
		5_3 (? 4_3)	6833·04 6843·51	0·99254 0·99560	$5^{\bar{7}}_{3j}$	
Mn I	8	4_2	41232·09 41534·98	2·36861 2·37457	$1p$	M. A. Catalan, *Phil. Trans.* vol. CCXXIII, p. 127 (1922)
		6_2	7431·57 7448·47	2·15730 2·16166	$3p$	
		4_3 (? 3_3)	13224·93 13231·30	1·11942 1·12011	$2d$	
Mn II	7	4_2	85193·5 85634·0	1·73011 1·73596		Deduced from lines given by Catalan (*loc. cit.*) assuming $1s$ = 119,000 (see footnote previous page)
		4_3	44431·6 44459·9	0·85688 0·85788		
Mo I	7	5_2	33239·94 33946·07	3·18304 3·20203	$2p$	C. C. Kiess, *Sci. Pap. Bur. Stand.* No. 474 (vol. XIX, p. 113) (1923)
		5_3 (? 3_3)	14590·46 14624·71	2·25753 2·26062	$3d$	

The p terms are all three-fold and the d terms five-fold.
$2R$ = Multiplicity of system.

not be at all large), so that Δq would be expected to be somewhat large. Actually all the points lie on or rather below the straight lines for penetrating orbits giving doublet terms, but the general agreement for the p terms is probably good enough to be significant. For the d terms, the term values for the neutral atoms suggest non-penetrating orbits, but the d term of Mn II indicates definitely a penetrating orbit and perhaps the agreement of the other d term separations can be taken as indicating that the corresponding orbits penetrate.

For the Mn I sextets and Cr I and Mo I quintets it seems doubtful if the p terms observed are members of a Rydberg sequence; the values of Δq for the p terms giving the "narrow triplet" series of these spectra are much too low for penetrating orbits.

§ 7. *Negative Separations.*

For a very great majority of terms of simple spectra, and perhaps for a majority of terms of complex spectra, the wave number of a nember of a multiple term decreases as the inner quantum number j ncreases. Sommerfeld* has called such terms "normal terms," and hose for which the variation of ν with j is in the opposite sense, ‘inverted terms." It is convenient to define the sign of $\Delta\nu$ as positive or a normal term, and negative for an inverted term.

For doublet spectra, some non-penetrating orbits give negative loublet separations, but some are definitely normal†; there are no egative separations for terms belonging to sequences corresponding o penetrating orbits, but the non-series d terms of Cu I and Zn II how abnormally large negative separations (see Table II). The small ncrease in these terms and in the doublet separations on going from u I to Zn II compared to the increases for the p and d terms shows that the orbits lie in a region in which the ratio of the effective nuclear harge in the two atoms is not very much greater than 1, which onfirms the suggestion already made by Bohr, that they are 3_3 ircular orbits lying probably just within the boundary of the core‡. The reaction between the core and such orbits must be expected to be abnormal.

Table II.

	First p term		First series d term		Non-series d term	
	ν	$\Delta\nu$	ν	$\Delta\nu$	ν	$\Delta\nu$
Cu I	31524 31773	248	12366 12373	7	51106 49063	-2043
Zn II	95537 96410	873	47931 47982	51	82169 79450	-2719

In triplet spectra negative separations are rare; an interesting case is furnished by the d terms of Al II. For this spectrum the f terms behave like the d terms of Al I, i.e. they show an abnormal increase of q and of Δq within the sequence; the interpretation probably is that the first f orbit lies entirely outside the core, but the perihelia of the other f orbits lie in the region covered by the one outermost orbit already present in the core. The d terms might be expected to show a similar phenomenon, but instead they show an almost constant value of q within the sequence, and small negative

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[*] A. Sommerfeld, *Atombau und Spectrallinien*, 4th Ed. p. 685. Negative separations and deviations from Landé's "Interval Rule" have been discussed recently in a paper by Stoner (*Phil. Mag.* vol. XLIX, p. 1289 (1925), § 9) from the point of view of the magnetic theory of the origin of multiple terms.

[†] See A. Sommerfeld, *loc. cit.*

[‡] N. Bohr, *Ann. der Phys. loc. cit.* pp. 268–9, footnote. On the recent theory of Heisenberg (see below), however, the interpretation of these terms is rather different, and the negative sign of the separation is explained. (See Hund, *Zeit. f. Phys.* vol. XXXIII, p. 345 (1925).)

separations. A possible interpretation is that though the *f* orbits penetrate, the *d* orbits do not; this is in violent disagreement with anything that is possible with a central field, but this would probably be true of any interpretation of *d* and *f* terms of this spectrum*.

It is certainly the case for doublet and triplet spectra, as far as they are known at present, that all terms to which the Landé formula can be applied are normal terms, and it is possible that this is general.

§ 8. *Heisenberg's Theory of Multiplet Spectra.*

In a recent paper†, Heisenberg has suggested an interpretation of the multiple terms of optical spectra; it involves some modifications in the significance of the magnitudes of the separations, and so is directly related to the subject of this paper; it will be considered shortly here, especially that part of it which concerns doublet spectra.

He points out that in thinking of the atom as a core and a series electron in mutual interaction, instead of as a single system, we introduce a certain non-mechanical "duplexity‡," so that for one stationary state of the core and one of the series electron, there are two values of the energy of mutual interaction, and so two stationary states of the whole atom.

He considers first doublet spectra, and for quantitative work on term values, and particularly on doublet separations, suggests that the duplexity should be expressed in the following way. The energy of the "quasi-mechanical" orbit with given quantum numbers *n* and *k* is thought of as divided into two parts, the "intrinsic energy" (Eigenenergie) and the "interaction energy" (Wechselwirkungsenergie). The latter is considered to arise from the interaction of the core and series electron, and the former is supposed to depend on *k* only through the relativity correction. The duplexity is introduced in the idea that the energy of a stationary state of the atom as a whole may be the sum of values of the intrinsic energy and interaction energy for quasi-mechanical orbits with different values of *k*. The scheme of values of *k* suggested by Heisenberg for the different terms of a doublet spectrum is:

Term	s	p_2	p_1	d_2	d_1	f_2	f_1	...
Intrinsic energy, k	1		2		3		4	...
Interaction energy, k	$\frac{1}{2}$	$\frac{3}{2}$		$\frac{5}{2}$		$\frac{7}{2}$		...

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* Another interpretation of the *d* terms of Al II has been given recently by Schrödinger (*Ann. der Phys.* vol. LXXVII, p. 43). The principal quantum numbers of the later *f* terms are increased by 1 so that the quantum defect is small and negative; this is ascribed to a negative polarisation of the core due to an effect of the nature of anomalous dispersion.

† W. Heisenberg, *Zeit. f. Phys.* vol. XXXII, p. 841 (1925).

‡ German:—Duplizität.

The p, d ... doublet separations then appear as differences of e intrinsic energies of orbits with different values of k, and, as eady mentioned, Heisenberg supposes that the intrinsic energy ly depends on k through the relativity correction. This correction creases with decreasing k (for given n), and is comparatively large r penetrating orbits, so that the magnitude of the doublet separation epends chiefly on whether the orbit with the smallest value of k ncerned penetrates or not; but the magnitude of the term depends whether the orbit with a value of k greater by $\frac{1}{2}$ than this pene- ates. Thus it might happen that the orbit with $k = 2$ penetrated ut the orbit with $k = \frac{5}{2}$ did not; the value of the d term would then iggest a non-penetrating orbit, but the doublet separation would iggest a penetrating one. Heisenberg points this out and suggests at the d terms of Rb I and Cs I may be examples.

It seems at first sight that here may be an explanation of the ifficulty already mentioned, that often the doublet separations of erms corresponding to non-penetrating orbits are large compared o the relativity effects to be expected for such orbits; and it seems easonable to suggest that a large doublet separation only shows that he orbit with the smallest value of k concerned penetrates. Thus for example a large d term separation only indicates that an orbit with $k = 2$ penetrates—and presumably therefore that orbits with $k = \frac{3}{2}$ giving the "interaction energy" part of the p terms penetrate.

If this is so, the interpretation of the doublet separations must be revised somewhat, but still it will be necessary to alter Bohr's assignment of quantum numbers in certain cases. For example, the d term separations of Cu I, Ag I, Au I on this view show that the p orbits penetrate. The separations of f terms are not well enough known for similar conclusions about d orbits to be made.

It seems likely that if all the orbits concerned penetrate (e.g. for d terms the orbits with $k = 2$ and 3 for the intrinsic energies and $k = \frac{5}{2}$ for the interaction energy), the doublet separation will probably be in fairly close agreement with the Landé formula, and that when none of them penetrate the relativity separation in a field point charge C is to be expected.

At present the relation between the Landé formula and the doublet separation when only one or two of the orbits concerned penetrate is not clear; at first sight it looks as if Δq should be larger than the value given by the Landé formula, whereas the observations for Cu I etc. show that it is smaller. Also it is not yet clear how negative separations in simple spectra or the doublet separations of a sequence like the d sequences of Al I, Ga I, and In I are to be explained.

A further important part of Heisenberg's paper deals with triplet and multiplet separations. The separations of a multiplet are not ascribed to one electron only, but to contributions from all electrons not in complete groups or s orbits. The validity of extending Landé's formula to triplets, and still more to the multiplets of complex spectra, without further theoretical investigation, appears somewhat doubtful.

§ 9. *Summary.*

Landé has indicated a connection between the "relativity" doublets of X-ray spectra, and the doublets and triplets of optical spectra, and so has obtained a formula for the separation of optical doublets and triplets. This formula can only be expected to hold if the orbit of the series electron penetrates into the core, so it may be possible to obtain evidence on the question whether an orbit penetrates from the doublet or triplet separation of the corresponding term.

From this evidence it is concluded that, except for lithium-like atoms, *p* terms of all known spectra correspond to penetrating orbits; this disagrees with Bohr's assignment of quantum numbers for those terms of the spectra of neutral atoms of the Cu and Zn sub-groups. For *d* terms the evidence is not so definite, but it seems possible that for Cs I and Tl I the *d* terms correspond to penetrating orbits, in disagreement with Bohr's assignment. It is shown that the Landé formula seems to hold approximately for separations in multiplet spectra when the terms belong to a sequence of the Rydberg type.

Ideas suggested by Heisenberg in a recent paper would involve some modifications of the interpretation of the magnitudes of doublet separations, but at least the conclusions as to the *p* orbits penetrating still stand.

On the Theorem of Pappus. By Mr M. H. A. NEWMAN, St John's College.

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PAPPUS' THEOREM. *If A, B, C are three points on a line, and A', B', C' are three points on another line intersecting the first, then the points of intersection of* (BC', $B'C$), *of* (CA', $C'A$), *and of* (AB', $A'B$) *lie on a line.*

1. The axioms necessary for the construction of a proof of Pappus' Theorem, regarded as a theorem in the geometry of projective space of three dimensions, fall into three groups:

I. *Axioms of Incidence*;

II. *Axioms of Order*, giving the properties of the relation "between," and establishing the order type of the projective line as cyclical and dense in itself;

III. *An Axiom of Continuity.*

It is customary in treatises on projective geometry* to adopt in Group III the *Axiom of Dedekind*, which states that if the points of a segment are divided into two classes, L and R, which have each at least one member, and are such that no member of L lies between two points of R, nor *vice versa*, then there is a point of the segment, not an end-point, which is neither between two points of L nor between two points of R. This axiom, however, assumes considerably more than is necessary for the proof of the theorem. Sufficient for this purpose is the following:

AXIOM OF ARCHIMEDES. *Let A, X, P_1, P_0 be points in order on a line. Let P_2 be $(A, P_1)/P_0$, P_3 be $(A, P_2)/P_1$, ... Then there is a number, n, such that A, P_n, X, P_0 are in order on the line.*

($(A, P_1)/P_0$ is written for "the harmonic conjugate of P_0 with respect to A and P_1.")

A proof of the theorem using this more economical axiom is extant†, which proceeds by establishing a symbolic correspondence between Pappus' Theorem and the commutativeness of multiplication for certain algebraical symbols. This proof, although well suited for investigations of the foundations of geometry, in which the consequences of denying certain of the axioms have to be examined, is far removed in method from the ordinary procedure of projective geometry, and could not be inserted in a normal presentation of the subject without seriously interrupting the argument. The object of the present note is to provide a direct proof of Pappus' Theorem making use only of the axioms of Groups I and II, and of the axiom of Archimedes.

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* Cf. e.g. Whitehead, *The Axioms of Projective Geometry*; Enriques, *Geometria Proiettiva*; Baker, *Principles of Geometry*, vol. I.

† Cf. Hilbert, *Grundlagen der Geometrie*, 5te Auflage, § 14.

2. By means of the Axioms of Incidence and Order it can easily be shown[*] that Pappus' Theorem is deducible from the following:

THEOREM. *Let A, B, C be three points on a line, L, and let a one-one correlation, μ, of the points of L with themselves be given such that*

(i) *If P, Q, R are any three points of L, μP, μQ, μR lie in the same sense on L as P, Q, R;*

(ii) *If $(PQRS)$ is any harmonic range on L, $(\mu P, \mu Q, \mu R, \mu S)$ is harmonic;*

(iii) *μA is A, μB is B, μC is C;*

then, if P is any point of L, μP is P.

In the proof which will now be given of this theorem the axioms of Group I and Group II will be assumed (as they are given, for example, in Whitehead, *op. cit.*, Axioms I–XVIII, or Baker, *op. cit.* pp. 4 and 95), together with the *Axiom of Archimedes*, as stated in the previous paragraph.

3. For brevity we make the following conventions:

Let a point O be taken on L, and, if X and Y are any other two points, let "*the segment XY*" mean "the segment XY not containing O," "*Z is between X and Y*" mean "Z belongs to the segment XY, and is not X or Y." Let two other fixed points, H and K, be taken and let H be declared *greater than* K. Then given two points, X, Y, neither of which is O, "$X > Y$" means that in proceeding along the line from O, X and Y are encountered in the same order as H and K.

There are then two cases of Archimedes' Axiom, viz.

Ax. *a*. *If X is between A and P_0, P_1 is between X and P_0, and P_n is $(A, P_{n-1})/P_{n-2}$, there is a number n such that P_n is between A and X.*

Ax. *b*. *If P_0 is between A and X, P_1 is between P_0 and X, and P_n is $(A, P_{n-1})/P_{n-2}$, there is a number n such that X is between A and P_n.*

"*(X_n) is the harmonic continuation of $(A, C)/B$*" means that X_1 is B, X_2 is C, X_3 is $(A, X_2)/X_1$, …, X_n is $(A, X_{n-1})/X_{n-2}$.

If (P_n) is an increasing[†] sequence of points on L, and A is such that a point exists which is both less than A and greater than all the points P_n, we write $A \rangle\rangle (P_n)$. $A \langle\langle (P_n)$ is similarly defined if (P_n) is a decreasing sequence.

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[*] Cf. Baker, *op. cit.* pp. 47 and 128.

[†] In the strict sense.

4. Lemma 1. *If* (P_n) *is an increasing sequence of points and* $A \rangle\rangle (P_n)$, *while* $D < P_n$ *for large values of* n, *there is a point* B, $\rangle\rangle (P_n)$, *such that* $(B, D)/A < P_n$ *when* n *is large.*

Take any point X_2 which is greater than D, but less than P_n for large values of n. Let (X_n) be the harmonic continuation of $(A, X_2)/D$, X_r the first member of (X_n) which is greater than P_n for all values of n. Let C be a point greater than X_{r-1} but less than P_n for large values of n. The required point B is $(AC)/D$. For since r is at least 3, $D \leqslant X_{r-2}$, and therefore $B > (A, X_{r-1})/X_{r-2}$ which is X_r: i.e. $B \rangle\rangle (P_n)$.

Lemma 2. *If* P_n *is an increasing sequence and* A *and* B *are both* $\rangle\rangle (P_n)$, $(A > B)$, *there is a point* D, *less than* P_n *for large values of* n, *such that* $(B, D)/A \rangle\rangle (P_n)$.

Take any point X_2 less than B but $\rangle\rangle (P_n)$. Let (X_n) be the harmonic continuation of $(A, X_2)/B$, X_r the first member of (X_n) which is less than some P_n. Let D be a point greater than X_r but less than some P_n. Since $r \geqslant 3$, $B \geqslant X_{r-2}$. Hence $(B, D)/A > (X_{r-2}, X_r)/A$ which is X_{r-1}, and this is greater than P_n for all values of n.

Clearly A may be replaced by the origin in Lemma 1, provided there is a point greater than all the points P_n; and in Lemma 2 in all cases.

5. Lemma 3. *If* P_n *is an increasing sequence of points on* L, *and there is a point* A *such that* $P_n < A$ *for all values of* n, *a decreasing sequence*, (Q_n), *can be found such that* $P_n < Q_n$ *for all values of* n, *but not more than one point* X *exists such that, for all values of* n, $P_n < X < Q_n$.

On L take any point S less than P_0. Let $(X_n{}^{(r)})$ $(n = 1, 2, \ldots)$ be the harmonic continuation of $(S, P_r)/P_{r-1}$, Y_r the first member of $(X_n{}^{(r)})$ which $\rangle\rangle (P_n)$. Let Q_1 be Y_1, Q_2 the first member of (Y_r) which is less than Q_1, Q_3 the first member of (Y_r) which is less than Q_2, etc. Clearly, given r, there is a number s, not greater than r, such that Q_s exists and is not greater than Y_r.

Let K be any point $\rangle\rangle (P_n)$. By Lemma 1, Z_1 and Z_2 can be found such that $Z_2 \rangle\rangle (P_n)$, $Z_1 < P_n$ for large values of n, and (SZ_1Z_2K) is harmonic. Suppose n_0 such that $P_{n_0} > Z_1$, and let $X_m^{(n_0+1)}$ be the first of the series of harmonic continuations of $(S, P_{n_0+1})/P_{n_0}$ which is $\rangle\rangle (P_n)$. Since $X_{m-1}^{(n_0+1)} < Z_2$ and $X_{m-2}^{(n_0+1)} > Z_1$, $X_m^{(n_0+1)} < K$. But $X_m^{(n_0+1)}$ is in fact Y_{n_0+1}. Hence for some number, s, not greater than $n_0 + 1$, Q_s exists and is less than K.

We have shown, then, that given any point K, $\rangle\rangle (P_n)$, there is a point, Q_r, which is less than K. It follows that the series (Q_r) does not terminate; for Q_r itself $\rangle\rangle (P_n)$.

This completes the proof of the lemma.

Pairs of series such as (P_n) and (Q_n), of which (P_n) is increasing and Q_n is decreasing, while $P_n < Q_n$ for every n, but which are such that no two fixed points X and Y exist such that $P_n < X < Y < Q_n$ for every n, will be called *facing sequences.*

6. *The Main Theorem.*

We shall prove that there is no point P of the line L such that $A, P, \mu P, B, C$ or $A, P, B, \mu P, C$ are in order. The remaining cases are then obtained by permuting A, B and C among themselves.

Let C be taken as origin, and let the definition of *greater than* be arranged so that $B > A$. Section (i) of the enunciation (paragraph 2 above) may then be replaced by

(i *a*). *If* $P > Q$, *then* $\mu P > \mu Q$.

Suppose P_0 to be a point of AB such that $\mu P_0 > P_0$. Let P_1 be μP_0, P_n be μP_{n-1}, P_{-1} be $\mu^{-1} P_0$, P_{-n} be $\mu^{-1} P_{-n+1}$, where $\mu^{-1} P$ is the point Q such that μQ is P. From (i *a*) it follows that (P_n) is an increasing and (P_{-n}) a decreasing sequence, and that if P is such that, for some large enough value of n, $P_{-n} < P < P_n$, then $\mu P > P$; for if $P_k < P < P_{k+1}$, $P_{k+1} < \mu P < P_{k+2}$. Hence

$$A < P_{-n} < P_n < B$$

for all (positive) values of n.

We shall write "*P belongs to* $\{P_0\}$" for "P is such that, for some large enough value of n, $P_{-n} < P < P_n$." Clearly, if P belongs to $\{P_0\}$ so does $\mu^k P$, where k is any positive or negative integer.

Let (Q_n) be an increasing sequence facing (P_{-n}) on L. By the remark just made there are no integers k and n, and no positive integer m, such that $\mu^k Q_m > P_n$.

(μQ_n) is also an increasing sequence facing P_{-n}. For it is certainly increasing, and $\mu Q_m \ll (P_{-n})$ for all values of m. But if X and Y exist such that, for all values of n, $\mu Q_n < X < Y < P_{-n}$, then $Q_n < \mu^{-1} X < \mu^{-1} Y < P_{-n-1} < P_{-n}$ for all values of n, contrary to the assumption that (Q_n) faces (P_{-n}).

Let R_{2n} be Q_n, R_{2n-1} be μQ_n. The sequence (R_n), all members of which $\ll (P_{-n})$, is not in general an increasing sequence, but *given* n_0, *a number* n_1 *can be found such that* $R_n > R_{n_0}$ *when* $n > n_1$. For since the odd and even terms, taken separately, form two increasing sequences the truth of this statement follows if it can be shown that there is a number r such that $R_{n_0+2r+1} > R_{n_0}$. But if this is not so, every term of the same parity as R_{n_0+1} is less than R_{n_0}, which $\ll (P_{-n})$, contrary to the fact that the sequences (R_{2n}) and (R_{2n+1}) both face (P_{-n}).

Let C_n be $(P_n, R_n)/C$, n taking positive values. Since $P_m > P_n$ when $m > n$, there corresponds to each n_0 a number n_1 such that $C_n > C_{n_0}$ when $n > n_1$. By Lemma 1 a point X can be found in $\{P_0\}$ such that both $(C, X)/Q_0$ and $(C, X)/\mu Q_0 \gg (P_n)$. Since every

R_n is either not less than Q_0 or not less than μQ_0, every C_n is less han X. By Lemma 2 a point Y can be found in $\{P_0\}$ such that $C, Y)/P_0$ is less than some Q_n. Since $R_n > Q_{n_0}$ for any fixed value f n_0 when n is large enough, and $P_n > P_{n_0}$ when $n > 0$, $C_n > Y$ for ll large enough values of n. From these two facts it follows that there is a largest number, n, such that an infinity of members of (C_m) are not less than P_n. Let m_0 be such that $C_m < P_{n+1}$ when $m > m_0$. There is a number, m_1, greater than m_0, such that $C_{m_1} \geqslant P_n$ and a number m_2 such that $C_m > C_{m_1}$ when $m > m_2$, i.e. there is an *even number*, $2m_3$, greater than m_0, such that $C_{2m_3} > P_n$. But C_{2m_3} is $(P_{2m_3}, Q_{m_3})/C$ and therefore, by (ii) of the enunciation, μC_{2m_3} is $(P_{2m_3+1}, \mu Q_{m_3})/C$ which is C_{2m_3+1}. Hence, by the property of points belonging to $\{P_0\}$, $C_{2m_3+1} > \mu P_n$, which is P_{n+1}.

This contradiction shows that the original hypothesis, $\mu P_0 > P_0$, is untenable.

On Differentiation with respect to a Function. By Mr E. C. FRANCIS, Peterhouse.

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Thirty years ago, in a paper on continued fractions*, Stieltjes published a definition of the integral which bears his name. His replacement of the variable of integration x by a more general "base function" $\phi(x)$—a change which throws so much light upon other theories of integration—received at first little attention, but has later sprung into greater prominence; so much so that Professor Hildebrandt, in summarizing these various theories in a paper to the American Mathematical Society†, makes the statement that "it [the Stieltjes Integral] seems destined to play the central rôle in the integrational and summational processes of the future." Yet even now the integral and the allied theory of differentiation with respect to a function have been subjected to little detailed analysis, and the possibilities of extension have been only touched upon. It is the object of this present paper to establish certain results which are of some value in themselves and which prepare the way for an attack upon the integral.

A theory of differentiation of a function $F(x)$ with respect to a function $\phi(x)$ depends essentially upon the study of the limits of an incrementary ratio of the form $v(F)/v(\phi)$ and so must be preceded by and correlated with a Theory of Variation. This we attempt to do. For various reasons—in part only the consideration of subsequent applications to the theory of the Stieltjes Integrals—the functions $F(x)$, $\phi(x)$ are restricted, and we concentrate upon a *strictly increasing* base function $\phi(x)$ and a "*smooth*" function $F(x)$ (i.e. a function whose discontinuities are of the first species only). Such a restriction enables us to define generalized derivatives and to establish some interesting properties, in particular extensions of Rolle's Theorem and the Mean Value Theorem of ordinary differential calculus; and finally to sum them up in a Generalized Variation Theorem which proves under very general conditions that if a derivative $D(x)$ is such that

$$H \leqslant D(x) \leqslant K$$

at points of a set E, then

$$v(\phi) \,.\, H \leqslant v(F) \leqslant v(\phi) \,.\, K,$$

where variations on E are considered.

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* T. J. Stieltjes, "Recherches sur les Fractions Continues," *Ann. de la Fac. des Sc. de Toulouse*, vol. VIII (1894).

† Prof. T. H. Hildebrandt, "On Integrals related to, and extensions of, Lebesgue Integrals," *Bull. Amer. Math. Soc.* vol. XXIV (1918).

The paper concludes with an extension of this theorem to "special" derivatives with regard to a set.

§ 1. VARIATION.

A useful definition of the variation of a function $F(x)$ on a set of points E—we shall denote this by $v(F \mid E)$ or by $v(F)$ or $v(E)$ when there is no ambiguity—must reduce to the measure mE in the case $F(x) \equiv x$. If $F(x)$ is continuous this at once suggests

$$v(F \mid ab) = F(b) - F(a),$$

and upon this basis Denjoy has dealt with the variation of a continuous function on certain simple sets*. If, however, $F(x)$ is discontinuous the lack of the absolutely additive property is sufficient to condemn the definition.

Considerations which are apparent enough make far the simplest case that of *an increasing function* $F(x)$. Here we distinguish between open and closed intervals (ab) and $\langle ab\rangle$; and defining

$$v(F \mid \langle ab\rangle) = F(b+0) - F(a-0),$$
$$v(F \mid (ab)) = F(b-0) - F(a+0),$$

we can extend our definition via open and closed sets to give for any set E upper and lower variations $v^*(E)$ and $v_*(E)$. If these are equal we define $v(E)$ equal to each and say that the set E is *measurable* (F) (or *measurable* when there is no ambiguity). The whole theory is exactly analogous to de la Vallée Poussin's Theory of Measure†.

Extension to functions of bounded variation is obvious‡, but both the form of definition adopted and Denjoy's work on continuous functions suggest a further extension. Provided $F(x \pm 0)$ exist at all points x we can define $v(F \mid I)$ as above, I being any interval; and we are led to single out such functions. We shall call them *smooth* functions. (They include as special cases continuous functions and functions of bounded variation.) A useful general definition of $v(F \mid E)$ is not possible even in this case, but

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* A. Denjoy, "Mémorie sur la Totalization des Nombre Dérivés non-sommables," *Annales de l'École Normale Supérieure* (1916).

† Ch. de la Vallée Poussin, *Intégrales de Lebesgue.*

‡ There is an ambiguity here which is usually allowed to pass unnoticed. $F(x)$, if of bounded variation, can be split up into the difference of two increasing functions $F_1(x)$, $F_2(x)$, but *this can be done in many ways.* One definite way is indicated in all developments of the theory of functions of bounded variation and this is best regarded as standard. If we denote the corresponding functions by $F_1(x)$, $F_2(x)$, we can prove the following Consistency Theorem (often tacitly assumed):

If $G_1(x)$, $G_2(x)$ are any two increasing functions such that

$$F(x) = G_1(x) - G_2(x),$$

and if a set E is measurable (G_1) and (G_2): then E is measurable (F_1) and (F_2) and

$$v(F_1 \mid E) - v(F_2 \mid E) = v(G_1 \mid E) - v(G_2 \mid E).$$

Since the subject is somewhat removed from the main object of the paper I omit the proof.

we can extend Denjoy's method to treat the case when E is a *simple set*—i.e. is formed of a finite or infinite number of intervals u_n or is the complement of such intervals. (Open and closed sets, in particular, are simple.) If $\Sigma \mid v(u_n) \mid$ is convergent we say that $F(x)$ is of *definite variation* on E and define vE as follows:

$$\text{if } \; E = \Sigma u_n \qquad \text{we define } vE = \Sigma v u_n;$$
$$\text{if* } E = I - \Sigma u_n \; \text{ we define } vE = vI - \Sigma v u_n.$$

For most purposes we are concerned with the value of $F(x)$ in the neighbourhood of a point rather than at the point itself and we define a *strictly smooth* function as a smooth function for which $F(x) = \frac{1}{2}\{F(x+0) + F(x-0)\}$. Any given smooth function can be replaced by the obvious "corresponding" strictly smooth function without altering its variation on any set.

The properties of the variation of an increasing function (or of a function of bounded variation) will be taken as known. They include in particular the "absolutely additive" property and that expressed by the equation

$$v(\lim E_n) = \lim (vE_n),$$

which holds if the measurable† sets E_n have a limit set. We can clearly define "measurable (F)" functions: their properties, too, are deduced exactly as when $\phi(x) \equiv x$.

In the case of smooth functions which are not necessarily of bounded variation the properties are considerably restricted, but the following results present no difficulty:

A. *A smooth function is bounded.*

B. *If F_1, F_2, F_3 are functions such that $F_1 = F_2 + F_3$, and if any two of them are strictly smooth and of definite variation on a simple set S, so is the third and*

$$v(F_1 \mid S) = v(F_2 \mid S) + v(F_3 \mid S).$$

The following "absolutely additive" property follows at once by transfinite induction:

If (i) *$F(x)$ is smooth,*
(ii) *(I) can be split up into a denumerable set of intervals (J),*
(iii) *$v(F \mid J)$ has always the same sign,*

then

$$v(F \mid I) = \Sigma v(F \mid J).$$

We have also the product theorem:

If F_1, F_2 are strictly smooth and of definite variation on S then $F_1 F_2$ is smooth and of definite variation on S. ($F_1 F_2$ is strictly smooth if F_1 and F_2 are positive.)

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[*] Throughout this paper I will denote an interval.

[†] I.e. measurable with respect to the base function in question.

§ 2. DERIVATIVES.

We can now proceed to a definition of derivatives with respect to a function. To be useful this should be intimately connected with our theory of variation, should have at least the additive property, and should reduce to the ordinary definition in the case when the base function $\phi(x) \equiv x$. A failure to comply with the last requirement condemns a definition given by P. J. Danniell*.

Difficulty will clearly be caused if the denominator of our incrementary ratio can become zero; and we are led to restrict the base function $\phi(x)$ to be strictly increasing†.

The function to be differentiated, $F(x)$, we can take to be smooth, and we define as follows four generalized derivatives, denoted (after Young) by $F^+(x)$, $F_+(x)$, $F^-(x)$, $F_-(x)$. When these are all equal we say that a differential coefficient $F_\phi{}'(x)$ (or $F'(x)$ if there is no ambiguity) exists and is equal to each of them. We consider three cases:

$$\text{(i)}\ \left\{\begin{aligned}\phi(x+0)&=\phi(x-0)\\ F(x+0)&=F(x-0)\end{aligned}\right. \qquad \left.\begin{aligned} F^+(x) &= \overline{\lim_{h\to+0}} \\ F_+(x) &= \underline{\lim_{h\to+0}} \\ F^-(x) &= \overline{\lim_{h\to-0}} \\ F_-(x) &= \underline{\lim_{h\to-0}}\end{aligned}\right\}\left(\frac{F(x+h)-F(x)}{\phi(x+h)-\phi(x)}\right).$$

$$\text{(ii)}\ \ \phi(x+0) \neq \phi(x-0) \qquad \begin{aligned} F^+(x) = F_+(x) = F^-(x) &= F_-(x) \\ &= \frac{F(x+0)-F(x-0)}{\phi(x+0)-\phi(x-0)}.\end{aligned}$$

$$\text{(iii)‡}\ \left\{\begin{aligned}\phi(x+0)&=\phi(x-0)\\ F(x+0)&\neq F(x-0)\end{aligned}\right. \qquad \begin{aligned} F^+(x) = F_+(x) = F^-(x) &= F_-(x) \\ &= \begin{cases}+\infty \text{ if } F(x+0) > F(x-0),\\ -\infty \text{ if } F(x+0) < F(x-0).\end{cases}\end{aligned}$$

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[*] P. J. Danniell, "On differentiation with regard to a function of limited variation," *Trans. Amer. Math. Soc.* vol. XIX (1918).

Danniell defines two derivatives (upper and lower) by

$$\left.\begin{aligned}\overline{D}(F(x)) &= \overline{\lim_{h\to0}}\\ \underline{D}(F(x)) &= \underline{\lim_{h\to0}}\end{aligned}\right\}\left(\frac{F(x+h)-F(x-h)}{\phi(x+h)-\phi(x-h)}\right).$$

If $\overline{D}=D$, $F(x)$ has a differential coefficient equal to each.

But on this definition the function which is equal to x when $x \geq 0$, and to $-x$ when $x<0$, has differential coefficient zero at the origin.

[†] I.e. $\phi(x_1) > \phi(x_2)$ if $x_1 > x_2$.

[‡] For many purposes $F(x)$ is "continuous with ϕ" (i.e. (iii) is excluded).

§ 3. ELEMENTARY PROPERTIES.

The following results are easily proved:

C. *A constant has everywhere differential coefficient zero.*

D. *The base function has everywhere differential coefficient unity.*

E. *If D denotes any derivative and c a positive constant, then*

$$D(cF) = c(DF).$$

F. *If $F'(x)$ exists and c is any constant, then $(cF)' = c \, . \, F'$.*

G. *If $F_1(x)$, $F_2(x)$ are strictly smooth and $F(x) = F_1(x) + F_2(x)$, then*

$$F_1^+(x) + F_2^+(x) \geqslant F^+(x) \geqslant \begin{Bmatrix} F_1^+(x) + F_{2+}(x) \\ F_{1+}(x) + F_2^+(x) \end{Bmatrix}$$
$$\geqslant F_+(x) \geqslant F_{1+}(x) + F_{2+}(x).$$

H. *If $F_1(x)$, $F_2(x)$ are strictly smooth and have differential coefficients this is also true of*

$$F(x) \; (= F_1(x) + F_2(x)),$$

and we have $$F'(x) = F_1'(x) + F_2'(x).$$

Further if $G(x) = F_1(x) \, . \, F_2(x)$, $G(x)$ is smooth and has a differential coefficient ($G(x)$ is strictly smooth if F_1, F_2 are positive), and

$$G'(x) = F_1'(x) F_2(x) + F_1(x) F_2'(x).$$

I. *If $F_\psi'(x)$, $\psi_\phi'(x)$ both exist and are finite then $F_\phi'(x)$ exists and is equal to their product.*

J. *If strictly smooth functions $F(x)$, $G(x)$ are such that*

$$F_\phi'(x) = G_\psi'(x),$$

then each of these expressions is equal to $(F+G)'_{\phi+\psi}$.

K. *$F'(x)$ and $\lim\limits_{I \to 0} \left(\dfrac{v(F \mid I)}{v(\phi \mid I)} \right)$, where I is any interval containing the point x, exist together and are equal.*

This theorem shows the close connection between the theories of differentiation and variation.

§ 4. ROLLE'S THEOREM AND ITS CONSEQUENCES.

We are now in a position to extend certain theorems well known in ordinary differential calculus. To do so we make use of the following lemma due to Lebesgue*:

If every point of an interval I is the left-hand end-point of an interval J, then I can be completely covered by at most a denumerable set of non-overlapping intervals J.

We require the following extension:

If every point of a closed set Q is the left-hand end-point of an

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* Lebesgue, *Leçons sur l'Intégration*, p. 63.

nterval J, then Q can be completely covered by at most a denumerable et of non-overlapping intervals J.

This follows easily from the original lemma. For let a, b be he end-points of Q. Associate with each point x falling in a black nterval $(a_n b_n)$ of Q the interval $\langle x b_n \rangle$. $\langle ab \rangle$ can therefore be covered by a finite or denumerable set of intervals—some intervals J and some intervals of type $\langle x b_n \rangle$. Intervals of the latter type contain no points of Q. Omit them and the extended lemma is proved.

L. *Rolle's Theorem:*

If $D(x)$ is any given derivative of $F(x)$ and I an interval, other than a single point, for which $v(F \mid I) \geqslant 0$, then there is some point ξ of I at which $D(\xi) \geqslant 0$.

For simplicity we take $D(x) = F_+(x)$. The theorem follows *a fortiori* for $F^+(x)$, and is proved for $F^-(x)$, $F_-(x)$ by interchanging left and right.

We may take $v(F \mid I) > 0$:
for if $v(F \mid I) = 0$ then either $F(x)$ is constant in I (and the theorem is true) or there is some point x of I at which one of $F(x \pm 0)$ is different from $F(a)$ (where a stands for $a \pm 0$ according as I is open or closed at its left-hand end a). There is therefore an interval contained in I on which $v(F) > 0$.

We may take I closed:
for by our definition of variation I contains a closed interval I' such that $v(F \mid I') > 0$.

Making these assumptions we suppose the theorem untrue. With each point x of I we can associate an interval $J(x) \equiv \langle x x' \rangle$, as small as we please, and such that

$$v(F \mid J(x)) < 0,$$

for

(i) if $F(x)$ is discontinuous at x

$$F(x+0) < F(x-0),$$

we can therefore choose $x' > x$ and as near x as we please such that

$$F(x'-0) < F(x-0).$$

(ii) if $F(x)$ is continuous at x there is no difficulty. Our definition of $F_+(x)$ shows at once that we can choose x' as required.

We can now apply Lebesgue's Lemma. This shows that I can be covered by at most a denumerable infinity $J_1, J_2, \ldots$ of such intervals $J(x)$, non-overlapping and extending to a point $b' > b$ and as near to b as we please. Only one of these intervals extends beyond b.

By a theorem already stated

$$v(F \mid \langle ab' \rangle) = \Sigma v(F \mid J_n),$$

since the terms are all negative.

Hence $$v(F \mid \langle ab' \rangle) \leqslant 0.$$

This is true for b' as near b as we please. Hence

$$v(F \mid \langle ab \rangle) \leqslant 0,$$

and we have a contradiction; which proves the theorem.

M. *The Generalized Mean Value Theorem:*

If $D(x)$ is any given derivative of $F(x)$ and I any interval, then there are points ξ, ξ' contained in I and such that

$$D(\xi').v(\phi \mid I) \leqslant v(F \mid I) \leqslant D(\xi).v(\phi \mid I).$$

We may consider $F(x)$, $\phi(x)$ to be *strictly* smooth, for the substitution of the corresponding strictly smooth functions affects neither derivatives nor variation.

Define $\psi(x)$ by

$$\psi(x) = F(x).v(\phi \mid I) - (\phi(x) - \phi(\alpha)).v(F \mid I),$$

where α stands for $a \pm 0$ according as I is open or closed at a. $\psi(x)$ is strictly smooth and $v(\psi \mid I) = 0$.

By Theorems **C**, **D**, **F**, **G**

$$D(\psi(x)) = D(F(x)).v(\phi \mid I) - v(F \mid I).$$

But by the Generalized Rolle's Theorem there are points ξ, ξ' of I such that

$$D(\psi(\xi)) \geqslant 0 \geqslant D(\psi(\xi')).$$

This proves the theorem.

As a corollary we note the following:

N. *If at each point of an interval I, $F(x)$ has a median or extreme derivative zero on a given side, then $F(x)$ is constant in I.*

For suppose the given side is the right and that $v(F \mid I_1) > 0$ for some interval I_1 contained in I. By the last theorem there is a point ξ in I, at which

$$F_+(\xi) \geqslant \frac{v(F \mid I_1)}{v(\phi \mid I_1)} > 0.$$

Hence every median or extreme right derivative is greater than zero, which contradicts our hypothesis. An exactly similar argument for the other possible cases proves the theorem.

The Mean Value Theorem now serves to prove a more general result which embraces it as a special case:

O. *The Generalized Variation Theorem:*

If $D(x)$ is any given derivative of $F(x)$ and E a set measurable (ϕ) *on which*

$$H \leqslant D(x) \leqslant K^*;$$

then $$H.v(\phi \mid E) \leqslant v(F \mid E) \leqslant K.v(\phi \mid E),$$

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* By this statement we mean that if one only of these inequalities is true, then the corresponding inequality of the result holds. We are *not* assuming $D(x)$ bounded.

rovided that either

(i) *E is an interval I.*

(ii) *E is an open set O on which $F(x)$ is of definite variation.*

(iii) *$F(x)$ is of bounded variation* and E is measurable (F).*

Case (i) is contained in the Generalized Mean Value Theorem.

Case (ii). If $O = \Sigma u_n$ we have from case (i)

$$Hv(\phi \mid u_n) \leqslant v(F \mid u_n) \leqslant Kv(\phi \mid u_n).$$

But by our hypothesis both $\Sigma \mid v(\phi \mid u_n) \mid$ and $\Sigma \mid v(F \mid u_n) \mid$ are convergent. Hence

$$Hv(\phi \mid O) \leqslant v(F \mid O) \leqslant Kv(\phi \mid O).$$

Case (iii). This presents considerably more difficulty. The theorem is untrue unless we put some restriction upon $F(x)$. We begin with the case when E is a closed set Q (black intervals u_n). From our assumption that $F(x)$ is of bounded variation we note that ΣW_n is convergent where $W_n = \text{osc}(F \mid u_n)$.

We may assume $H < D(x) < K$: for if the theorem is true in this case, then, under the given conditions we know that for every $H' < H$, $K' > K$

$$H'v(\phi \mid E) \leqslant v(F \mid E) \leqslant K'v(\phi \mid E).$$

Thus $$Hv(\phi \mid E) \leqslant v(F \mid E) \leqslant Kv(\phi \mid E),$$

and the theorem is proved.

For simplicity we shall consider only the case $F_+(x) > H$. The others are treated by similar methods.

We proceed to show that every point x of Q is the left-hand end-point of an interval $J(x) = \langle xx' \rangle$, as small as we please, such that

$$v(F \mid J(x)) > H \,.\, v(\phi \mid J(x)) \qquad \text{......(1)}.$$

There are two cases:

(*a*) x a point of discontinuity of $\phi(x)$ or of $F(x)$.

Therefore

$$F(x+0) - F(x-0) > H\{\phi(x+0) - \phi(x-0)\}.$$

There is therefore some point $x' > x$ and as near x as we please, such that

$$F(x'-0) - F(x-0) > H\{\phi(x'-0) - \phi(x-0)\}.$$

(*b*) x a point of continuity of $\phi(x)$ and of $F(x)$.

The result is now an immediate consequence of our definition of $F_+(x)$.

Hence in all cases $J(x)$ can be chosen so that (1) holds: but this result is insufficient as it stands. We must still further restrict our choice of $J(x)$ by means of our assumptions as to $F(x)$.

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* This condition is made less stringent by later results in §§ 5, 6

Choose positive numbers $\epsilon_1, \epsilon_2, \ldots$ of sum less than ϵ. We can find N such that

$$\sum_{N+1}^{\infty} W_n < \epsilon.$$

Now $J(x)$ can be as small as we please.

If x is the left-hand end-point a_n of a black interval u_n where $1 \leqslant n \leqslant N$, choose x' such that

$$|F(x'-0) - F(x+0)| < \epsilon_n.$$

In all other cases choose $J(x)$ so small as to contain no portion of the intervals $u_1, u_2, \ldots u_N$.

The extension of Lebesgue's Lemma already proved now enables us to choose a denumerable (or finite) set of non-overlapping intervals $J(x)$ to contain every point of Q. For each of these

$$v(F \mid J(x)) > Hv(\phi \mid J(x)) \qquad \ldots\ldots(1).$$

Further, we see by choice of our intervals and by the fact that the variation of a function of bounded variation is absolutely additive, that if

$$v(F \mid J) = v(F \mid QJ) + \eta \qquad \ldots\ldots(2),$$

then

$$\begin{aligned}\Sigma \mid \eta \mid &\leqslant \epsilon_1 + \epsilon_2 + \ldots + \epsilon_N + \epsilon \\ &< 2\epsilon \qquad \ldots\ldots(3).\end{aligned}$$

But by the same absolutely additive property

$$v(F \mid Q) = \Sigma v(F \mid QJ).$$

Combined with equations (2) and (3) above we have therefore

$$\begin{aligned}v(F \mid Q) + 2\epsilon &> \Sigma v(F \mid J) \\ &> H \,.\, \Sigma v(\phi \mid J) \\ &\geqslant H \,.\, v(\phi \mid Q).\end{aligned}$$

Since ϵ is arbitrary we deduce the result required at once:

$$v(F \mid Q) \geqslant H \,.\, v(\phi \mid Q).$$

The theorem is now proved for a closed set. The extension to the general set E contemplated in case (iii) is immediate. For we know that for every closed set Q contained in E

$$H \,.\, v(\phi \mid Q) \leqslant v(F \mid Q) \leqslant K \,.\, v(\phi \mid Q).$$

Since $v(\phi \mid E)$, $v(F \mid E)$ both exist it follows that

$$H \,.\, v(\phi \mid E) \leqslant v(F \mid E) \leqslant K \,.\, v(\phi \mid E).$$

P. As a corollary we may notice that *if* $F(x)$ *is an increasing function, and if, on any set* E, *a derivative* $D(x)$ *of* $F(x)$ *is such that*

$$H \leqslant D(x) \leqslant K,$$

then

$$H \,.\, v_*(\phi \mid E) \leqslant v_*(F \mid E) \leqslant K \,.\, v_*(\phi \mid E).$$

Q. If, under the same conditions, *$F(x)$ is a function of bounded variation only, we have*

$$v_*(F \mid E) \leqslant Kv_*(\phi \mid E),$$
$$v^*(F \mid E) \geqslant Hv_*(\phi \mid E).$$

§ 5. SPECIAL DERIVATIVES.

It happens often that we are concerned only with the values which our functions $F(x)$, $\phi(x)$ take *at points of a set E*. We therefore introduce the idea of a *special derivative on E* which shall take into account only such values.

At a point x of a set E we define four special derivatives $(F^+(x))_E, (F_+(x))_E, (F^-(x))_E, (F_-(x))_E$ by repeating the definitions of § 2 with the added proviso that in case (i) the points $(x+h)$ must belong to E. Thus when $\phi(x)$ is discontinuous at x all four special derivatives are defined (and are equal), while if x is the limit point of points of E on either side the two corresponding special derivatives exist.

The extension to special differential coefficient is obvious: and the Theorems **C**, **D**, **E**, **F**, **G**, **H**, **I**, **J** are seen still to be true for special derivatives.

We shall conclude this paper by proving the following extended form of the Variation Theorem:

R. *If $F(x)$ is a function of bounded variation and $D(x)$ is any given special derivative of $F(x)$ on a closed set Q which is such that*

$$H \leqslant D(x) \leqslant K^*$$

at all points of Q; then

$$H \, . \, v(\phi \mid Q) \leqslant v(F \mid Q) \leqslant K \, . \, v(\phi \mid Q).$$

We can extend the previous proof to deal with this case.

No modification is needed until we come to choose intervals $J(x)$ in case (*b*); and here the difficulty occurs only when x is a point of continuity of both $F(x)$ and $\phi(x)$ and is the left-hand end-point a_n of a black interval u_n of Q. We know nothing of the right-hand derivatives at such a point x. We therefore choose $\langle xx' \rangle$ so small that x' lies in u_n and

$$| v(F \mid \langle xx' \rangle) | < \epsilon_n.$$

Equation (1) may now be false for such intervals $\langle xx' \rangle$.

As before we have

$$v(F \mid J) = v(F \mid QJ) + \eta \qquad \text{......(2)},$$
$$\Sigma \mid \eta \mid < 2\epsilon \qquad \text{......(3)},$$
$$v(F \mid Q) = \Sigma v(F \mid QJ) \qquad \text{......(4)},$$

so that
$$v(F \mid Q) + 2\epsilon \geqslant \Sigma v(F \mid J)$$
$$> H \, . \, \Sigma' v(\phi \mid J) - \epsilon,$$

[*] See note on p. 930.

where Σ' extends over those J intervals for which (1) is true,

$$\geqslant H \,.\, v\,(\phi \mid Q) - \epsilon,$$

since at most a denumerable set of points a_n of Q, at each of which $\phi\,(x)$ is continuous, are excluded from the intervals of Σ'. Hence as before

$$v\,(F \mid Q) \geqslant H \,.\, v\,(\phi \mid Q).$$

§ 6. A FURTHER EXTENSION. TOTAL VARIATION ON A CLOSED SET.

The *result* of Theorem **R** is concerned only with values of $F\,(x \pm 0)$ and of $\phi\,(x \pm 0)$ at points x of Q. The *hypotheses* include the assumption that $F\,(x)$ is of bounded variation—an assumption which involves values of $F\,(x)$ at other points x. We are naturally led to suppose that the theorem can be improved, as is indeed the case.

We define the phrase $F\,(x)$ *is of bounded total variation on* Q in the case of a function $F\,(x)$ for which points of Q are discontinuities of the first species only (i.e. which is *smooth on* Q), as meaning that, if I is any interval containing Q, and if I is divided in any way into a finite number of distinct intervals

$$I = I_1 + I_2 + \ldots + I_n,$$

then

(i) $F\,(x)$ *is of definite variation on* Q.

(ii) $\Sigma \mid v\,(QI_r) \mid$ *is bounded for all divisions of the type contemplated.*

The upper bound of the sum in (ii) we define as the *total variation of* $F\,(x)$ *on* Q.

It can now be shown that Theorem **R** is true if the condition "$F\,(x)$ *is of bounded variation*" is replaced by the less stringent one that "$F\,(x)$ *is of bounded total variation on* Q."

The Lebesgue-Stieltjes Integral. By Mr E. C. FRANCIS, Peterhouse.

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It has long been realized that a Stieltjes integral can be defined and developed on the same lines as a Lebesgue integral, using in place of the measure of a set E the variation of an increasing function $\phi(x)$ upon E; but only the more obvious—and less useful—properties of such an integral seem to have been stated. The original integral defined by Stieltjes[*] possesses the remarkable Integration by Parts property that if either of $\int_a^b f d\phi$ and $\int_a^b \phi\, df$ exists, so does the other, and their sum is equal to

$$f(b)\phi(b) - f(a)\phi(a).$$

Stieltjes' definition does not lend itself to systematic analysis and has been largely replaced by definitions similar to Darboux's definition of the "Riemann integral"; but for the integrals so produced this Integration by Parts result—in which lies the chief claim to importance of the Stieltjes integral—has usually either been ignored or been proved in a form shorn of its generality[†]. For the Lebesgue-Stieltjes integral it is, in fact, true under most general conditions[‡]. An attempt has here been made to give a systematic sketch of the development of the integral, leading up to three main theorems—Change of Variable, Integration by Parts, and the Second Mean Value Theorem—theorems which display in striking manner the fundamental character of this conception of integration with respect to a function.

Making use of the theory of variation and of differentiation with respect to a function contained in an earlier paper[§], I have first proved the fundamental differential properties of the *LS*-integral; and have used these as a basis for the theorem on change

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[*] T. J. Stieltjes, "Recherches sur les Fractions Continues," *Ann. de la Fac. des Sc. de Toulouse*, vol. VIII (1894).

[†] E.g. W. H. Young, "Integration with respect to a Function of Bounded Variation," *Proc. L. M. S.* vol. XIII (1913), proves that if u and v are positive increasing functions such that both u has an *RS* (Riemann-Stieltjes) integral with respect to v and v with respect to u, then

$$\int u\,dv + \int v\,du = uv + \text{constant},$$

where the constant is zero if either function is continuous.

[‡] The Integration by Parts theorem proved in this paper has also been given—proved by quite different methods—by P. J. Danniell, "On differentiation with regard to a function of limited variation," *Trans. Amer. Math. Soc.* vol. XIX, p. 353 (1918).

[§] E. C. Francis, "On Differentiation with respect to a Function," *Proc. Camb. Phil. Soc.* vol. 22, p. 924.

of variable. The Integration by Parts property has been proved without making use of the idea of differentiation, by establishing the connection between the *LS*-integral and the original Stieltjes integral.

§ 1. THE ORIGINAL STIELTJES-INTEGRAL.

Some reasons have already been given for the fact that the original Stieltjes-integral (or, as we shall call it, the *S*-integral) has something more than an historical interest. This integral has been dealt with somewhat exhaustively in a recent paper by S. Pollard[*], and we shall confine ourselves here to a statement, for reference, of the definition and of the Integration by Parts property.

Definition. Take any division of (ab) *by a finite number of points*

$$a = x_0 < x_1 < x_2 \dots < x_n = b.$$

Let ξ_ι *be any point such that* $x_{\iota-1} \leqslant \xi_\iota \leqslant x_\iota$.
Consider the sum $\mathcal{S} = \Sigma f(\xi_\iota)\{\phi(x_\iota) - \phi(x_{\iota-1})\}$.
If $\mathcal{S}$ *tends to a limit, for all possible* ξ_ι, *as* max $|x_\iota - x_{\iota-1}| \to 0$, *then* $f(x)$ *is integrable* (S) *on* (ab) *and*

$$S\int_a^b f(x)\,d\phi(x) = \lim \mathcal{S}.$$

As an immediate consequence of this definition we have the remarkable result:

If $S\int_a^b f d\phi$ *exists so does* $S\int_a^b \phi\, df$ *and*

$$S\int_a^b f d\phi + S\int_a^b \phi\, df = f(b)\phi(b) - f(a)\phi(a).$$

It is upon this formula that we base our proof of an even more general theorem for the *LS*-integral.

§ 2. THE LEBESGUE-STIELTJES INTEGRAL.

We define and develop a Lebesgue-Stieltjes integral $LS\int_E f d\phi$ by straightforward methods, following closely Young's development[†] of the Lebesgue-integral. Since *variation* is involved we must restrict $\phi(x)$, and shall, in the first place, take it to be an *increasing function*, later extending our theory to embrace a function of bounded variation. $f(x)$ we assume *bounded*.

Definition. Consider any division of E *into a finite number of measurable* (ϕ) *sets* $E_1, E_2 \dots E_n$. *Let* M_ι, m_ι *be the upper and lower bounds of* $f(x)$ *in* E_ι. *Define the Partial Sums* $S = \Sigma M_\iota \,.\, v(\phi \mid E_\iota)$.

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[*] S. Pollard, "The Stieltjes Integral and its Generalizations," *Q. J.* vol. XLIX, 1920–21.

[†] W. H. Young, "On the general theory of integration," *Phil. Trans. R. S.* (A), 204, pp. 221–252 (1904).

$s = \Sigma m_\iota . v(\phi \mid E_\iota)$. *It is easily seen that every* $S \geqslant$ *every* s, *so that the sums* S *have a lower bound* I *and the sums* s *have an upper bound* I', *where* $I \geqslant I'$. *If* $I = I'$ *we say that* $f(x)$ *is integrable* (ϕ) *on* E *and define*

$$LS \int_E f d\phi = I = I'.$$

A measurable function is easily shown to be integrable and the converse can be proved.

For some purposes it is convenient to use only integrals over an interval. This is always possible, for it is easy to establish the equivalence of the integral above to $\int_I f_1 d\phi$, where the interval I contains E and

$$f_1(x) \begin{cases} = f(x) \text{ on } E, \\ = 0 \text{ elsewhere.} \end{cases}$$

The *LS*-integral thus defined for bounded functions $f(x)$ can be extended, as in the case of the *L*-integral, to certain *unbounded functions*. We say that $f(x)$ is *summable* on E in each of the following three cases:

(i) $f(x) \geqslant 0$ on E and $\lim\limits_{n\to\infty} \left[LS \int_E f_n d\phi \right]$ exists, where

$$f_n(x) \begin{cases} = f(x) \text{ when } f(x) \leqslant n, \\ = n \text{ elsewhere.} \end{cases}$$

(ii) $f(x) \leqslant 0$ on E and $-f(x)$ is summable on E.

(iii) $f(x)$ is the difference of two functions each positive and summable on E.

We give the obvious interpretation to $LS \int_E f d\phi$ in each case.

§ 3. ELEMENTARY PROPERTIES.

The following properties of the *LS*-integral extended as above are deduced by straightforward methods, and proofs will be omitted:

A. *If* E *is measurable* $\int_E d\phi$ *exists and is equal to* $v(\phi \mid E)$.

B. *If* $\int_E f d\phi$ *exists so does* $\int_E cf d\phi$ *and*

$$\int_E cf d\phi = c \int_E f d\phi,$$

c being a constant.

C. *The First Additive Theorem. If* $\int_E f d\phi$ *exists and* E *can be* *split up into two measurable sets* E_1, E_2, *then* $\int_{E_1} f\,d\phi$ *and* $\int_{E_2} f\,d\phi$ *both exist and are together equal to* $\int_E f\,d\phi$.

D. *The Second Additive Theorem. If* $\int_{E_1} f\,d\phi$, $\int_{E_2} f\,d\phi$ *both exist, and if* E_1, E_2 *are distinct sets with sum* E, *then* $\int_E f\,d\phi$ *exists and is equal to the sum of the other two integrals.*

E. *The f-Sum Theorem. If* $\int_E f_1\,d\phi$, $\int_E f_2\,d\phi$ *both exist, then* $\int_E (f_1+f_2)\,d\phi$ *exists and is equal to their sum.*

F. *The* ϕ-*Sum Theorem. If* $\int_E f\,d\phi_1$, $\int_E f\,d\phi_2$ *both exist, then* $\int_E f\,d(\phi_1+\phi_2)$ *exists and is equal to their sum.*

G. *The integral is "absolutely convergent." If* $\int_E f\,d\phi$ *exists so does* $\int_E |f|\,d\phi$ *and* $\int_E |f|\,d\phi \geqslant \left|\int_E f\,d\phi\right|$.

H. *If* $\int_E f_1\,d\phi$, $\int_E f_2\,d\phi$ *both exist, and* $f_1(x) \geqslant f_2(x)$ *on* E, *then* $\int_E f_1\,d\phi \geqslant \int_E f_2\,d\phi$.

I. *The First Mean Value Theorem* is a special case of **H**.

If $\int_E f\,d\phi$ *exists and* $f(x) \geqslant m$ *on* E, *then* $\int_E f\,d\phi \geqslant mv(\phi\,|\,E)$: a similar result holding for the reverse inequality.

J. *Lebesgue's Theorem.* (True in a more general case than that stated):

If $f_1(x), f_2(x), \ldots$ *is a bounded sequence of measurable functions which tend to* $f(x)$ *on* E, *then* $\int_E f\,d\phi$ *exists and* $= \lim\limits_{n\to\infty} \int_E f_n\,d\phi$.

K. *The First Absolutely Additive Theorem. If* $\int_E f\,d\phi$ *exists and* E *is split up into a denumerable infinity of measurable subsets* $E_1, E_2, \ldots$, *then* $\int f\,d\phi$ *exists over each of these sets and*

$$\int_E f\,d\phi = \Sigma \int_{E_n} f\,d\phi.$$

L. *The Absolute Continuity Property. If* $LS\int_E f d\phi$ *exists, then given any positive number* ϵ *there is an* η $(=\eta(\epsilon))$ *such that for all measurable subsets* S *of* E *for which* $v(\phi \mid S) < \eta$ *we shall have*

$$\left|\int_S f d\phi\right| < \epsilon.$$

An indefinite integral $F(x)$ is defined as follows at points x of an interval $\langle ab\rangle$ on which $f(x)$ is summable:

$$F(x-0) = \int_{\langle ax)} f d\phi,$$

$$F(x+0) = \int_{\langle ax\rangle} f d\phi,$$

$$F(x) = \tfrac{1}{2}\{F(x+0) + F(x-0)\}.$$

The notation is free from ambiguity since we can show without difficulty that

$$F(x \pm 0) = \lim_{\xi \to x \pm 0} F(\xi).$$

We have the following properties.

M. *$F(x)$ is of bounded variation.*

N. *$F(x)$ has discontinuities of the first species only and these only where $\phi(x)$ is discontinuous.* It is thus a "*strictly smooth*" *function** *continuous with* ϕ†.

O. *If E is any measurable*‡ *subset of* $\langle ab\rangle$, *then*

$$v(F \mid E) = LS\int_E f d\phi.$$

This theorem is of considerable importance in what follows. It is clearly true when E is an interval and extends at once to the case of an open or a closed set. For the general set it can be deduced without difficulty from the definition of measurability and the absolute continuity property **L**.

The case when $\phi(x)$ is a function of bounded variation.

Since a function of bounded variation is the difference of two increasing functions it is possible to extend our definition of $LS\int_E f d\phi$ to the case when $\phi(x)$ is such a function.

The new integral retains the majority of the properties of the old. Theorems **A** to **F**, **J**, **K**, **M**, **N** and **O** all remain true.

G, **I** and **L** may be modified in a fairly obvious manner by introducing a function $\bar{\phi}(x)$ to represent the "total variation" of $\phi(x)$.

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* E. C. Francis, *loc. cit.*

† E. C. Francis, *ibid.*

‡ I.e. measurable with respect to ϕ.

§ 4. THE INTEGRATION BY PARTS FORMULA.

We shall deduce the "Integration by Parts" result for the LS-integral by the straightforward process—free from all appeal to differential properties—of connecting the LS-integral with the S-integral (of which we have already noticed the Integration by Parts property). We establish the following theorem:

P. *If $\phi(x)$ is continuous and of bounded variation, and $S\int_a^b f d\phi$ exists, then $LS\int_a^b f d\phi$*[*] *exists and has the same value.*

Let
$$S\int_a^b f d\phi = J.$$

With the notation of § 1 and § 2, we know that, given $\epsilon > 0$, we can find a division $x_1, \ldots x_n$ such that
$$|\mathscr{S} - J| < \epsilon$$
for this division and for every choice of ξ_ι. Hence
$$|\Sigma M_\iota \{\phi(x_\iota) - \phi(x_{\iota-1})\} - J| \leqslant \epsilon,$$
$$|\Sigma m_\iota \{\phi(x_\iota) - \phi(x_{\iota-1})\} - J| \leqslant \epsilon,$$
and so
$$S - s \leqslant 2\epsilon.$$

This is a sufficient condition that $LS\int_a^b f d\phi$ exists and is equal to J.

We have therefore at once the following result:

Q. *If $f(x)$, $\phi(x)$ are both continuous and of bounded variation in I, then*
$$LS\int_I f d\phi + LS\int_I \phi df = v(f\phi \mid I).$$

The assumption of continuity, however, is unnecessary and we proceed to the general theorem:

R. *Integration by Parts Formula:*

If $f(x)$, $\phi(x)$ are strictly smooth and of bounded variation on I, then
$$LS\int_I f d\phi + LS\int_I \phi df = v(f\phi \mid I).$$

From the Sum and Additive Theorems it is seen to be sufficient prove the theorem when $f(x)$, $\phi(x)$ are monotonic. In this case can resolve them in a manner suggested by Lebesgue*, giving

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[*] When $\phi(x)$ is continuous we have seen that the LS-integral is continuous. It is therefore possible to revert to the simpler notation $LS\int_a^b f d\phi$ in place of $LS\int_{\langle ab\rangle} f d\phi$ or $LS\int_{(ab)} f d\phi$.

$$f(x) = f'(x) + f''(x),$$
$$\phi(x) = \phi'(x) + \phi''(x),$$

here $f'(x)$, $\phi'(x)$ are monotonic continuous functions; and $f''(x)$, $''(x)$ are typified by $\phi''(x)$, where

$$\phi''(x-0) = \sum_{\langle ax)} o_n, \qquad \phi''(x+0) = \sum_{\langle ax\rangle} o_n;$$

o_2, ... being the oscillations of $\phi(x)$ over the denumerable (or nite) set of its discontinuities q_1, q_2, ... denumerated in order of agnitude of the corresponding o's.

Again applying the Sum and Addition Theorems it is sufficient o prove the theorem in the three following cases:

(*a*) $f(x)$, $\phi(x)$ continuous and monotonic.

(*b*) $f(x)$ continuous and monotonic: $\phi(x)$ of type $\phi''(x)$.

(*c*) $f(x)$, $\phi(x)$ both of type $\phi''(x)$.

Case (*a*). The theorem has just been proved (**Q**).

Case (*b*). For simplicity suppose $\phi''(x)$ increasing. Let $\bar{M}$ be the upper bound of $f(x)$ in I. Given $\epsilon > o$, choose N such that

$$\sum_{N+1}^{\infty} o_n < \frac{\epsilon}{5\bar{M}} \qquad \text{......(1).}$$

Let q be any point of I, then

$$\int_q f d\phi = f(q) \, . \, o(q),$$

$$\int_q \phi \, df = o,$$

and $$v(f\phi \mid q) = f(q) \, . \, o(q),$$

where $o(q) = \operatorname{osc}(\phi \mid q)$. Hence if E denotes the set $q_1, q_2 \dots q_N$ together with those end-points of I (if any) at which I is closed, we have on adding

$$\int_E f d\phi + \int_E \phi \, df = v(f\phi \mid E).$$

Remove the set E from I. There remain a finite number of open intervals J {$= (hk)$, say}.

By the first mean value theorem

$$\left| \int_J f d\phi \right| \leqslant \bar{M} . v(\phi \mid J) \qquad \text{......(3),}$$

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* Lebesgue, *Leçons sur l'Intégration*, p. 58.

and $$\int_J \phi\, df = \phi(h+o)\, v(f \mid J) + \eta \qquad \text{......(4)},$$

where $$|\eta| \leqslant 2\bar{M}.v(\phi \mid J) \qquad \text{......(5)}.$$

Also $$v(f\phi \mid J) = f(k)\,\phi(k-o) - f(h)\,\phi(h+o)$$
$$= \phi(h+o).v(f \mid J) + \theta \qquad \text{......(6)},$$

where $$|\theta| \leqslant 2\bar{M}.v(\phi \mid J) \qquad \text{......(7)}.$$

Combining equations (3), (4), (5), (6) and (7) we have

$$\left| \int_J f\,d\phi + \int_J \phi\, df - v(f\phi \mid J) \right| \leqslant 5\bar{M}.v(\phi \mid J),$$

and, summing this for all intervals J, and remembering equation (2), we have

$$\Big| \int_I f\,d\phi + \int_I \phi\, df - v(f\phi \mid I) \leqslant 5\bar{M}.\Sigma v(\phi \mid J)$$
$$< \epsilon$$

by equation (1). Since ϵ is arbitrary the left side must be zero and the theorem is proved for case (*b*).

Case (*c*). Let $\omega_1, \omega_2, \ldots, p_1, p_2, \ldots$ bear the same relation to $f(x)$ as do $o_1, o_2, \ldots, q_1, q_2, \ldots$ to $\phi(x)$. Given $\epsilon > o$ we can choose N so large that

$$\sum_{N+1}^{\infty} o_n < \frac{\epsilon}{4\bar{M}} \qquad \text{......(8)},$$

$$\sum_{N+1}^{\infty} |\omega_n| < \frac{\epsilon}{4\bar{\mu}} \qquad \text{......(9)},$$

where $\bar{M}$, $\bar{\mu}$ are the upper bounds of $|f(x)|$, $|\phi(x)|$. [$\phi(x)$ is supposed increasing as before: $f(x)$ may be either increasing or decreasing.]

Let x be any point of I. Using the strictly smooth properties we have:

$$\int_x f\,d\phi = \tfrac{1}{2}[f(x+o) + f(x-o)][\phi(x+o) - \phi(x-o)],$$

$$\int_x \phi\, df = \tfrac{1}{2}[f(x+o) - f(x-o)][\phi(x+o) + \phi(x-o)],$$

$$v(f\phi \mid x) = f(x+o)\,\phi(x+o) - f(x-o)\,\phi(x-o).$$

Hence if E denotes the set $q_1, q_2, \ldots, q_N, p_1, p_2 \ldots p_N$, together with those end-points of I (if any) at which I is closed, we have on adding:

$$\int_E f\,d\phi + \int_E \phi\, df = v(f\phi \mid E) \qquad \text{......(10)}.$$

emove the set E from I. There remain a finite number of open tervals J [such as (hk)] and we have

$$\int_J f\,d\phi \leqslant \bar{M}.v(\phi \mid J),$$

$$\int_J \phi\,df \leqslant \bar{\mu} \mid v(f \mid J) \mid,$$

$$\mid v(f\phi \mid J) \mid \leqslant \bar{M}.v(\phi \mid J) + \bar{\mu}. \mid v(f \mid J) \mid.$$

ence

$$\left| \int_J f\,d\phi + \int_J \phi\,df - v(f\phi \mid J) \right| \leqslant 2\bar{M}.v(\phi \mid J) + 2\bar{\mu}. \mid v(f \mid J) \mid.$$

dding we have, on making use of equations (8), (9), (10),

$$\left| \int_I f\,d\phi + \int_I \phi\,df - v(f\phi \mid I) \right| \leqslant 2\bar{M}.\Sigma v(\phi \mid J) + 2\bar{\mu}.\Sigma \mid v(f \mid J) \mid$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon.$$

Since ϵ is arbitrary the left side must be zero and the theorem is proved for case (*c*), and therefore for all the cases.

Only in case (*c*) had we to make use of the "strictly smooth" condition; and case (*c*) appears only when *both* $f(x)$ *and* $\phi(x)$ *are discontinuous.* We have therefore:

S. *If* $f(x)$, $\phi(x)$ *are any functions of bounded variation, never discontinuous together,*

$$\int_I f\,d\phi + \int_I \phi\,df = v(f\phi \mid I).$$

The main Theorem **R** suggests at once the following:

T. *If* $f(x)$, $\phi(x)$ *are strictly smooth and of bounded variation, and if* E *is any set measurable* (f) *and* (ϕ), *then*

$$\int_E f\,d\phi + \int_E \phi\,df = v(f\phi \mid E).$$

This surprisingly general theorem is true, but not immediate, for we cannot make use at once of the facts that, if $f(x)$ is monotonic,

$$\int_E f\,d\phi = \int_I \bar{f}\,d\phi,$$

where $$\bar{f}(x) \begin{cases} = f(x) \text{ on } E, \\ = 0 \text{ elsewhere,} \end{cases} \quad \text{etc.,}$$

for $\bar{f}(x)$, $\bar{\phi}(x)$, defined in this way, are no longer strictly smooth. The theorem may be proved by considering the changes in the various terms on making $\bar{f}(x)$, $\bar{\phi}(x)$ strictly smooth, but it is m easier to proceed as follows:

If E is an open set the theorem follows at once from Theorem S and the absolutely additive properties of variation and of t *LS*-integral.

For any set E measurable (f) *and* (ϕ) there is an open set containing E and such that

$$v(f \mid O-E), \quad v(\phi \mid O-E), \quad v(f\phi \mid O-E)$$

are as small as we please*. But the theorem is true for O. By th absolute continuity property it is therefore true for E.

As in Theorem **S** we may dispose of the "strictly smooth" condition if one of $f(x)$, $\phi(x)$ is continuous.

§ 5. Differential Properties.

We first prove the following Lemma:

U. *If $\phi(x)$ is an increasing function, $\int_E f d\phi$ exists, and $f(x) > k$ on E; then either $v(\phi \mid E) = 0$ or $\int_E f d\phi > k \cdot v(\phi \mid E)$.*

Take any strictly decreasing sequence of numbers $k_1, k_2 \ldots$ which tend to k. Put $E_n = E\{f \geqslant k_n\}$.

Then either $v(\phi \mid E_n) = 0$ for all n, in which case $v(\phi \mid E) = 0$; or some $v(\phi \mid E_n) > 0$ in which case

$$\int_{E_n} f d\phi \geqslant k_n \cdot v(\phi \mid E_n) > k \cdot v(\phi \mid E_n).$$

But
$$\int_{E-E_n} f d\phi \geqslant k \cdot v(\phi \mid E - E_n).$$

Hence
$$\int_E f d\phi > k \cdot v(\phi \mid E).$$

Using the definitions of derivatives with regard to a strictly increasing function given in the paper quoted we can now extend the fundamental theorems of the integral calculus from the Lebesgue to the *LS*-integral. Throughout we assume—unless otherwise stated—$\phi(x)$ to be increasing, $LS\int_I f d\phi$ to exist, and $F(x)$ to be the corresponding indefinite integral. The proofs apply whether $f(x)$ is bounded or not.

V. *The First Fundamental Theorem:*

$F'(x)$ exists and is equal to $f(x)$ almost everywhere (ϕ).

Take $D(x)$ to be any derivative of $F(x)$. $D(x)$ is measurable since it is the limit of measurable functions.

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* We have supposed $f(x)$, $\phi(x)$ increasing. For other cases the modification required is obvious.

The set of points in the neighbourhood of which $\phi(x)$ is stationary clearly form an open set O on which $v(\phi \mid O) = v(F \mid O) = 0$. If Δ is the set of those end-points of intervals of O at which $\phi(x)$ is continuous, then Δ is finite or denumerable, so that

$$v(\phi \mid \Delta) = v(F \mid \Delta) = 0.$$

Neglecting the sets O, Δ we are therefore left with a set G at each point of which $D(x)$ has a definite meaning.

Now let E_k be the set of points of G at which

$$D(x) > k > f(x),$$

k being some rational number. E_k is measurable since $D(x)$, $f(x)$ and G are all measurable, and by **O** $v(F \mid E_k)$ exists and is equal to $\int_{E_k} f d\phi$.

Hence, by our Lemma **U**, either $v(\phi \mid E_k) = 0$ or

$$v(F \mid E_k) < k.v(\phi \mid E_k),$$

and the latter alternative is impossible since by the Generalized Variation Theorem*

$$v(F \mid E_k) \geqslant k.v(\phi \mid E_k).$$

But the sum of the sets E_k for all rational values of k is the set E on which $D(x) > f(x)$. Hence

$$v(\phi \mid E) = 0.$$

Similarly if $\bar{E}$ is the set on which $f(x) > D(x)$,

$$v(\phi \mid \bar{E}) = 0,$$

and the theorem is proved, for the points at which $F'(x)$ either does not exist or is not equal to $f(x)$ are contained in E, $\bar{E}$, O, and Δ, all sets of measure zero.

There follows immediately:

W. *The Second Fundamental Theorem:*

If $D(x)$ is any derivative of $F(x)$, and if $F'(x)$ is considered only where it exists, then $D(x)$ and $F'(x)$ are summable and

$$\int_I D(x)\, d\phi = \int_I F'(x)\, d\phi = \int_I f d\phi = v(F \mid I).$$

These theorems have been stated with reference to an interval I. In virtue of a remark made in § 2, whereby $\int_E f d\phi$ can be regarded as an integral over an interval, they are true when E replaces I.

Our methods serve to prove the following extension of a well-known theorem on Increasing Functions. We take $\phi(x)$ to be strictly increasing.

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* E. C. Francis, *loc. cit.*

X. *If* $\Psi(x)$ *is an increasing function on* I *and* $D(x)$ *any of its derivatives, then* $D(x)$ *is summable* (ϕ) *and*

$$\int_I D(x)\,d\phi \leqslant v(\Psi \mid I).$$

As before $D(x)$ is measurable.

Put
$$\{D(x)\}_n \begin{cases} = D(x) & \text{when } D(x) \leqslant n, \\ = n & \text{when } D(x) > n. \end{cases}$$

Then $\int_I \{D\}_n\,d\phi$ exists and defines an increasing indefinite integral $\Phi_n(x)$ which, by **V**, has a differential coefficient $\{D\}_n$ except on a set E where $v(\phi \mid E) = 0$.

Hence $\{\Phi_n(x) - \Psi(x)\}$ has a derivative $\leqslant 0$ at points of CE (the complement of E). Hence we deduce from Theorem **Q** of the earlier paper

$$v_*(\Phi_n - \Psi \mid CE) \leqslant 0,$$
$$v_*(\Phi_n \mid CE) \leqslant v^*(\Psi \mid CE).$$

But $v(\Phi_n \mid E)$ exists and is equal to $LS\int_E \{D\}_n\,d\phi = 0$. Hence

$$v(\Phi_n \mid I) = v(\Phi_n \mid CE) = v_*(\Phi_n \mid CE) \leqslant v^*(\Psi \mid CE) \leqslant v(\Psi \mid I).$$

i.e.
$$LS\int_I \{D\}_n\,d\phi \leqslant v(\Psi \mid I).$$

The theorem is therefore proved.

As a corollary we have:

Y. *An increasing function has almost everywhere a summable differential coefficient.*

For let $D(x)$ be any derivative of an increasing function $\Psi(x)$. Let $F(x)$ be the indefinite integral corresponding to the integral $\int_I D\,d\phi$, which we know to exist. It follows from **X** that for any sub-interval J of I

$$v(F \mid J) \leqslant v(\Psi \mid J),$$

and therefore that, if D' be *any* derivative of $\Psi(x)$,

$$D' \geqslant \text{the corresponding derivative of } F(x).$$

But by **V** $F(x)$ has almost everywhere a differential coefficient D. Hence, almost everywhere

$$D' \geqslant D.$$

Similarly, almost everywhere

$$D \geqslant D',$$

i.e. the derivatives of $\Psi(x)$ are almost everywhere equal to one another.

The last part of the theorem follows at once since $D(x)$ is summable.

§ 6. CHANGE OF VARIABLE.

We make use of the differential properties of the *LS*-integral to prove the following important theorem:

Z. *If $\phi(x)$ is of bounded variation and $\Psi(x)$ is the indefinite integral corresponding to $\int_I \psi(x)\,d\phi(x)$, then*

$$\int_E f(x)\,d\Psi(x) = \int_E f(x)\,\psi(x)\,d\phi(x),$$

provided that the right-hand side exists.

By a remark in § 2 it is sufficient to consider the case when E is an interval I.

We shall first assume:

(i) $f(x)$ bounded $\geqslant 0$.

(ii) $\psi(x) > 0$.

(iii) $\phi(x)$ is strictly increasing.

$\Psi(x)$ is therefore strictly increasing.

Since $f(x)\,\psi(x)$ is measurable (ϕ) and $\psi(x)$ is positive and measurable (ϕ) it follows that $f(x)$ is measurable (ϕ).

But, by definition of Ψ, if E is any set measurable (ϕ), E is also measurable (Ψ). Hence $f(x)$ is measurable (Ψ), and, since $f(x)$ is bounded, $\int_E f\,d\Psi$ exists.

Let $F(x)$, $\mathcal{F}(x)$ be the indefinite integrals corresponding to $\int f\psi\,d\phi$ and $\int f\,d\Psi$ respectively. Both are strictly increasing functions.

By Theorem **V**, except on a set E_1 for which $v(\phi \mid E_1) = 0$, $F_\phi'(x)$ and $\Psi_\phi'(x)$ exist and

$$F_\phi'(x) = f(x)\,\psi(x),$$
$$\Psi_\phi'(x) = \psi(x).$$

Hence by a result in our theory of differentiation* we deduce that, except on E_1, $F_\Psi'(x)$ exists and

$$F_\Psi'(x) = f(x).$$

This is therefore true almost everywhere (Ψ) for

$$v(\Psi \mid E_1) = LS\int_{E_1} \psi\,d\phi = 0.$$

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* E. C. Francis, *loc. cit.*

Hence by **X**

$$v(F \mid I) \geqslant \int_I f(x)\, d\Psi = v(\mathscr{F} \mid I) \qquad \text{......(1).}$$

By an exactly similar argument $\mathscr{F}_\phi'(x)$ exists and

$$\mathscr{F}_\phi'(x) = f(x)\, \psi(x)$$

except on a set E_2 on which $v(\Psi \mid E_2) = 0$, and therefore

$$\int_{E_2} \psi\, d\phi = 0.$$

Since $\psi(x) > 0$ it follows from the lemma **U** that

$$v(\phi \mid E_2) = 0.$$

Hence, almost everywhere (ϕ)

$$\mathscr{F}_\phi'(x) = f(x)\, \psi(x),$$

and again applying **X** we have

$$v(\mathscr{F} \mid I) \geqslant \int_I f(x)\, \psi(x)\, d\phi(x) = v(F \mid I) \qquad \text{......(2).}$$

Equations (1) and (2) prove the theorem in the case considered.

The relaxation of our assumptions one by one, thus extending our result by means of the sum and addition theorems to the general case, is almost trivial.

(*a*) *If* $f(x) \geqslant 0$ *but is unbounded*,

$$\int_E f_n\, d\Psi = \int_E f_n\, \psi\, d\phi,$$

where
$$f_n(x) \begin{cases} = f(x) & \text{when } f(x) \leqslant n, \\ = n & \text{when } f(x) > n, \end{cases}$$

$$\leqslant \int_E f\psi\, d\phi$$

which exists by hypothesis. Hence $\int_E f\, d\Psi$ exists and we can complete the theorem for this case either by repeating the remainder of the argument used above, or by quoting the theorem on limits under the integral sign.

(*b*) *If* $f(x)$ *takes negative values* we can express it as the difference of two positive functions, and the theorem follows at once from the f-Sum Theorem **E**.

(*c*) *If* $\psi(x) \geqslant 0$ *but takes the value zero on a set* G.

G is measurable (ϕ) and therefore measurable (Ψ), and

$$v(\Psi \mid G) = \int_G \psi\, d\phi = 0.$$

Both $\int_G f d\Psi$ and $\int_G f\psi d\phi$ are therefore zero and the theorem still holds.

(*d*) *If* $\psi(x)$ *takes negative values.*

We can express $\psi(x)$ as the difference of two positive functions:

$$\psi(x) = \psi_1(x) - \psi_2(x),$$

where $\psi_1(x)$ is zero when $\psi(x)$ is negative, and $\psi_2(x)$ is zero when $\psi(x)$ is positive.

Now by Theorem **G** $\int_E |f|\,|\psi|\,d\phi$ and $\int_E |\psi|\,d\phi$ exist so that $\int_E f\psi_1 d\phi$, $\int_E f\psi_2 d\phi$, $\int_E \psi_1 d\phi$, $\int_E \psi_2 d\phi$ all exist.

Taking $\Psi_1(x)$, $\Psi_2(x)$ as the indefinite integrals corresponding to the last two integrals we have

$$\int_E f\psi d\phi = \int_E f\psi_1 d\phi - \int_E f\psi_2 d\phi$$

$$= \int_E f d\Psi_1 - \int_E f d\Psi_2 \text{ by case } (c)$$

$$= \int_E f d\Psi \text{ by the } \phi\text{-Sum Theorem } \mathbf{F}.$$

(*e*) *If* $\phi(x)$ *is increasing in the broad sense only* its constant stretches form a denumerable (or finite) set of intervals over each of which both $\int f\psi d\phi$ and $\int f d\Psi$ are zero; so the theorem remains true.

(*f*) *If* $\phi(x)$ *is of bounded variation only* the theorem follows at once from the ϕ-Sum Theorem **F**.

The theorem is therefore completely proved.

There is one interesting case—that when $f(x)$ is bounded and measurable (ϕ)—when the existence of the integral

$$\int_E f(x)\,\psi(x)\,d\phi(x)$$

is known*. We then have in all cases

$$\int_E f d\Psi = \int_E f\psi d\phi.$$

In particular this is true if $f(x)$ is a function of bounded variation.

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* If $\psi(x)$ is not positive we can express it as the difference of two positive functions $\psi_1(x)$, $\psi_2(x)$. $f(x)\,\psi_1(x)$, $f(x)\,\psi_2(x)$ are measurable (ϕ) and it is easily seen that the existence of $\int_E |f\psi_1|\,d\phi$, $\int_E |f\psi_2|\,d\phi$, and therefore of $\int_E f\psi_1 d\phi$, $\int_E f\psi_2 d\phi$, follows at once from that of $\int_E \psi_1 d\phi$, $\int_E \psi_2 d\phi$.

§ 7. THE SECOND MEAN VALUE THEOREM.

Combining the formula **T** for Integration by Parts with that for change of variable **Z**, we have:

α. *If $f(x)$ is a strictly smooth function of bounded variation, while $\psi(x)$ is summable (ϕ) on E and defines an indefinite integral $\Psi(x)$, then*

$$\int_E f\psi d\phi = v(f\Psi \mid E) - \int_E \Psi df.$$

β. In the case when E is an interval I ($\langle ab \rangle$, say, for convenience) and $f(x)$ *is monotonic* this can be re-written as the *Second Mean Value Theorem*

$$\int_I f\psi d\phi = f(a-o)(P-\Psi(a-o)) + f(b+o)(\Psi(b+o)-P).$$

where P is some number lying between the upper and lower bounds of $\Psi(x)$ on I.

For by the first mean value theorem

$$\int_I \Psi df = P\{f(b+o) - f(a-o)\}.$$

γ. In the case when $f(x)$ *is strictly smooth, positive, and decreasing*, we have *Bonnet's Form*

$$\int_I f\psi d\phi = f(a-o)\{P - \Psi(a-o)\}.$$

Both these theorems assume a more simple form when $\phi(x)$ is *continuous*. We then have

δ. *If $f(x)$ is a monotonic function, while $\psi(x)$ is summable (ϕ) on (ab), then*

$$\int_a^b f\psi d\phi = f(a \pm o)\int_a^\xi \psi d\phi + f(b \pm o)\int_\xi^b \psi d\phi,$$

where any of the ambiguous signs may be taken, and $a \leqslant \xi \leqslant b$.

ε. *If also $f(x)$ is decreasing and positive*

$$\int_a^b f\psi d\phi = f(a \pm o)\int_a^X \psi d\phi,$$

where $$a \leqslant X \leqslant b.$$

The Enumeration of the Partitions of Multipartite Numbers. By Major P. A. MacMahon, Sc.D., St John's College.

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Introduction.

This paper is a study of a new method of enumeration of the partitions of multipartite numbers.

Incidentally an algebraic function, which is derived from the repetitional exponents of partitions of unipartite numbers, presents itself. The generating function which enumerates the partitions of unipartite numbers is expressible in terms of these functions and finds in such expression its fullest connection with the divisors of numbers. There are also similarly derived functions connected directly with bipartite, tripartite, etc. numbers. It has not been necessary to study these for the purposes of this paper.

An arithmetical function $\sigma_0^{(f)}(n)$ which enumerates what I have called the factor-divisor of n has appeared. It is the multipartite generalization of the well-known arithmetical function $\sigma_0(n)$ which enumerates the divisions of the unipartite number n. It gives rise to an allied arithmetical function $j(n)$ which possesses properties analogous to those which appertain to the ϕ-function of Euler. A short table which enumerates the partitions of some bipartite and tripartite numbers is appended.

§ 1. *The Dirichlet Series Generating Function.*

1·1. This subject has been dealt with by the author* and two methods explained for carrying out the evaluation. The matter has acquired a fresh interest from the discovery of the new Generating Function, viz. the ordinary Dirichlet series

$$\eta(s) = \prod_2^\infty (1 - n^{-s})^{-1}.$$

1·2. The development of this may be written

$$\Sigma p_{\pi_1 \pi_2 \ldots \pi_k} n^{-s},$$

where the prime number form of n is

$$p_1^{\pi_1} p_2^{\pi_2} \cdots p_k^{\pi_k},$$

and $\underline{p_{\pi_1 \pi_2 \ldots \pi_k}}$ denotes the number of partitions of the multipartite number $(\pi_1 \pi_2 \ldots \pi_k)$; or alternatively

$$\Sigma fc(n) n^{-s},$$

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* *Combinatory Analysis*, vol. I, pp. 264 *et seq.*

where $fc(n)$ denotes the number of ways of factorizing the integer n. Thus for $n = 12 = 2^2.3$, p_{21} enumerates the four partitions of the bipartite number $(\overline{21})$, these being

$$(\overline{21}),\quad (\overline{20}, \overline{01}),\quad (\overline{11}, \overline{10}),\quad (\overline{10}, \overline{10}, \overline{01}),$$

and $fc(12)$ enumerates the four factorizations

$$12,\quad 2 \times 6,\quad 3 \times 4,\quad 3 \times 2^2.$$

If we expand the function $\eta(s)$ to the power l we find

$$1{\cdot}3.\quad \{\eta(s)\}^l = \Sigma\Sigma\binom{\nu_1+l-1}{\nu_1}\binom{\nu_2+l-1}{\nu_2}\cdots \binom{\nu_k+l-1}{\nu_k}(n_1^{\nu_1}n_2^{\nu_2}\dots n_k^{\nu_k})^{-s}\ \dots\dots(\text{A}),$$

where $n_1^{\nu_1}n_2^{\nu_2}\dots n_k^{\nu_k}$ is a factorization of n, and the summation is for all such factorizations of n and for all integer values of $n > 1$.

1·31. To every integer n there appertains the number

$$\Sigma\binom{\nu_1+l-1}{\nu_1}\binom{\nu_2+l-1}{\nu_2}\cdots\binom{\nu_k+l-1}{\nu_k}\ \dots\dots(\text{A})'.$$

where the summation is for every factorization of n and involves $fc(n)$ terms. Before proceeding to the arithmetical interpretation of this numerical quantity reference is made to Bouniakowski*, who interpreted the powers of Riemann's Zeta Function

$$\sum_1^\infty n^{-s},$$

and to E. B. Elliott† who also threw light upon the powers of $\zeta(s)$. We write for comparison

$$\{\zeta(s)\}^l = \sum_n\binom{\pi_1+l-1}{\pi_1}\binom{\pi_2+l-1}{\pi_2}\cdots\binom{\pi_k+l-1}{\pi_k}(p_1^{\pi_1}p_2^{\pi_2}\dots p_k^{\pi_k})^{-s}$$
$$\dots\dots(\text{B}).$$

1·32. When $l = 2$, the numerical coefficient in (B) enumerates the divisors of n. In (A) the coefficient enumerates those divisors of n which appertain to the factorization $n_1^{\nu_1}n_2^{\nu_2}\dots n_k^{\nu_k}$ of n. In other words the divisors enumerated are drawn from the powers and products of powers of $n_1, n_2 \dots n_k$ that present themselves in the factorization; unity is of course included. The sum in (A)′ enumerates the divisors appropriate to each factorization and adds the results. We write it $\sigma_0^{(f)}(n)$.

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* *Mém. Ac. Sc. St Pétersbourg* (7). 4, 1862, No. 2, 35 pp.

† *Proc. Lond. Math. Soc.* 34, 1901, pp. 3–15.

We may write

$$\{\zeta(s)\}^2 = \Sigma\sigma_0(n)\, n^{-s}; \quad \{\eta(s)\}^2 = \Sigma\sigma_0^{(f)}(n)\,.\,n^{-s}.$$

When $n = 12$, the factorizations are

$$12, \quad 2 \times 6, \quad 3 \times 4, \quad 2^2 \times 3,$$

and give $$\sigma_0^{(f)}(12) = 2 + 4 + 4 + 6 = 16.$$

1·33. When $l = 3$, we know that (Bouniakowski and Elliott, *l.c.*)

$$\{\zeta(s)\}^3 = \sum_n \sigma_{00}(n)\, n^{-s},$$

where $\sigma_{00}(n) = \Sigma\sigma_0(d)$, d a divisor of n.

We may write $$\{\eta(s)\}^3 = \sum_n \sigma_{00}^{(f)}(n)\,.\,n^{-s},$$

wherein $\sigma_{00}^{(f)}(n) = \Sigma\sigma_0^{(f)}(d)$, d a divisor of n.

1·34. And generally from the definitions

$$\left.\begin{aligned}\sigma_{00\ldots k\text{ times}}(n) &= \Sigma\sigma_{00\ldots k-1\text{ times}}(d)\\ \sigma_{00\ldots k\text{ times}}^{(f)}(n) &= \Sigma\sigma_{00\ldots k-1\text{ times}}^{(f)}(d)\end{aligned}\right\}\ d \text{ a divisor of } n,$$

we may write $$\{\zeta(s)\}^l = \Sigma\sigma_{00\ldots l-1\text{ times}}(n)\,.\,n^{-s},$$

$$\{\eta(s)\}^l = \Sigma\sigma_{00\ldots l-1\text{ times}}^{(f)}(n)\,.\,n^{-s}.$$

Ex. gr. When $n = 12$, $\sigma_0^{(f)}(12)$ enumerates the divisors which occur in the factorizations

$$12, \quad 2.6, \quad 3.4, \quad 2^2.3,$$

viz.

$$1,\ 12, \quad 1,\ 2,\ 6,\ 2 \times 6, \quad 1,\ 3,\ 4,\ 3 \times 4, \quad 1,\ 2,\ 3,\ 2^2,\ 2 \times 3,\ 2^2 \times 3,$$

altogether 16 in number. To obtain $\sigma_{00}^{(f)}(12)$ we enumerate the factor-divisors of these 16 numbers; we find

$$1 + 2; \quad 1 + 2 + 2 + 4; \quad 1 + 2 + 2 + 4; \quad 1 + 2 + 2 + 3 + 4 + 6,$$

giving $$3 + 9 + 9 + 18 = 39$$

in correspondence with

$$\binom{3}{2} + \binom{3}{2}\binom{3}{2} + \binom{3}{2}\binom{3}{2} + \binom{3}{2}\binom{4}{2} = 39$$

as required by the theory.

1·4 It is clear that by carrying out the multiplication to produce $\{\eta(s)\}^2$ we connect $\sigma_0^{(f)}(n)$ and $fc(n)$ in the manner

$$\sigma_0^{(f)}(n) = \Sigma fc(d) fc\left(\frac{n}{d}\right),$$

where d is a divisor of m.

§ 2. *The Bipartite Theory.*

2·1. Euler's function for the enumeration of the partitions of ordinary or unipartite numbers is

$$F(x) = 1/(1-x)(1-x^2)(1-x^3) \dots \text{ad inf.} = \Sigma p_n x^n$$

which has the advantage of being in close contact with elliptic functions.

That for bipartite numbers $\overline{n_1 n_2}$ is

$$1/(1-x)(1-y).(1-x^2)(1-xy)(1-y^2).(1-x^3)(1-x^2y)$$
$$\times (1-xy^2)(1-y^3). \dots = \Sigma p_{n_1 n_2} x^{n_1} y^{n_2}.$$

which can be expressed in the form

2·11. $F(x) F(y) F(xy) F(x^2y) F(xy^2) \dots F(x^a y^b) \dots,$

wherein a, b are relatively prime integers.

2·12. If $n_2 \nless 1$ we are only concerned with y to the power unity; the generating function becomes

$$1/1-x.1-x^2.1-x^3. \dots 1-y.1-xy.1-x^2y. \dots,$$

and since $1/1-y.1-xy.1-x^2y. \dots = 1 + \frac{y}{1-x} + \dots$

the coefficient of y is

$$1/(1-x)^2(1-x^2)(1-x^3) \dots \text{ad inf.} = p_{01} + p_{11}x + p_{21}x^2 + p_{21}x^3 + \dots$$

and thence $p_{n1} = \sum_0^n p_n$, where $p_0 = 1$.

2·13. If $n_2 \nless 2$, the generating function is, effectively

$$1/(1-x)(1-x^2)(1-x^3) \dots (1-y)(1-xy)(1-x^2y) \dots$$
$$(1-y^2)(1-xy^2)(1-x^2y^2) \dots$$

which is

$$F(x)\left\{1 + \frac{y}{1-x} + \frac{y^2}{(1-x)(1-x^2)} + \dots\right\}\left\{1 + \frac{y^2}{1-x} + \dots\right\},$$

wherein the coefficient of y^2 is

$$F(x)\left\{\frac{1}{1-x} + \frac{1}{(1-x)(1-x^2)}\right\} = p_{02} + p_{12}x + p_{22}x^2 + \dots + p_{n2}x^n + \dots$$

which leads easily to the formula

2·14. $p_{n2} = 2p_n + 2p_{n-1} + 3p_{n-2} + 3p_{n-3} + 4p_{n-4} + 4p_{n-5} + \dots$

to $n+1$ terms.

We observe that the factor of $F(x)$, above, consists of two terms. Moreover, the unipartite number 2 has two partitions 2, 11 which we may write so as to exhibit the frequency with which the parts 1, 2 occur in the partitions in the form

$$2^1, \quad 1^2$$

and we will term these exponents "frequency exponents."

We next note that if we associate with the frequency exponents 1, 2 the algebraic fractions

$$\frac{1}{1-x}, \quad \frac{1}{(1-x)(1-x^2)} \text{ respectively,}$$

there is a simple correspondence between the generating function and the partitions of the unipartite number 2.

2·2. If $n_2 \nless 3$, we proceed in a similar manner. For brevity we write after Cayley

$$(1-x^k) = (\mathbf{k}),$$

and find in the first place

$$F(x)\left\{1+\frac{y}{(\mathbf{1})}+\frac{y^2}{(\mathbf{1})(\mathbf{2})}+\frac{y^3}{(\mathbf{1})(\mathbf{2})(\mathbf{3})}\right\}\left\{1+\frac{y^2}{(\mathbf{1})}\right\}\left\{1+\frac{y^3}{(\mathbf{1})}\right\},$$

whence taking out the coefficient of y^3

$$F(x)\left\{\frac{1}{(\mathbf{1})}+\frac{1}{(\mathbf{1})^2}+\frac{1}{(\mathbf{1})(\mathbf{2})(\mathbf{3})}\right\}$$
$$= p_{03} + p_{13}x + p_{23}x^2 + p_{33}x^3 + \ldots + p_{n3}x^n + \ldots.$$

We now remark that, in the repetitional notation, the partitions of 3 are

$$3^1, \quad 2^1 1^1, \quad 1^3,$$

and that we can derive the left-hand side of the identity just obtained by associating the algebraic fractions

$$\frac{1}{(\mathbf{1})}, \quad \frac{1}{(\mathbf{1})(\mathbf{2})}, \quad \frac{1}{(\mathbf{1})(\mathbf{2})(\mathbf{3})}$$

with the repetitional exponents 1, 2, 3 respectively.

It is also clear that this correspondence persists and that we may in general write down the co-factor of $F(x)$ by associating $1/(\mathbf{1})(\mathbf{2})\ldots(\mathbf{k})$ with the repetitional exponents k which occur in the partitions of the unipartite number n_2.

Thus the reader will verify the formula

$$F(x)\left\{\frac{1}{(\mathbf{1})}+\frac{1}{(\mathbf{1})^2}+\frac{1}{(\mathbf{1})(\mathbf{2})}+\frac{1}{(\mathbf{1})^2(\mathbf{2})}+\frac{1}{(\mathbf{1})(\mathbf{2})(\mathbf{3})(\mathbf{4})}\right\}$$
$$= p_{04} + p_{14}x + p_{24}x^2 + \ldots + p_{n4}x^n + \ldots$$

as being derived from the frequencies of the partitions

$$4^1, \quad 3^1 1^1, \quad 2^2, \quad 2^1 1^2, \quad 1^4.$$

We have thus the cardinal theorem.

2·3. "If a partition of the unipartite number n_2 be

$$1^{c_1}2^{c_2}3^{c_3}\ldots s^{c_s},$$

$$F(x)\,\Sigma\,\frac{1}{\mathbf{(1)}\,\mathbf{(2)}\ldots(\mathbf{c}_1)\,\mathbf{(1)}\,\mathbf{(2)}\ldots(\mathbf{c}_2)\,\mathbf{(1)}\,\mathbf{(2)}\ldots(\mathbf{c}_3)\ldots\ldots\mathbf{(1)}\,\mathbf{(2)}\ldots(\mathbf{c}_s}$$

$$= p_{0n_2} + p_{1n_2}x + p_{2n_2}x^2 + \ldots + p_{n_1n_2}x^{n_1} + \ldots,$$

where the summation is for every partition

$$1^{c_1}2^{c_2}\ldots s^{c_s} \text{ of } n_2,$$

and $$F(x) = 1/\mathbf{(1)}\,\mathbf{(2)}\,\mathbf{(3)}\ldots \text{ ad inf."}$$

§ 3. *Study of the Frequencies of Partitions.*

3·1. I study the partition

$$n_1^{c_1}n_2^{c_2}\ldots n_s^{c_s} \text{ of the number } n,$$

$$\Sigma c_t n_t = n,\quad n_1 < n_2 \ldots < n_s,\quad c_t > 0,\quad t = 1, 2, \ldots s$$

with particular reference to the frequency integers $c_1, c_2, \ldots c_s$.

I write

$$\frac{1}{\prod\limits_1^\infty (1 - x^n)} = \prod_1^\infty\left(1 + \frac{x^n}{1 - x^n}\right)$$

$$= 1 + X_1 + X_2 + \ldots + X_3 + \ldots.$$

where $$X_s = \Sigma\,\frac{x^{c_1+2c_2+\ldots+sc_s}}{(1 - x^{c_s})(1 - x^{c_{s-1}+c_s})\ldots(1 - x^{c_1+c_2+\ldots+c_s})}$$

$$= \Sigma\,\frac{x^{c_s}}{1 - x^{c_s}}\cdot\frac{x^{c_{s-1}+c_s}}{1 - x^{c_{s-1}+c_s}}\cdots\frac{x^{c_1+c_2+\ldots+c_s}}{1 - x^{c_1+c_2+\ldots+c_s}},$$

where the summation is for every ordered set of s integers, each > 0, between which there may be any number of equalities.

It is obvious that the integer n possesses a number of partitions which have the set of frequencies $c_1, c_2, \ldots c_s$ and that these are enumerated by the coefficients of x^n in the development of X_s.

3·2. We may write *in extenso*, taking $1 - x^s$ to be denoted by $\mathbf{(s)}$,

$$\frac{1}{\prod\limits_1^\infty(1 - x^n)} = 1 + \frac{x}{\mathbf{(1)}} + \frac{x^2}{\mathbf{(2)}} + \frac{x^3}{\mathbf{(3)}} + \frac{x^4}{\mathbf{(4)}} + \frac{x^5}{\mathbf{(5)}} + \frac{x^6}{\mathbf{(6)}} + \ldots$$

$$+ \frac{x^3}{\mathbf{(1)}\,\mathbf{(2)}} + \frac{x^4}{\mathbf{(1)}\,\mathbf{(3)}} + \frac{x^5}{\mathbf{(1)}\,\mathbf{(4)}} + \frac{x^6}{\mathbf{(1)}\,\mathbf{(5)}} + \ldots$$

$$+ \frac{x^5}{\mathbf{(2)}\,\mathbf{(3)}} + \frac{x^6}{\mathbf{(2)}\,\mathbf{(4)}} + \ldots$$

$$+ \frac{x^6}{\mathbf{(1)}\,\mathbf{(2)}\,\mathbf{(3)}} + \ldots$$

$$= 1 + \sum_1^\infty\Sigma\,\frac{x^n}{(\mathbf{n}_1)\,(\mathbf{n}_2)\,(\mathbf{n}_3)\ldots},$$

where $(n_1 n_2 n_3 \ldots)$ is any partition of n into unequal parts and in the first place the summation is for all such partitions.

Ex. gr. We write down at once the next set of terms

$$\frac{x^7}{1-x^7}+\frac{x^7}{(1-x)(1-x^6)}+\frac{x^7}{(1-x^2)(1-x^5)}+\frac{x^7}{(1-x^3)(1-x^4)}$$
$$+\frac{x^7}{(1-x)(1-x^2)(1-x^4)}.$$

We note at once that

$$X_1=\Sigma\sigma_0(x)\,x^n,$$

where $\sigma_0(n)$ enumerates the divisors of n.

3·3. We have before us what we may regard as a generalization of the notion of a divisor of an integer and we may expect to obtain expressions for X_2, X_3, ... which involve the arithmetical functions

$$\sigma_0(n),\quad \sigma_1(n),\quad \sigma_2(n),\ \ldots,$$

where $\sigma_k(n)$ denote the sum of the kth powers of the divisors of n. To reach such expressions I consider the symmetric functions of the elements

$$\frac{x}{1-x},\quad \frac{x^2}{1-x^2},\quad \frac{x^3}{1-x^3},\ \ldots$$

which occur in the expression of X_1.

I write $\left(\frac{x}{1-x},\ \frac{x^2}{1-x^2},\ \frac{x^3}{1-x^3},\ \ldots\right)=(\alpha,\beta,\gamma,\ldots)$,

or alternatively $(\alpha_1,\alpha_2,\alpha_3,\ldots)$,

$$(\Sigma\alpha,\ \Sigma\alpha\beta,\ \Sigma\alpha\beta\gamma,\ \ldots)=(b,c,d,\ldots),$$

or alternatively $(a_1, a_2, a_3, \ldots)$ in the usual notation of symmetric functions of a single system of quantities.

Also $\Sigma\alpha^k=s_k$.

It is clear that we have

$$(X_1, X_2, X_3, \ldots)=(\Sigma\alpha,\ \Sigma\alpha\beta,\ \Sigma\alpha\beta\gamma,\ \ldots)$$

and that we obtain expressions for $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, ... by first of all calculating expressions for the sums of powers $s_2, s_3, s_4, \ldots$ and then making use of the formulae which express $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, ... in terms of $s_1, s_2, s_3, \ldots$.

We have already before us

$$s_1=X_1=\Sigma\sigma_0(n)\,x^n,$$

and we next see that

$$s_2=\left(\frac{x}{1-x}\right)^2+\left(\frac{x^2}{1-x^2}\right)^2+\left(\frac{x^3}{1-x^3}\right)^2+\ldots,$$

and since $$\left(\frac{x^t}{1-x^t}\right)^2 = \sum_{k=1}^{\infty} kx^{kt},$$

we see that $$s_2 = \sum_1^{\infty}{}_n \{\sigma_1(n) - \sigma_0(n)\} x^n.$$

3·31. Similarly, in succession,

$$s_3 = \Sigma \frac{1}{2!} \{\sigma_2(n) - 3\sigma_1(n) + 2\sigma_0(n)\} x^n,$$

$$s_4 = \Sigma \frac{1}{3!} \{\sigma_3(n) - 6\sigma_2(n) + 11\sigma_1(n) - 6\sigma_0(n)\} x^n,$$

……………………………………………………

$$s_k = \Sigma \frac{1}{(k-1)!} \left\{\sigma_{k-1}(n) - \binom{k}{2} \sigma_{k-2}(n) + \ldots + (-)^{k-1}(k-1)!\,\sigma_0(n)\right\} x^n,$$

the coefficients being those in the development of

$$(x-1)(x-2)\ldots(x-k+1).$$

In fact we may write symbolically

$$s_2 = \Sigma(\sigma - 1)(n)x^n,$$

$$s_3 = \Sigma \frac{1}{2!}(\sigma-1)(\sigma-2)(n)x^n$$

$$\vdots$$

$$s_k = \Sigma \frac{1}{(k-1)!}(\sigma-1)(\sigma-2)\ldots(\sigma-k+1)(n).x^n,$$

where after multiplication of the symbol σ, σ^k is to be replaced by σ_k.

Thus in general $s_k = \Sigma \binom{\sigma-1}{k-1}(n).x^n$ symbolically.

We now find

$$\begin{aligned} 2!\,X_2 = 2!\,a_2 &= s_1{}^2 - s_2 \\ &= \{\Sigma\sigma_0(n)x^n\}^2 - \Sigma\{\sigma_1(n) - \sigma_0(n)\}x^n \\ &= \Sigma_n \left\{\sum_{m=1}^{n-1} \sigma_0(m)\sigma_0(n-m) - \sigma_1(n) + \sigma_0(n)\right\} x^n. \end{aligned}$$

etc.

3·32. The inverse formulae for the expression of $\Sigma\sigma_k(n)x^n$ in terms of $s_1, s_2, s_3, \ldots$ are

$$\Sigma\sigma_0(n)x^n = s_1,$$

$$\Sigma\sigma_1(n)x^n = s_1 + s_2,$$

$$\Sigma\sigma_2(n)x^n = s_1 + 3s_2 + 2s_3,$$

$$\Sigma\sigma_3(n)x^n = s_1 + 7s_2 + 12s_3 + 6s_4,$$

……………………………………

here the table of coefficients

$$\begin{array}{llll} 1 & & & \\ 1 & 1 & & \\ 1 & 3 & 2 & \\ 1 & 7 & 12 & 6 \\ \multicolumn{4}{c}{\dots\dots\dots\dots} \end{array}$$

s such that if $u_{x,y}$ is the number in the xth column and yth row

$$u_{x,y} = x u_{x,y-1} + (x-1)\, u_{x-1,y-1}.$$

If we write $$X_k = \Sigma \tau^{(k)}(n)\, x^n,$$

ve have the new arithmetical function

$$\tau^{(k)}(n).$$

3·321. We have obtained the expressions

$$\tau^{(1)}(n) = \sigma_0(n),$$

$$\tau^{(2)}(n) = \frac{1}{2!}\left\{\sum_{m=1}^{n-1} \sigma_0(m)\,\sigma_0(n-m) - \sigma_1(n) + \sigma_0(n)\right\},$$

and we have the means for the calculation of the expression of $\tau^{(k)}(n)$.

It must involve the arithmetical function $\sigma_t(m)$ where t takes the values $0, 1, 2, \dots k-1$.

When we restrict the numbers $c_1, c_2, \dots$ to be unity we enumerate partitions into unrepeated parts and the result gives

$$(1+x)(1+x^2)(1+x^3)\dots = 1 + \frac{x}{(1)} + \frac{x^3}{(1)(2)} + \frac{x^6}{(1)(2)(3)} + \dots$$

as is well known.

3·4. I now write

$$\Sigma \frac{x^{c_1+2c_2+\dots+sc_s}}{(1-x^{c_s})(1-x^{c_{s-1}+c_s})\dots(1-x^{c_1+c_2+\dots+c_s})} = \phi_1(c_1, c_2, \dots c_s),$$

where the summation is for all permutations of $c_1, c_2, \dots c_s$.

3·41. I note the identities

$$\phi_1(c_1, c_2) = \phi_1(c_1+c_2)\{\phi_1(c_1) + \phi_1(c_2)\} = \phi_1(c_1)\,\phi_1(c_2) - \phi_1(c_1+c_2),$$

so that we have the additive theorem

3·42. $$\phi_1(c_1+c_2) = \frac{\phi_1(c_1)\,\phi_1(c_2)}{1+\phi_1(c_1)+\phi_1(c_2)}.$$

3·43. Also

$$\phi_1(c_1, c_2, c_3) = \phi_1(c_1+c_2+c_3)\{\phi_1(c_2, c_3) + \phi_1(c_3, c_1) + \phi_1(c_1, c_2)\},$$

and generally

3·44. $$\phi_1(c_1, c_2, \dots c_s) = \phi_1(c_1+c_2+\dots+c_s)\,\Sigma\phi_1(c_2, c_3, \dots c_s),$$

where the summation is for the s terms obtained by the circular substitution.

3·45. Also

$$\phi_1 (c_1 + c_2 + \dots + c_s)$$
$$= \frac{\phi_1 (c_1)\, \phi_1 (c_2) \dots \phi_1 (c_s)}{\{1 + \phi_1 (c_1)\}\{1 + \phi_1 (c_2)\} \dots \{1 + \phi_1 (c_s)\} - \phi_1 (c_1)\, \phi_1 (c_2) \dots \phi_1 (c}$$

3·46. $\phi_1 (c_1, c_2, c_3) = \phi_1 (c_1 + c_2 + c_3) \{\phi_1 (c_2)\, \phi_1 (c_3) + \phi_1 (c_3)\, \phi_1 (c$
$+ \phi_1 (c_1)\, \phi_1 (c_2) - \phi_1 (c_2 + c_3) - \phi_1 (c_3 + c_1) - \phi_1 (c_1 - c$

§ 4. *Tripartite, etc. Numbers.*

4·1. The generating function has for denominator

$$(1 - x)(1 - y)(1 - z)$$
$$\times (1 - x^2)(1 - xy)(1 - xz)(1 - y^2)(1 - yz)(1 - z^2)$$
$$\times \dots\dots\dots\dots$$

For the tripartite $\overline{n11}$ this denominator is effectively composed of the four infinite products

$$(1 - x)(1 - x^2)\dots; \quad (1 - y)(1 - xy)(1 - x^2y)\dots;$$
$$(1 - z)(1 - xz)(1 - x^2z)\dots; \quad (1 - yz)(1 - xyz)(1 - x^2yz)\dots;$$

and the effective expansion of the algebraic fraction is thus

$$F(x)\left\{1 + \frac{y}{(\mathbf{1})}\right\}\left\{1 + \frac{z}{(\mathbf{1})}\right\}\left\{1 + \frac{yz}{(\mathbf{1})}\right\},$$

where $(\mathbf{1})$ stands for $(1 - x)$ (after Cayley).

Herein the coefficient of yz is

$$F(x)\left\{\frac{1}{(\mathbf{1})} + \frac{1}{(\mathbf{1})^2}\right\} = \sum_0^\infty p_{n11} x^n.$$

We remark that the multipartite number $\overline{11}$ has the two partitions which, with the frequency exponents inserted, are

$$11^1, \quad 10^1 01^1,$$

and we obtain a correspondence between these and the co-factor of $F(x)$ by associating $1/(\mathbf{1})$ with the exponent 1.

4·2. The resulting formula is

$$p_{n11} = (n + 2) + (n + 1)\, p_1 + np_2 + (n - 1)\, p_3 + \dots \text{ to } n + 1 \text{ terms.}$$

Similarly we may calculate the series for $p_{nn_2n_3}$, where n_2, n_3 are any assigned integers.

As an example consider the series for p_{n22}.

The partitions of the bipartite $\overline{22}$ are

$$\overline{22}^1;\quad \overline{21}^1\overline{01}^1;\quad \overline{12}^1\overline{10}^1;\quad \overline{20}^1\overline{02}^1;\quad \overline{11}^2;$$
$$\overline{20}^1\overline{01}^2;\quad \overline{02}^1\overline{10}^2;\quad \overline{11}^1\overline{10}^1\overline{01}^1;\quad \overline{10}^2\overline{01}^2;$$

and we are led to the formula

$$F(x)\left\{\frac{1}{(1)}+3\frac{1}{(1)^2}+\frac{1}{(1)(2)}+2\frac{1}{(1)^2(2)}+\frac{1}{(1)^3}+\frac{1}{(1)^2(2)^2}\right\}$$
$$=p_{022}+p_{122}x+\ldots+p_{n22}x^n+\ldots$$

for the calculation of p_{n22}.

4·3. For multipartite numbers in general we can now enumerate the Cardinal Theorem:

"Let $\overline{n_2n_3\ldots n_s}$ be any multipartite number and

$$\overline{a_1a_2\ldots a_s}^{\pi_1}\,\overline{b_1b_2\ldots b_s}^{\pi_2}\ldots$$

one of its partitions. Then

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}$$
$$\times\Sigma\left\{\frac{1}{(1-x)(1-x^2)\ldots(1-x^{\pi_1})}\cdot\frac{1}{(1-x)(1-x^2)\ldots(1-x^{\pi_2})}\cdot\ldots\right\}$$
$$=\sum_{n_1=0}^{\infty}p_{n_1n_2\ldots n_s}x^{n_1},$$

where the summation on the left-hand side is in regard to every partition of the multipartite number

$$\overline{n_2n_3\ldots n_s}."$$

§ 5. *The Factor-divisor Function* $\sigma_0^{(f)}(n)$.

5·1. I refer to § 1 and write

$$J(x)=j(1)\frac{x}{1-x}+j(2)\frac{x^2}{1-x^2}+\ldots=\Sigma j(n)\frac{x^n}{1-x^n}.$$

5·11. Then $$J(x)=\Sigma\sigma_0^{(f)}(n)\,x^n$$

if $$\sigma_0^{(f)}(n)=\Sigma j(d),\ d \text{ a divisor of } n^*.$$

---

* This result is not difficult to arrive at, but we may if we please derive it from

$$\Sigma j(n)\frac{x^n}{1-x^n}=\Sigma\sigma_0^{(f)}(n)\,x^n$$

by referring to Laguerre, *Bull. Soc. Math. France*, 1, 1872–3, pp. 77–81, who showed that if

$$\sum_1^{\infty}f(n)\frac{x^n}{1-x^n}=\sum_1^{\infty}F(n)\,x^n,$$

then $$F(n)=\Sigma f(d),\ d \text{ a divisor of } n.$$

I refer to Dedekind* who proved that if

$$F(n) = \Sigma f(d),$$

where d ranges over the divisors of n,

$$f(n) = F(n) - \Sigma F\left(\frac{n}{p_1}\right) + \Sigma F\left(\frac{n}{p_1 p_2}\right) \dots,$$

where $p_1, p_2, \dots$ are the distinct prime factors of n.

5·2. Applying this theorem we get

$$j(n) = \sigma_0^{(f)}(n) - \Sigma\sigma_0^{(f)}\left(\frac{n}{p_1}\right) + \Sigma\sigma_0^{(f)}\left(\frac{n}{p_1 p_2}\right) - \dots.$$

5·3. Thence

$$J(x) = \Sigma\sigma_0^{(f)}(x)\frac{x^n}{1-x^n} - \Sigma\sigma_0^{(f)}\left(\frac{n}{p_1}\right)\frac{x^n}{1-x^n} + \Sigma\sigma_0^{(f)}\left(\frac{n}{p_1 p_2}\right)\frac{x^n}{1-x^n} - \dots$$

wherein after the first term the summations are double, i.e. in regard (i) to the distinct prime factors of n, (ii) in regard to n.

5·4. Now by inversion we obtain

$$\sum_1^\infty \sigma_0^{(f)}(n)\frac{x^n}{1-x^n} = \sum_1^\infty J(x^n).$$

The arithmetical function $j(n)$ has been expressed in terms of the functions $\sigma_0^{(f)}(n)$. It will be remarked that it is $j(n)$ and not $\sigma_0^{(f)}(n)$ that possesses properties analogous to those possessed by allied arithmetical functions.

5·5. Thus if n_1, n_2 be relatively prime integers

$$j(n_1)\,j(n_2) = j(n_1 n_2),$$

whilst the result

$$\Sigma j\left(\frac{n}{d}\right) = \sigma_0^f(n)$$

seems to be the analogue of Gauss' property of the ϕ-functions.

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* *Journ. für Math.* 54, 1857, pp. 21, 25.

Enumeration of Bipartite Partitions.

11	**21**	**31**	**41**	**51**	**61**	**71**	**81**	**91**	**101**
2	4	7	12	19	30	45	67	97	139
	22	**32**	**42**	**52**	**62**	**72**	**82**	**92**	**102**
	9	16	29	47	77	118	181	267	392
		33	**43**	**53**	**63**	**73**	**83**	**93**	**103**
		31	57	97	162	257	401	608	907
			44	**54**	**64**	**74**	**84**	**94**	**104**
			109	189	323	522	831	1279	1941
				55	**65**	**75**	**85**	**95**	**105**
				339*	589	975	1576	2472	3737
					66	**76**	**86**	**96**	**106**
					1043	1752	2876	4571	7128
						77	**87**	**97**	**107**
						2998	4987	8043	12693
							88	**98**	**108**
							8406	13748	21938

Enumeration of Tripartite Partitions.

111	**211**	**311**	**411**	**511**	**611**	**711**	**811**	**911**	**1011**
5	11	21	28	64	105	165	254	381	562
	221	**321**	**421**	**521**	**621**	**721**	**821**	**921**	
	26	52	98	171	289	467	737	1131	
		331	**431**	**531**	**631**	**731**	**831**		
		109	212	382	662	1097	1768		

222	**322**	**422**	**522**	**622**
66	137	269	484	843
	332	**432**	**532**	
	300	606	1129	

---

* This number is given, erroneously, as 336 in *Combinatory Analysis*, vol. I, p. 269. Read 'The multipartite **55** has 339 partitions.'

[To replace p. 963 (Vol. XXII, Part 6), which is cancelled.]

Enumeration of Bipartite Partitions.

11	*21*	**31**	*41*	**51**	*61*	**71**	*81*	**91**	*101*
2	*4*	**7**	*12*	**19**	*30*	**45**	*67*	**97**	*139*
	22	**32**	*42*	**52**	*62*	**72**	*82*	**92**	*102*
	9	**16**	*29*	**47**	*77*	**118**	*181*	**267**	*392*
		33	*43*	**53**	*63*	**73**	*83*	**93**	*103*
		31	*57*	**97**	*162*	**257**	*401*	**608**	*907*
			44	**54**	*64*	**74**	*84*	**94**	*104*
			109	**189**	*323*	**522**	*831*	**1279**	*1941*
				55	*65*	**75**	*85*	**95**	*105*
				339*	*589*	**975**	*1576*	**2472**	*3737*
					66	**76**	*86*	**96**	*106*
					1043	**1752**	*2876*	**4571**	*7128*
						77	*87*	**97**	*107*
						2998	*4987*	**8043**	*12693*
							88	**98**	*108*
							8406	**13748**	*21938*

Enumeration of Tripartite Partitions.

111	*211*	**311**		*411*		**511**		*611*		**711**		*811*	**911**	*1011*
5	*11*	**21**		*28*		**64**		*105*		**165**		*254*	**381**	*562*
	221	**321**		*421*		**521**		*621*		**721**		*821*	**921**	
	26	**52**		*98*		**171**		*289*		**467**		*737*	**1131**	
		331		*431*		**531**		*631*		**731**		*831*		
		109		*212*		**382**		*662*		**1097**		*1768*		
			222		*322*		**422**		*522*		**622**			
			66		*137*		**269**		*484*		**843**			
					332		**432**		*532*					
					300		**606**		*1129*					

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* This number is given, erroneously, as 336 in *Combinatory Analysis*, vol. I, p. 269.

On the substitution of Wallis's Postulate of Similarity for Euclid's Postulate of Parallels. By Professor M. J. M. Hill, Peterhouse.

[*Received* 20 April, *read* 4 May 1925.]

INTRODUCTION.

1. In the year 1663 John Wallis, Savilian Professor of Geometry in the University of Oxford, delivered a lecture in which he claimed that he had *proved* Euclid's Postulate of Parallels according to the strictest laws of demonstration after Euclid's manner*.

In reality he had not, and as it is now known, he could not have, *proved* Euclid's Postulate. He had attempted to deduce Euclid's Postulate from the following Postulate:

"*Any figure being given, it is always possible to construct a similar figure of arbitrary magnitude.*"

When Wallis used the words "similar figure," it cannot be doubted that he had in mind Euclid's definition of similar figures, viz.:

"Similar rectilineal figures are such as have their angles severally equal and the sides about the equal angles proportional."

But Wallis in his argument uses of this definition nothing but the following:

"*If any triangle be known to exist, then it is always possible to construct another triangle of arbitrary magnitude which has the same angles.*"

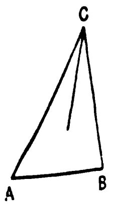
It should be particularly noticed that Wallis makes no use in his argument of that part of Euclid's definition of similar figures which depends on the concept of ratio.

2. *Initial Assumptions:*

(i) A straight line through C in the angle ACB meets AB.

(ii) If A, B, C, A' be angles and $A = A'$ and $A + B < A' + C$, then $B < C$.

(iii) Hilbert's Axiom of Congruence, IV. 4, quoted in the next article.



3. Euclid's Theory of Parallels is contained in his Postulate of Parallels and in the 27th, 28th, and 29th Propositions of his first book. The 27th proposition depends on the 16th, and that again on the 4th, which deals with a case of congruent triangles, so that Euclid's theory of parallel lines depends on a proposition concerned with *areas*. Hilbert's proof of Euc. I. 27 does not depend on I. 16, but does depend on I. 4.

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* See p. 674 of the 2nd volume of Wallis's *Collected Works*.

The distinguishing feature of the deduction of the theory of parallel ines from Wallis's Postulate of Similarity here given is that it does not depend on any property of congruent AREAS. It does assume that at any point a straight line can be drawn making with a given straight line through that point on a given side an angle equal to a given angle (Hilbert's Axiom of Congruence, IV. 4). The actual construction of the angle requires the use of Euc. I. 4, but the demonstration of the theory of parallels requires only the knowledge that the angle referred to in Hilbert's Axiom exists.

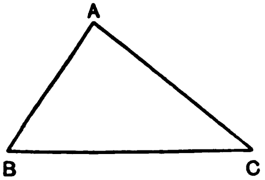
It may be a matter of opinion whether Euclid's Postulate or Wallis's is most readily comprehended by beginners in Geometry, but inasmuch as the idea of parallelism is quite distinct from that of an area, there is an advantage in keeping the latter out of the discussion of the former*.

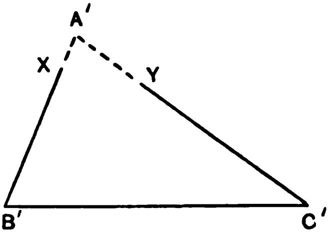
THE THEORY OF PARALLEL LINES DEDUCED FROM WALLIS'S POSTULATE OF SIMILARITY.

4. Suppose then the triangle ABC is given. Take any length $B'C'$.

Let $B'X$, $C'Y$ be straight lines drawn on the same side of $B'C'$ so that $C'\hat{B}'X$ is equal to $C\hat{B}A$, and $B'\hat{C}'Y$ is equal to $B\hat{C}A$.

Then it is assumed, in accordance with the postulate of similarity, that $B'X$, $C'Y$, if produced, will meet in some point A' and that the angle $B'A'C'$ will be equal to the angle BAC.

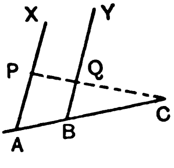




From the preceding there follow immediately as Corollaries:

(i) If two angles of one triangle be respectively equal to two angles of another triangle, then the third angles of the triangles are equal.

(ii) If the angles CAX, CBY are equal, a line CQ, meeting BY in Q, meets AX in some point P. (For proof consider the construction of the triangle CAP equiangular with CBQ.)



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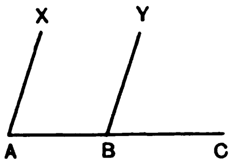
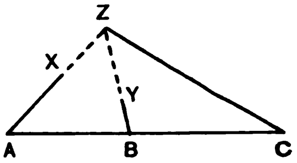
* See also Note 2 at the end of this paper.

PROPOSITION I*.

5. *If two straight lines AX, BY meet a transversal ABC so as to make the angle XAB equal to the angle YBC then AX, BY cannot meet.*

If possible let them meet at Z. Join CZ. Then the triangles AZC, BZC have the angles ZAC, ZCA equal to ZBC, ZCB respectively and therefore by Art. 4, Cor. (i) the remaining angles AZC, BZC should be equal. This is impossible. Hence AX, BY cannot meet. This proposition is Euc. I. 27, from which Euc. I. 28 is not distinct so far as any question of parallelism is concerned.



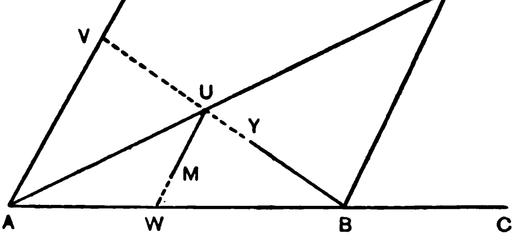


Another way of putting this proposition is to say that

It is impossible to construct a triangle which has two of its angles together equal to two right angles.

PROPOSITION II.

6. *If two straight lines XA, YB meet the straight line ABC so as to make the interior angles XAB, YBA on the same side of ABC together less than two right angles, then AX, BY will meet on that side of ABC on which the two interior angles are situated.*



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* This proof is due to Professor Nunn (see *Mathematical Gazette*, May 1922).

At B make the angle CBZ equal to the angle BAX.

Then the angles CBZ and ZBA are together equal to two right angles.

That is, the angles XAB, ZBA are together equal to two right angles.

But BAX and ABY are together less than two right angles.

Hence XAB and ABY are together less than XAB and ABZ.

Hence by the second initial assumption, Art. 2, the angle ABY is less than the angle ZBA.

Hence BY lies in the angle ABZ.

If therefore the point A be joined to any point T on BZ, then BY being inside the angle ABT must cut AT in some point U between A and T. (See the assumption (i) in Art. 2.)

At U make the angle AUM equal to the angle ATB.

Then by Prop. I, UM cannot meet TB.

Hence UM cannot lie in the angle BUT, but must lie in the angle AUB, and must meet AB in some point W between A and B. (See Art. 2, assumption (i).)

Now in the triangles AUW, ATB the angles at A are the same and the angles at U and T are equal.

Hence by Art. 4, Cor. (i), the angles AWU, ABT are equal.

Hence the angle UWB is equal to the angle CBZ, which is equal to the angle BAX.

Now consider the straight lines UW, AX, which make equal angles with AB. Therefore, by Art. 4, Cor. (ii), BU which meets UW also meets AX in some point V, i.e. AX, BY meet.

This is Euclid's Postulate of Parallels. It is here shown that it can be deduced as a consequence of Wallis's Postulate of Similarity.

PROPOSITION III.

7. *If a straight line fall on two straight lines in the same plane, which do not meet, it must make the alternate angles equal.*

This is one of the forms in which Euc. I. 29 is stated, and may be proved in Euclid's manner, or it may be regarded as a Corollary to Prop. II.

PROPOSITION IV.

8. *Through the point A, not situated on the straight line BC, only one straight line can be drawn in the plane ABC, which does not meet BC.*

This proposition also may be regarded as a Corollary to Prop. II.

PROPOSITION V.

9. *The exterior angle of a triangle is greater than either of the interior and opposite angles* (Euc. I. 16)*.

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* The proof of this proposition is included here because Euclid uses it in his proof of I. 27.

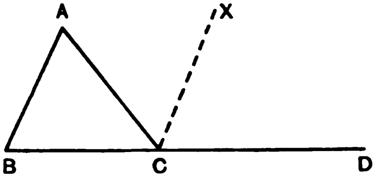
Let the side BC of a triangle ABC be produced to D.

Through C draw CX on the same side of BCD as A is, and so that the angle DCX is equal to the angle CBA.

Then by Prop. I, CX cannot meet BA or BA produced.

Hence, by assumption (i) in Art. 2, CX cannot fall inside the angle BCA, and therefore must fall inside the angle DCA, as shown by the dotted line in the figure.

Hence the angle ACD is greater than the angle DCX and so it is greater than the angle CBA.



10. From this proposition there follows immediately the proposition that

Any two angles of a triangle are together less than two right angles (Euc. I. 17).

1. OTHER DEMONSTRATIONS OF THE EQUIVALENCE OF THE TWO POSTULATES.

So far as I am aware seven demonstrations other than that given in Art. 6 above have been *attempted*.

(i) Wallis, 1663 (*l.c.*), uses the Principle of Continuity.

(ii) Saccheri, 1733 (*Euclides ab omni naevo vindicatus*, p. 84 of Engel and Stäckel's translation), uses the principle of continuity and the theorem that the external angle of a triangle is greater than either of the interior and opposite angles, and so assumes the infinity of the straight line. (It is to be noted that Saccheri assumes the existence of only two triangles having the same angles.)

(iii) W. K. Clifford, 1885 (*Common Sense of the Exact Sciences*, pp. 71–72), uses the Principle of Continuity.

(iv) T. P. Nunn (*Mathematical Gazette*, May 1922, p. 71) gives *the outline of a proof*.

(v) *A Report on the Teaching of Geometry in Schools* prepared for the Mathematical Association (1923), p. 39 (II), uses an argument involving the Principle of Continuity.

(vi) M. J. M. Hill (*Mathematical Gazette*, December 1923, pp. 410–413). The proof given in Art. 6 above is a simplification of this proof.

(vii) Krishnaswami Ayyangar (*Mathematical Gazette*, October 1924, pp. 191–194), in using Wallis's Postulate, makes use of that part of it which involves the concept of ratio as well as that part of it which is concerned with the equality of angles, and he also uses the property of the external angle of a triangle (Euc. I. 16).

There may be other demonstrations than the above, but I have been unable to find them, if they exist.

The proofs marked (iv) and (vi) above are the only proofs, in addition to that given in Art. 6, which pass directly from the Postulate of Similarity to the Postulate of Parallels without introducing into the argument any extraneous hypothesis, for I do not regard the assumptions mentioned in Art. 2 as being of this nature.

2. COMPARISON OF THE TWO POSTULATES.

(i) Bonola (*Non-Euclidean Geometry*, p. 16) takes the view that the idea of form, independent of the dimensions of the figure, constitutes a hypothesis, which is certainly not more evident than the Postulate of Euclid.

(ii) Carnot, in a note (p. 481) to his *Géométrie de Position* (1803), affirms that the theory of parallels is allied to the principle of similarity, the evidence for which is on the same plane as that for equality, and that if this idea is once admitted, it is easy to establish the said theory rigorously, without having recourse to the idea of infinity.

(iii) Laplace (1824) (*Œuvres*, T. VI, Livre V, Ch. V, p. 472) takes the same view as Carnot, but his deduction of the Postulate of Similarity from the properties of space is not in accord with the Theory of Relativity.

Unfortunately neither Carnot nor Laplace explains in what way he deduced the Postulate of Parallels from that of Similarity; nor does De Morgan, who, in the article on Similar Figures in the *Penny Cyclopaedia*, says that the two postulates are equivalent. Bonola alone amongst these writers refers to Wallis.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1924—1925.

ANNUAL GENERAL MEETING.

October 27, 1924.

In the Comparative Anatomy Lecture Room.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:

Prof. J. T. Wilson.

Vice-Presidents:

Mr J. Barcroft.
Mr C. T. Heycock.
Dr G. T. Bennett.

Treasurer:

Mr F. A. Potts.

Secretaries:

Dr F. W. Aston.
Mr J. Gray
Mr F. P. White.

Members of the Council who continue:

Mr T. C. Nicholas.
Mr J. M. Wordie.
Mr G. I. Taylor.
Mr H. McCombie.
Mr J. B. S. Haldane.

New Members of the Council:

Dr A. B. Appleton.
Dr C. D. Ellis.
Mr F. F. Blackman
Mr C. T. R. Wilson.
Mr J. E. Littlewood.
Mr G. Udny Yule.
Mr E. A. Milne.

The following were elected Associates of the Society:

J. C. Boyce, Trinity College.
Miss H. B. Fell.
G. P. Harnwell, Trinity College.
G. A. C. Herklots, Trinity Hall.
H. A. Murray, Trinity College.
D. C. Rose, Gonville and Caius College.
D. L. Thomson, Gonville and Caius College.
B. W. Tucker, Gonville and Caius College.

The following Communications were made to the Society:

1. The hydrogen ion concentration of the waters of Lake Lucerne. By Mr J. T. SAUNDERS, Christ's College.

2. The process of cell division in the eggs of Echinoderms. By Mr J. GRAY, King's College.

3. Oogenesis of *Lithobius*. By V. NATH, Trinity Hall. (Communicated by Mr J. Gray.)

4. Studies on Insect Metamorphosis—1. Prothetely in Mealworms (*Tenebrio molitor*) and other Insects. Effects of different temperatures. By Dr H. SINGH PRUTHI, Peterhouse. (Communicated by Mr J. Gray.)

5. Chlorocruorin. By Mr H. MUNRO FOX, Gonville and Caius College.

6. Characteristics of complexes of conics in space of four dimensions. By Mr C. G. F. JAMES, Trinity College.

7. On integrals developable about a singular point of a Hamiltonian system of differential equations. By Dr T. M. CHERRY, Trinity College.

8. A helical method of focussing β-rays. By R. A. R. TRICKER, B.A., Trinity College. (Communicated by Prof. Sir E. Rutherford.)

9. Some methods of estimating the successive ionisation potentials of any element. By Mr D. R. HARTREE, St John's College.

10. Why wireless electric waves can bend round the earth. By Prof. Sir JOSEPH LARMOR, St John's College.

November 10, 1924.

In the Anatomy School.

Prof. J. T. WILSON, PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:

F. G. Maunsell, B.A., Trinity College.

The following were elected Associates of the Society:

M. L. Anson, King's College.
R. G. Dickinson.
L. E. Grenfell, Downing College.
W. T. Richards, Trinity College.
A. Weinstein.
F. W. Whitehouse, St John's College.

The following Communication was made to the Society:

The Natural and Artificial Disintegration of Elements. By Prof. Sir **Ernest Rutherford**.

November 24, 1924.

In the Cavendish Laboratory.

Prof. J. T. **Wilson, President, in the Chair.**

The President announced the death of Corrado Segre, Professor of Higher Geometry in the University of Turin, and Honorary Fellow of this Society.

The following were elected Fellows of the Society:

T. Alty, Ph.D., Trinity College.
R. Cooper, B.A., Trinity College.
J. S. Dunn, B.A., St John's College.
J. E. Jones, Ph.D., Trinity College.
R. K. Schofield, B.A., Trinity College.
A. M. Taylor, B.A., Trinity College.
W. A. Wooster, B.A., Peterhouse.

The following were elected Associates of the Society:

I. Bowen, Trinity College.
P. Bracelin, Emmanuel College.
I. C. Jones, Gonville and Caius College.
A. F. Kovarik.
G. Owen, Gonville and Caius College.

The following Communications were made to the Society:

1. Acceleration of β-particles in strong electric fields such as those of thunderclouds. By Mr C. T. R. **Wilson**, Sidney Sussex College.

2. The determination of the frequency of an alternating current supply by the vibrations of rods. By Dr G. F. C. **Searle**, Peterhouse.

3. Photographic plates for the detection of mass rays. By Dr F. W. **Aston**, Trinity College.

4. On the current induced in a wireless telegraph receiving antenna. By Mr E. B. MOULLIN, Downing College. (Communicated by Mr E. V. Appleton.)

5. Note on the chemical constant of chlorine. By R. R. S. COX, B.A., Christ's College. (Communicated by Mr R. H. Fowler.)

6. The decay constant of radium emanation. By L. BASTINGS. (Communicated by Prof. Sir E. Rutherford.)

7. The kinetic theory of metallic conduction. By E. L. DAVIES and Mr G. H. LIVENS, Jesus College.

December 8, 1924.

In the Cavendish Laboratory.

Prof. NEWALL, IN THE CHAIR.

The following were elected Fellows of the Society:

T. A. A. Broadbent, B.A., St John's College.
F. H. Constable, B.A., St John's College.
G. S. Mahajani, B.A., St John's College.
C. G. Nobbs, B.A., St John's College.
T. Omer-Cooper, B.A., Fitzwilliam Hall.
J. H. Quastel, Ph.D., Trinity College.
J. A. Steers, M.A., St Catharine's College.

The following was elected an Associate of the Society:

Miss R. R. Sass, Newnham College.

The following Communications were made to the Society:

1. The bursting of soap-bubbles in a uniform electric field. By Mr C. T. R. WILSON, Sidney Sussex College, and Mr G. I. TAYLOR, Trinity College.

2. Dissociative equilibrium in an external field of force. By Mr E. A. MILNE, Trinity College.

3. (*a*) Some examples of trajectories defined by differential equations of a generalized dynamical type. (*b*) Integrals developable about a singular point of a Hamiltonian system of differential equations. Part II. By Dr T. M. CHERRY, Trinity College.

4. Note on the Lorentz group. By Mr J. BRILL, St John's College.

5. Note on the harmonic conic. By Mr E. H. NEVILLE, Trinity College.

6. Thermodynamics and quantum theory. By Mr R. HARGREAVES, St John's College.

January 19, 1925.

In the Cavendish Laboratory.

Prof. J. T. **Wilson, President, in the Chair.**

The President announced the death of Dr G. D. Liveing, Professor of Chemistry 1861–1908, aged 97 years, a Fellow of the Society since 1852.

The following were elected Fellows of the Society:

J. D. Cockcroft, B.A., St John's College.
C. N. H. Lock, M.A., Gonville and Caius College.
R. W. Paine, B.A., Trinity College.
L. S. Penrose, B.A., St John's College.
P. A. Taylor, B.A., Emmanuel College.

The following were elected Associates of the Society:

J. W. Hinton, Emmanuel College.
Miss M. Bowen.
Miss L. T. Hayner.
G. H. Briggs, Emmanuel College.
Mrs G. H. Briggs.

The following Communications were made to the Society:

1. A transformation of Segre's figure in space of four dimensions and the equation of Kummer's surface. By Prof. H. F. **Baker**, St John's College.

2. On simplexes doubly incident with a quadric. By R. **Vaidyanathaswamy**. (Communicated by Prof. H. F. Baker.)

3. The classification of conicoids by their generators. By Mr H. G. **Green**, Gonville and Caius College. (Communicated by Mr H. T. H. Piaggio.)

4. On the positive flash in vacuum discharge tubes. (Preliminary note.) By Mr R. **Whiddington**, St John's College.

5. The structure of radiation. By E. C. **Stoner**, B.A., Emmanuel College.

6. The symmetric functions of which the General Determinant is a particular case. By Major P. A. **MacMahon**, St John's College.

February 2, 1925.

In the Cavendish Laboratory.

Prof. C. T. R. **Wilson, in the Chair.**

The following were elected Associates of the Society:

N. A. de Bruyne, Trinity College.
G. H. S. Bushnell, Downing College.
M. E. Hufford.
H. Moquin, Trinity Hall.

Prof. V. K. F. Bjerknes, of the Geophysical Institute, Bergen, gave an "Experimental demonstration of hydrodynamical action at a distance."

The following Communications were made to the Society:

1. A preliminary note on a direct determination of the distribution of intensity in the natural β-ray spectrum of *RaB* and *RaC*. By Dr L. F. **Curtiss**. (Communicated by Prof. Sir E. Rutherford.)

2. Note on the heating effect of the γ-rays from *RaB* and *RaC*. By Dr C. D. **Ellis**, Trinity College, and W. A. **Wooster**, B.A., Peterhouse.

3. The interpretation of β-ray absorption curves. By Dr C. D. **Ellis**, Trinity College, and M. **Bowman-Manifold**, B.A., Trinity College.

4. A thermostat for high temperatures. By Dr D. **Stockdale**, King's College. (Communicated by Mr C. T. Heycock.)

5. A method of finding the composition of adsorption films of mixed gases. By Mr W. G. **Palmer**, St John's College.

6. The spreading of oils and fats on water surfaces. By A. P. **Cary**, Christ's College, and Mr E. K. **Rideal**, Trinity Hall.

7. Triple binary forms: the complete system for a single (1, 1, 1) form with its geometrical interpretation. By Mr W. **Saddler**. (Communicated by Mr H. W. Turnbull.)

8. On the phrase "equally probable." By Mr W. **Burnside**, Pembroke College.

February 16, 1925.

In the Botany School.

Prof. J. T. **Wilson**, **President, in the Chair.**

The following were elected Fellows of the Society:

E. H. Callow, Ph.D., Christ's College.
H. G. Green, M.A., Gonville and Caius College.

The following was elected an Associate of the Society:

E. J. B. Willey, St John's College.

Prof. D. M. S. Watson, F.R.S., of University College, London, delivered a Lecture entitled "Orthogenesis."

March 2, 1925.

In the Comparative Anatomy Lecture Room.

Prof. J. T. **Wilson**, **President, in the Chair.**

The following were elected Fellows of the Society:

M. Dixon, M.A., Emmanuel College.
W. S. Farren, M.A., Trinity College.
E. A. Guggenheim, B.A., Gonville and Caius College.
R. Hill, M.A., Emmanuel College.
C. C. Hurst, Ph.D., Trinity College.

The following was elected an Associate of the Society:

A. B. Miller, Trinity College.

The following Communications were made to the Society:

1. Biology of the Suez Canal. By Mr H. **Munro Fox**, Gonville and Caius College.

2. On a group having the Lorentz group for a sub-group. By Mr J. **Brill**, St John's College.

3. On Transvectant series. By A. W. **Veater**. (Communicated by Mr W. M. Smart.)

4. A note on wireless signal strength measurements made during the solar eclipse of 24 January 1925. By Mr E. V. **Appleton**, St John's College, and M. A. F. **Barnett**, Clare College.

5. A geometrical treatment of the correspondence between lines in three-fold space and points of a quadric in five-fold space. By Mr H. W. **Turnbull**, Trinity College.

6. The origin of the potential differences between the interior and exterior of cells. By Mr J. B. S. **Haldane**, Trinity College.

7. Spermatogenesis of *Lithobius forficatus*. By V. **Nath**, Trinity Hall. (Communicated by Mr J. Gray.)

8. The mechanism of cell-division II. By Mr J. **Gray**, King's College.

9. The trichocysts of paramecium. By Mr J. T. **Saunders**, Christ's College.

May 4, 1925.

In the Cavendish Laboratory.

Prof. J. T. **Wilson, President, in the Chair.**

The following were elected Associates of the Society:

W. Ackermann.
N. Feather, Trinity College.

The following Communications were made to the Society:

1. A theoretical study of the stopping power of hydrogen atoms for α-particles. By Mr R. H. **Fowler**, Trinity College.

2. The action of the electrical counter. By K. G. **Emeléus**, B.A., St John's College. (Communicated by Dr J. Chadwick.)

3. An apparatus for the investigation of the effect of poisonous substances, and mixed vapours on catalytic activity. By F. H. **Constable**, B.A., St John's College.

4. Theory of statistical estimation. By Mr R. A. **Fisher**, Gonville and Caius College.

5. (1) On the idea of frequency.
   (2) On the representation of the modular group of order $\frac{1}{2}p(p^2-1)$ as a group of linear substitutions on $\frac{1}{2}(p-1)$ symbols, when p is a prime of the form $4n+3$.

By Mr W. **Burnside**, Pembroke College.

6. On pedal quadrics in non-euclidean hyperspace. By Mr J. P. **Gabbatt**, Peterhouse.

7. An extension of Wallace's, Miquel's and Clifford's theorems on circles. By Mr F. P. **White**, St John's College.

8. (1) The stability of rotating masses of liquid.
   (2) Note on a formula for Lamé functions.

By Prof. H. F. **Baker**, St John's College.

9. (1) On the substitution of Wallis's postulate of similarity for Euclid's postulate of parallels.
   (2) On the hypothesis of the obtuse angle.

By Dr M. J. M. **Hill**, Peterhouse.

10. Some formulae for scrolls and line systems in higher space. By C. G. F. **James**, B.A., Trinity College.

11. The temperature distribution in a transformer in which heat is generated at a uniform rate. By J. D. **Cockcroft**, B.A., St John's College.

May 18, 1925.

In the Cavendish Laboratory.

Prof. J. T. **Wilson**, **President, in the Chair.**

The following were elected Fellows of the Society:

Prof. Sir Humphry D. Rolleston, M.D., St John's College.
Prof. J. B. Buxton.
R. B. Braithwaite, B.A., King's College.
F. P. Ramsey, B.A., King's College.

The following was elected an Associate of the Society:

A. Pringle-Jameson.

The following Communication was made to the Society:

The passage of electrons through small apertures. By J. F. **Lehmann**, Trinity College, and T. H. **Osgood**, Emmanuel College. (Communicated by Prof. Sir Ernest Rutherford.)

July 20, 1925.

In the Botany School.

Prof. J. T. WILSON, PRESIDENT, IN THE CHAIR.

The following was elected an Associate of the Society:

Miss K. B. Blodgett, Newnham College.

The following Communications were made to the Society:

1. The Structure of Light. By Prof. Sir J. J. THOMSON, Trinity College.

2. Internal Constitution of the Stars. By Prof. A. S. EDDINGTON, Trinity College.

3. On groups of linear substitutions which contain irreducible metacyclical subgroups. By Mr W. BURNSIDE, Pembroke College.

4. On Hausdorff's proof of the extended Riesz-Fischer theorem. By Mr S. POLLARD, Trinity College.

5. On the theorem of Pappus. By Mr M. H. A. NEWMAN, St John's College.

6. Notes on optical constants. By Mr C. G. DARWIN, Christ's College.

7. The theory of the influence of magnetic fields on the stopping power of gases for α-particles. By R. DE L. KRONIG (Communicated by Mr R. H. Fowler.)

8. The Trichocysts of *Paramecium*. By Mr J. T. SAUNDERS, Christ's College.

9. Spermatogenesis of *Lithobius forficatus*. By V. NATH, Trinity Hall. (Communicated by Mr J. Gray.)

10. The effect of light on the vertical movement of aquatic organisms. By Mr H. MUNRO FOX, Gonville and Caius College.

11. (1) On Differentiation with respect to a function.
(2) The Lebesgue-Stieltjes integral.
By Mr E. C. FRANCIS, Peterhouse.

12. The transmutation of mercury into gold. By Dr H. NAGAOKA.

13. The β-ray spectrum of the natural L-radiation from radium B. By D. H. BLACK, Emmanuel College. (Communicated by Prof. Sir E. Rutherford.)

INDEX TO THE PROCEEDINGS VOL. XXII

with references to the Transactions

Accounts, statement of, for 1924, 814

Ackermann, W., Elected Associate 1925, May 4, 976

Adiabatic invariant, Burgers' theorem on (Thomas), 886

Algebraic equations, Graeffe's method for complex roots of (Brodetsky and Smeal), 83

Alpha particle counter (Appleton, Emeléus and Barnett), 434

Alpha particles, capture and loss of electrons by (Fowler), 253

Alpha particles, stopping power of hydrogen atoms for (Fowler), 793

Alpha particles, theory of the influence of magnetic fields on the stopping power of gases for (Kronig), 773

Alternating current supply by the vibrations of rods, frequency of an (Searle), 539

Alty, T., Elected Fellow 1924, November 24, 972

Angular momentum and electron impact (Blackett), 56

Anson, M. L., Elected Associate 1924, November 10, 972

Apparatus for the investigation of the effect of poisonous substances (Constable), 738

Apparatus for measuring the velocity of very rapid chemical reactions (Hartridge and Roughton), 426

Appleton, E. V., and Barnett, M. A. F., A note on wireless signal strength measurements made during the solar eclipse of January 24, 1925, 672

—— Emeléus, K. G., and Barnett, M. A. F., Some experiments with an α-particle counter, 434

Armitage, B. W. F., Elected Fellow 1924, February 18, 809

Aston, F. W., Photographic plates for the detection of mass rays, 548

Atomic number of a radioactive element at the moment of emission of the γ-rays (Ellis and Wooster), 844

Baker, H. F., On the generalisation of a theorem of Steiner, 28

Bannister, F. A., Elected Fellow 1923, November 26, 806

Barnett, M. A. F., Elected Associate 1924, May 5, 810

Barnett, M. A. F., *see* Appleton, E. V.

Barnett, M. A. F. and Appleton, E. V., A note on wireless signal strength measurements made during the solar eclipse of January 24, 1925, 672

Bastings, L., Elected Associate 1923, October 29, 804

—— The decay constant of radium emanation, 562

Bath, F., On five lines, in space of four dimensions, which lie upon a quadric threefold and the normal quartic curves of which they are chords, 189

—— and Richmond, H. W., On loci which have two systems of generating spaces, 319

Berry, A. J., The thallous thallic halides, 363

Beta particles, acceleration of, in strong electric fields such as those of thunderclouds (Wilson), 534

Beta particles from radium E, number of (Emeléus), 400

Beta-ray spectrum of the natural L-radiation from radium B (Black), 832

Beta-ray spectrum due to the natural L-radiation of radium B, analysis of (Black), 838

Beta-ray spectrum of Ra B and Ra C, direct determination of the distribution of intensity in the natural (Curtiss), 597

Beta-ray type of disintegration (Ellis and Wooster), 849

Beta-rays, helical method of focussing (Tricker), 454

Beta-rays of Ra C, high energy groups in the magnetic spectrum of the (Ellis), 369

Binary forms, triple (Saddler), 688

Bjerknes, V. K. F., Experimental demonstration of hydrodynamical action at a distance, 975

Black, D. H., Elected Associate 1924, May 19, 811

—— Some electrical properties of liquid sulphur, 393

—— The β-ray spectrum of the natural L-radiation from radium B, 832

Black, D. H., The analysis of the β-ray spectrum due to the natural L-radiation of radium B, 838

Blackett, P. M. S., Angular momentum and electron impact, 56

Blodgett, K. B., Elected Associate 1925, July 20, 978

Bohr's frequency condition (Dirac), 432

Boomer, E. H., Elected Associate 1923, October 29, 804

Bowen, I., Elected Associate 1924, November 24, 972

Bowen, M., Elected Associate 1925, January 19, 974

Boyce, J. C., Elected Associate 1924, October 27, 971

Bracelin, P., Elected Associate 1924, November 24, 972

Braddick, H. J. J., Elected Associate 1923, October 29, 804

Braithwaite, R. B., Elected Fellow 1925, May 18, 977

Briggs, G. H., Elected Associate 1925, January 19, 974

Brill, J., On the problem of four particles, 88

—— Note on the Lorentz group, 630

Bristowe, W. S., Elected Fellow 1924, February 4, 808

Broadbent, T. A. A., Elected Fellow 1924, December 8, 973

Brodetsky, S. and Smeal, G., On Graeffe's method for complex roots of algebraic equations, 83

Bruyne, N. A. de, Elected Associate 1925, February 2, 974

Bull, W. J., Elected Associate 1923, October 29, 804

Burgers' theorem on adiabatic invariants (Thomas), 886

Burgh, D. H. de, Elected Associate 1923, October 29, 804

Burnside, W., On errors of observation, 26

—— On the polygons inscribed in one conic and circumscribed to another, 163

—— The problems of random flight and conduction of heat, 167

—— On the phrase "equally probable," 669

—— On the idea of frequency, 726

—— On the representation of the modular group of order $\frac{1}{2}p\,(p^2-1)$ as a group of linear substitutions on $\frac{1}{2}\,(p-1)$ symbols, when p is a prime of the form $4n+3$, 779

—— On groups of linear substitutions which contain irreducible metacyclical subgroups, 788

Bushnell, G. H. S., Elected Associate 1925, February 2, 974

Buxton, J. B., Elected Fellow 1925, May 18, 977

Callow, E. H., Elected Fellow 1925, February 16, 975

Canonical forms of the quaternary cubic associated with arbitrary quadrics (Turnbull), 92

Carroll, D. C., Elected Fellow 1923, November 26, 806

Carter, F. W., Elected Fellow 1924, July 14, 812

Cary, A. P., Elected Associate 1923, October 29, 804

—— and Rideal, E. K., The spreading of oils and fats on water-surfaces, 975

Cashmore, A. E., Elected Fellow 1924, May 19, 811

Catalytic activity, apparatus for the investigation of the effect of poisonous substances, and mixed vapours on (Constable), 738

Characteristics of complexes of conics in space of four dimensions (James), 621

Chemical constant of chlorine (Cox), 491

Chemical reactions, apparatus for measuring the velocity of very rapid (Hartridge and Roughton), 426

Cherry, T. M., Integrals of systems of ordinary differential equations, 273

—— On Poincaré's theorem of "The non-existence of uniform integrals of dynamical equations," 287

—— On integrals developable about a singular point of a Hamiltonian system of differential equations, 325

—— Integrals developable about a singular point of a Hamiltonian system of differential equations. Part II, 510

—— Some examples of trajectories defined by differential equations of a generalised dynamical type. *See Transactions*, XXIII

—— On the form of the solution of the equations of dynamics. *See Transactions*, XXIII

Chlorine, chemical constant of (Cox), 491

Chords of a normal quartic curve (Bath), 189

Circles, an extension of Wallace's, Miquel's and Clifford's theorems on (White), 684

Circumcircle, extension of Wallace's pedal property of the (Richmond), 34

Clifford's theorems on circles, an extension of (White), 684

Cockcroft, J. D., Elected Fellow 1925, January 19, 974

—— The temperature distribution in a

transformer or other laminated core of rectangular cross section in which heat is generated at a uniform rate, 759

Coit, G. E., Elected Associate 1923, October 29, 804

Colloidal sols, protecting and sensitising (Rideal), 101

Complexes of conics and the Weddle surface (James), 201

Complexes of conics in space of four dimensions (James), 621

Conduction of heat, random flight and (Burnside), 167

Conics, complexes of, and the Weddle surface (James), 201

Conics in space of four dimensions, characteristics of complexes of (James), 621

Conics through fives of six points (White), 11

Constable, F. H., Elected Fellow 1924, December 8, 973

—— An apparatus for the investigation of the effect of poisonous substances, and mixed vapours on catalytic activity, 738

Cooper, R., Elected Fellow 1924, November 24, 972

Cox, R. R. S., Note on the chemical constant of chlorine, 491

Crowther, J. A., On a modified electrostatic oscillograph, 68

—— An oscillographic study of the production of X-rays by a Coolidge tube, 70

Cubic curves which are geodesic upon a quadric surface (Richmond), 42

Curtiss, L. F., A preliminary note on a direct determination of the distribution of intensity in the natural β-ray spectrum of Ra B and Ra C, 597

Curves, certain nets of plane (White), 1

Darwin, C. G., Notes on optical constants, 817, 824

—— The optical constants of matter. *See Transactions*, XXIII

Davies, E. L. and Livens, G. H., The kinetic theory of metallic conduction, 555

Davy, J. B., Elected Fellow 1924, July 14, 812

Dean, W. R., Elected Fellow 1923, November 26, 806

Determinant, general (MacMahon), 633

Dickinson, R. G., Elected Associate 1924, November 10, 972

Differential equations, Hamiltonian system of (Cherry), 325, 510

Differential equations, integrals of systems of ordinary (Cherry), 273

Differential invariants of the Laguerre group (Kubota), 113

Dirac, P. A. M., Elected Associate 1923, October 29, 804

—— Dissociation under a temperature gradient, 132

—— Note on the Doppler principle and Bohr's frequency condition, 432

Dissociative equilibrium in an external field of force (Milne), 493

Dixon, M., Elected Fellow 1925, March 2, 976

Doppler principle and Bohr's frequency condition (Dirac), 432

Dunn, J. S., Elected Fellow 1924, November 24, 972

Dymond, E. G., On the measurement of the critical potentials of gases, 405

Dynamical equations, Poincaré's theorem (Cherry), 287

Eddington, A. S., Internal constitution of the stars, 978

Einstein's quadratic form, electromagnetic equations as basis of (Hargreaves), 138

Electrical counter, action of the (Emeléus), 676

Electromagnetic equations as basis of Einstein's quadratic form (Hargreaves), 138

Electron impact, angular momentum and (Blackett), 56

Electrons by α-particles, capture and loss of (Fowler), 253

Electrons, passage of, through small apertures (Lehmann and Osgood), 731

Electrostatic oscillograph (Crowther), 68

Ellis, C. D., The high energy groups in the magnetic spectrum of the radium C β-rays, 369

—— and Wooster, W. A., Note on the heating effect of the γ-rays from Ra B and Ra C, 595

—— —— The atomic number of a radioactive element at the moment of emission of the γ-rays, 844

—— —— The β-ray type of disintegration, 849

Emeléus, K. G., The number of β-particles from Radium E, 400

—— The action of the electrical counter, 676

—— *See* Appleton, E. V.

"Equally probable," on the phrase (Burnside), 669

Equation of state of a gas (Jones), 105

Equilibrium, dissociative, in an external field of force (Milne), 493

Errors of observation (Burnside), 26

Euclid's postulate of parallels (HILL), 964
EVANS, U. R., The mechanism of the rusting of iron, 54

FARREN, W. S., Elected Fellow 1925, March 2, 976
FEATHER, N., Elected Associate 1925, May 4, 976
FELL, H. B., Elected Associate 1924, October 27, 971
Feuerbach's theorem, generalization of (GABBATT), 601
FISHER, R. A., Theory of statistical estimation, 700
Five lines, in space of four dimensions, which lie upon a quadric threefold and the normal quartic curves of which they are chords (BATH), 189
Flight, soaring, angle of incidence in (HANKIN and NORTH), 186
Four particles, the problem of (BRILL), 88
FOWLER, R. H., The statistical theory of dissociation and ionization by collision, with applications to the capture and loss of electrons by a-particles, 253
—— A theoretical study of the stopping power of hydrogen atoms for a-particles, 793
—— Assemblies of imperfect gases by the method of partition functions, 861
FRANCIS, E. C., On differentiation with respect to a function, 924
—— The Lebesgue-Stieltjes integral, 935
Frequency, on the idea of (BURNSIDE), 726
FREW, J. G. H., Elected Associate 1923, October 29, 805
Function, differentiation with respect to a (FRANCIS), 924
Functions, symmetric, of which the general determinant is a particular case (MACMAHON), 633

GABBATT, J. P., A generalization of Feuerbach's theorem, 601
—— On pedal quadrics in non-euclidean hyperspace, 751
Gamma rays from Ra B and Ra C, heating effect of the (ELLIS and WOOSTER), 595
Gamma rays, atomic number of a radioactive element at the moment of emission of the (ELLIS and WOOSTER), 844
Gas, equation of state of a (JONES), 105
Gas, steady flow of a, in two dimensions (LEES), 350
Gases, assemblies of imperfect, by the method of partition functions (FOWLER), 861
Gases, measurement of the critical potentials of (DYMOND), 405
Generating spaces, loci which have two systems of (RICHMOND and BATH), 319
Geodesics upon a quadric surface (RICHMOND), 42
GODWIN, H., Elected Fellow 1924, January 21, 808
Graeffe's method for complex roots of algebraic equations (BRODETSKY and SMEAL), 83
GREAVES, W. M. H., On the stability of the periodic states of the triode oscillator, 16
GREEN, H. G., Elected Fellow 1925, February 16, 975
GRENFELL, L. E., Elected Associate 1924, November 10, 972
Group, modular, of order $\frac{1}{2}p\,(p^2-1)$ as a group of linear substitutions on $\frac{1}{2}\,(p-1)$ symbols (BURNSIDE), 779
Groups of linear substitutions which contain irreducible metacyclical subgroups (BURNSIDE), 788
GUGGENHEIM, E. A., Elected Fellow 1925, March 2, 976
GURNEY, R. W., Elected Fellow 1924, February 18, 809

HALDANE, J. B. S., A mathematical theory of natural and artificial selection. *See Transactions*, XXIII
Halides, thallous thallic (BERRY), 363
Hamiltonian system of differential equations (CHERRY), 325, 510
Hankel transforms (TITCHMARSH), 67
HANKIN, E. H. and NORTH, J. D., On the angle of incidence in soaring flight, 186
HARGREAVES, R., The electromagnetic equations as basis of Einstein's quadratic form, 138
—— The quadratic form for radial acceleration, in the theory of relativity, 248
HARNWELL, G. P., Elected Associate 1924, October 27, 971
HARRISON, W. J., On the motion of spheres, circular and elliptic cylinders through viscous liquid. *See Transactions*, XXIII
HARTREE, D. R., The spectra of some lithium-like and sodium-like atoms, 409
—— Some methods of estimating the successive ionisation potentials of any element, 464
—— Doublet and triplet separations in optical spectra as evidence whether orbits penetrate into the core, 904

HARTRIDGE, H., and ROUGHTON, F. J. W., Improvements in the apparatus for measuring the velocity of very rapid chemical reactions, 426
Hausdorff's proof of the extended Riesz Fischer theorem (POLLARD), 777
HAYNER, L. T., Elected Associate 1925, January 19, 974
Helical method of focussing β-rays (TRICKER), 454
HERKLOTS, G. A. C., Elected Associate 1924, October 27, 971
HEYCOCK, C. T. and MILLS, W. H., The heat of combustion of ethylene, 808
HILL, M. J. M., On the substitution of Wallis's postulate of similarity for Euclid's postulate of parallels, 964
HILL, R., Elected Fellow 1925, March 2, 976
HINDLE, E., Elected Fellow 1923, November 26, 806
HINTON, J. W., Elected Associate 1925, January 19, 974
HIRST, H. S., Elected Associate 1923, October 29, 805
HUFFORD, M. E., Elected Associate, 1925, February 2, 974
HURST, C. C., Elected Fellow 1925, March 2, 976
HYDE, W. A., Elected Associate 1923, November 12, 806
Hydrogen atoms, theoretical study of the stopping power of, for α-particles (FOWLER), 793
Hyper-space, number of lines which meet four regions in (VAIDYANATHASWAMY), 49
HYSLOP, J., The integral expansions of arbitrary functions connected with integral equations, 169

Integral equations, integral expansions of arbitrary functions connected with (HYSLOP), 169
Integral expansions of arbitrary functions connected with integral equations (HYSLOP), 169
Integrals developable about a singular point of a Hamiltonian system of differential equations (CHERRY), 325, 510
Integrals of systems of ordinary differential equations (CHERRY), 273
Ionisation potentials, methods of estimating the successive (HARTREE), 464
Iron, mechanism of the rusting of (EVANS), 54

JAMES, C. G. F., Involutions on a normal quartic curve in space of four dimensions, 24
—— Complexes of conics and the Weddle surface, 201
JAMES, C. G. F., Characteristics of complexes of conics in space of four dimensions, 621
JONES, G. G., Elected Associate 1923, October 29, 804
JONES, I. C., Elected Associate 1924, November 24, 972
JONES, J. E., Elected Fellow 1924, November 24, 972
—— The equation of state of a gas, 105

KAPITZA, P. L., Elected Fellow 1923, November 26, 806
KEILIN, D., Elected Fellow 1923, December 3, 807
Kinetic theory of metallic conduction (DAVIES and LIVENS), 555
KOVARIK, A. F., Elected Associate 1924, November 24, 972
Kronecker's theorem (THOMAS), 886
KRONIG, R. de L., The theory of the influence of magnetic fields on the stopping power of gases for α-particles, 773
KUBOTA, T., On the differential invariants of the Laguerre group, 113

L-levels for radiation of varying wavelength, relative absorbing powers of the (SKINNER), 379
L-radiation, β-ray spectrum, due to the natural, of radium B (BLACK), 838
L-radiation from radium B, the β-ray spectrum of the natural (BLACK), 832, 838
Laguerre group, differential invariants of the (KUBOTA), 113
LAKE, P., Wegener's theory of continental drift, 806
LARMOR, J., An early formulation by Stokes of the theories of the rotatory polarizations of light, 76
—— Why wireless electric rays can bend round the earth (Abstract), 488
LAVERICK, L. W., Elected Associate 1923, December 3, 807
Lebesgue-Stieltjes integral (FRANCIS), 935
LEES, S., On a case of steady flow of a gas in two dimensions, 350
LEHMANN, J. F., Elected Associate 1923, November 12, 806
LEHMANN, J. F. and OSGOOD, T. H., the passage of electrons through small apertures, 731
LEVER, W. H., Elected Fellow 1924, March 3, 809*
Linear equations with an infinity of unknowns, a system of (TITCHMARSH), 282
Linear substitutions, groups of (BURNSIDE), 779, 788

Lines in threefold space and points of a quadric in fivefold space (Turnbull), 694

Lithium-like and sodium-like atoms, spectra of some (Hartree), 409

Littlewood, J. E., Trajectories of small horizontal velocity in a resisting medium, 217

—— Two notes on the Riemann Zeta-function, 234

—— On the zeros of the Riemann Zeta-function, 295

Livens, G. H. and Davies, E. L., The kinetic theory of metallic conduction, 555

Loci which have two systems of generating spaces (Richmond and Bath), 319

Lock, C. N. H., Elected Fellow 1925, January 19, 974

Lorentz group (Brill), 630

Lowry, T. M., A geometrical pattern having the symmetry of quartz, 808

Luck, J. M., Elected Associate 1924, January 21, 808

MacMahon, P. A., The symmetric functions of which the general determinant is a particular case, 633

—— The enumeration of the partitions of multipartite numbers, 951

—— Researches in the theory of determinants. *See Transactions*, xxiii

MacWillis, J., Elected Associate 1923, December 3, 807

Magnetic fields on the stopping power of gases for α-particles, theory of the influence of (Kronig), 773

Magnetic spectrum of the radium C β-rays, high energy groups in the (Ellis), 369

Mahajani, G. S., Elected Fellow 1924, December 8, 973

Martin, L. H., Elected Associate 1923, October 29, 805

Mass rays, detection of (Aston), 548

Maunsell, F. G., Elected Fellow 1924, November 10, 971

Metallic conduction, kinetic theory of (Davies and Livens), 555

Miller, A. B., Elected Associate 1925, March 2, 976

Mills, W. H. and Heycock, C. T., The heat of combustion of ethylene, 808

Mills, W. H. and Warren, E. H., The configuration of the ammonium radical, 809

Milne, E. A., Dissociative equilibrium in an external field of force, 493

Miquel's theorem on circles, an extension of (White), 684

Mirsky, A. E., Elected Associate 1924, May 5, 810

Modular group of order $\frac{1}{2}p(p^2-1)$ (Burnside), 779

Moquin, H., Elected Associate 1925, February 2, 974

Morrice, G. G., Elected Fellow 1924, July 14, 812

Moullin, E. B., On the current induced in a wireless telegraph receiving antenna, 567

Multipartite numbers, enumeration of the partitions of (MacMahon), 951

Murray, H. A., Elected Associate 1924, October 27, 971

Nagaoka, H., The transmutation of mercury into gold, 978

Nasini, A. G., Elected Associate 1924, February 18, 809

Nath, V., Elected Associate 1923, October 29, 805

Newall, H. F., Sun-spots, 809

Newman, M. H. A., On the theorem of Pappus, 919

Nobbs, C. G., Elected Fellow 1924, December 8, 973

Norrish, R. G. W., The mechanism of molecular activation, 809

North, J. D. and Hankin, E. H., On the angle of incidence in soaring flight, 186

Observation, errors of (Burnside), 26

Omer-Cooper, T., Elected Fellow 1924, December 8, 973

Optical constants (Darwin), 817, 824

Optical interference method of measuring Young's modulus for rods (Searle), 475

Oscillographic study of the production of X-rays by a Coolidge tube (Crowther), 70

Osgood, T. H., Elected Associate 1923, October 29, 805

—— and Lehmann, J. F., The passage of electrons through small apertures, 731

Owen, G., Elected Associate 1924, November 24, 972

Paine, R. W., Elected Fellow 1925, January 19, 974

Palmer, W. G., A method of finding the composition of adsorption films of mixed gases, 975

—— The stability of gas films adsorbed on tungsten, 809

Pappus' theorem (Newman), 919

Parsons, C. A., Elected Fellow 1923, November 26, 806

Partitions of multipartite numbers (MacMahon), 951

Pearson, K. A., Elected Fellow 1923, November 26, 806

Pedal quadrics in non-euclidean hyperspace (Gabbatt), 751
Penrose, L. S., Elected Fellow 1925, January 19, 974
Periodic states of the triode oscillator, stability of the (Greaves), 16
Phillips, F. C., Elected Fellow 1923, November 26, 806
Poincaré's theorem of "The non-existence of uniform integrals of dynamical equations" (Cherry), 287
Pollard, S., On Hausdorff's proof of the extended Riesz-Fischer theorem, 777
Polygons inscribed in one conic and circumscribed to another (Burnside), 163
Pope, W. J., The preparation of sulphuryl chloride, 808
Preece, C. T., Elected Fellow 1923, November 26, 806
Presswood, C., Elected Associate 1923, November 12, 806
Pringle-Jameson, A., Elected Associate 1925, May 18, 977
Proceedings at the Meetings held during the Session 1923–1924, 804
—— 1924–1925, 970
Puri, I. M., Elected Associate 1924, May 5, 810
Putterill, V. A., Elected Associate 1924, May 5, 810

Quadratic form for radial acceleration, in the theory of relativity (Hargreaves), 248
Quadric, simplexes doubly incident with a (Vaidyanathaswamy), 655
Quadric surfaces, three, vector algebra of eight associated points of (Turnbull), 481
Quadrics, pedal, in non-euclidean hyperspace (Gabbatt), 751
Quadric threefold, five lines, in space of four dimensions, which lie upon a (Bath), 189
Quartic curve in space of four dimensions, involutions on a (James), 24
Quartic curves in space of four dimensions, chords of (Bath), 189
Quastel, J. H., Elected Associate 1924, January 21, 808: Elected Fellow 1924, December 8, 973
Quaternary cubic associated with arbitrary quadrics, canonical forms of the (Turnbull), 92
Quayle, W. C., Elected Associate 1923, October 29, 805

Radial acceleration, quadratic form for, in the theory of relativity (Hargreaves), 248
Radiation, structure of (Stoner), 577
Radium B, natural X-ray spectrum of (Rutherford and Wooster), 834
Radium C β-rays, high energy groups in the magnetic spectrum of the (Ellis), 369
Radium emanation, decay constant of (Bastings), 562
Ramsey, F. P., Elected Fellow 1925, May 18, 977
Random flight and conduction of heat, problems of (Burnside), 167
Relativity, radial acceleration in the theory of (Hargreaves), 248
Resisting medium, trajectories in a (Littlewood), 217
Richards, W. T., Elected Associate 1924, November 10, 972
Richmond, H. W., On an extension of Wallace's pedal property of the circumcircle, 34
—— On the real twisted cubic curves which are geodesic upon a quadric surface, 42
—— and Bath, F., On loci which have two systems of generating spaces, 319
Rideal, E. K., On protecting and sensitising colloidal sols, 101
—— A note on protecting and sensitising colloidal sols, 809
—— and Casy, A. P., The spreading of oils and fats on water surfaces, 975
Riemann Zeta-function (Littlewood), 234
Riemann Zeta-function, zeros of the (Littlewood), 295
Riesz-Fischer theorem (Pollard), 777
Rolleston, H. D., Elected Fellow 1925, May 18, 977
Room, T. G., Elected Fellow 1923, November 26, 806
Rose, D. C., Elected Associate 1924, October 27, 971
Roughton, F. J. W. and Hartridge, H., Improvements in the apparatus for measuring the velocity of very rapid chemical reactions, 426
Rusting of iron, mechanism of the (Evans), 54
Rutherford, E., The natural and artificial disintegration of elements, 972
Rutherford, E. and Wooster, W. A., The natural X-ray spectrum of radium B, 834
Rutnam, D. R., Elected Fellow 1923, November 26, 806

Saddler, W., Triple binary forms; the complete system for a single (1, 1, 1) form with its geometrical interpretation, 688

Salaman, M. H., Elected Associate 1924, February 18, 809

Sass, R. R., Elected Associate 1924, December 8, 973

Schofield, R. K., Elected Fellow 1924, November 24, 972

Searle, G. F. C., An optical interference method of measuring Young's modulus for rods, 475

—— The determination of the frequency of an alternating current supply by the vibrations of rods, 539

Sen, B. M., On the applicability and deformation of surfaces, 243

Seth, T. N., Elected Associate 1923, October 29, 805

Simplexes doubly incident with a quadric (Vaidyanathaswamy), 655

Skinner, H. W. B., The relative absorbing powers of the *L*-levels for radiation of varying wave-length, 379

Smeal, G. and Brodetsky, S., On Graeffe's method for complex roots of algebraic equations, 83

Soap-bubbles, bursting of, in a uniform electric field (Wilson and Taylor), 728

Soaring flight, angle of incidence in (Hankin and North), 186

Space, lines in threefold, and points of a quadric in fivefold (Turnbull), 694

Spectra of some lithium-like and sodium-like atoms (Hartree), 409

Spectra, optical, doublet and triplet separations in (Hartree), 904

Spectrum, β-ray, due to the natural *L*-radiation of radium B (Black), 838

Spectrum of radium B, natural X-ray (Rutherford and Wooster), 834

Statistical estimation (Fisher), 700

Steers, J. A., Elected Fellow 1924, December 8, 973

Steiner, generalisation of a theorem of (Baker), 28

Stephenson, M., Elected Associate 1924, May 5, 810

Steward, G. C., Elected Fellow 1923, December 3, 807

Stieltjes integral (Francis), 935

Stockdale, D., A thermostat for high temperatures, 975

Stokes, early formulation by, of the theories of the rotatory polarizations of light (Larmor), 76

Stoneley, R., Elected Fellow 1923, November 26, 806

Stoner, E. C., The structure of radiation, 577

Successive ionisation potentials, methods of estimating the (Hartree), 464

Sulphur, liquid, electrical properties of (Black), 393

Surfaces, applicability and deformation of (Sen), 243

Symmetric functions of which the general determinant is a particular case (MacMahon), 633

Taylor, A. M., Elected Fellow 1924, November 24, 972

Taylor, G. I. and Wilson, C. T. R., The bursting of soap-bubbles in a uniform electric field, 728

Taylor, M., Elected Associate 1923, November 26, 806

Taylor, P. A., Elected Fellow 1925, January 19, 974

Temperature distribution in a transformer (Cockcroft), 759

Temperature gradient dissociation under a (Dirac), 132

Thallous thallic halides (Berry), 363

Thomas, L. H., An extended form of Kronecker's theorem with an application which shows that Burgers' theorem on adiabatic invariants is statistically true for an assembly, 596

Thomson, D. L., Elected Associate 1924, October 27, 971

Thomson, J. J., The structure of light, 978

Titchmarsh, E. C., Hankel transforms, 67

—— A system of linear equations with an infinity of unknowns, 282

Trajectories of small horizontal velocity in a resisting medium (Littlewood), 217

Transformer, temperature distribution in a (Cockcroft), 759

Tricker, R. A. R., A helical method of focussing β-rays, 454

Triode oscillator, stability of the periodic state of the (Greaves), 16

Triple binary forms (Saddler), 688

Tucker, B. W., Elected Associate 1924, October 27, 971

Turnbull, H. W., Canonical forms of the quaternary cubic associated with arbitrary quadrics, 92

—— On the vector algebra of eight associated points of three quadric surfaces, 481

—— A geometrical treatment of the correspondence between lines in threefold space and points of a quadric in fivefold space, 694

Turney, T. H., Elected Fellow 1923, November 26, 806

Vacuum discharge tubes, positive flash in (Whiddington), 574

VAIDYANATHASWAMY, R., On the number of lines which meet four regions in hyper-space, 49
—— On simplexes doubly incident with a quadric, 655
Vector algebra of eight associated points of three quadric surfaces (TURNBULL), 481
Velocity of very rapid chemical reactions, apparatus for measuring the (HARTRIDGE and ROUGHTON), 426

WADIA, J. H., Elected Associate 1923, October 29, 805
Wallace's pedal property of the circumcircle (RICHMOND), 34
Wallace's theorem on circles, an extension of (WHITE), 684
Wallis's postulate of similarity for Euclid's postulate of parallels, substitution of (HILL), 964
WARREN, E. H. and MILLS, W. H., The configuration of the ammonium radical, 809
WATSON, D. M. S., Orthogenesis, 975
Waves, penetration of, and their rays into a medium of gradually varying properties (WILCKEN), 124
WEBSTER, W. L., Elected Associate 1923, November 26, 806
Weddle surface (JAMES), 201
WEINSTEIN, A., Elected Associate 1924, November 10, 972
WELLS, G. P., Elected Fellow 1923, November 26, 806
WHIDDINGTON, R., On the positive flash in vacuum discharge tubes, 574
WHITE, F. P., On certain nets of plane curves, 1
—— The conics through fives of six points, 11
—— An extension of Wallace's, Miquel's and Clifford's theorems on circles, 684
WHITEHOUSE, F. W., Elected Associate 1924, November 10, 972
WHITFIELD, E. V., Elected Fellow 1923, November 26, 806
WILCKEN, J. A., On the penetration of waves and their rays into a medium of gradually varying properties, 124
WILLEY, E. J. B., Elected Associate 1925, February 16, 975
WILSON, A. H., Elected Associate 1923, October 29, 805
WILSON, C. T. R., The acceleration of β-particles in strong electric fields such as those of thunderclouds, 534
WILSON C. T. R. and TAYLOR, G. I., The bursting of soap-bubbles in a uniform electric field, 728
WILSON, W., Elected Associate 1923, October 29, 805
WILTON, J. R., Elected Fellow 1923, November 26, 806
Wireless electric rays, bending of, round the earth (LARMOR), 488
Wireless signal strength measurements made during the solar eclipse of January 24, 1925 (APPLETON and BARNETT), 672
Wireless telegraph receiving antenna, current induced in a (MOULLIN), 567
WOOSTER, W. A., Elected Fellow 1924, November 24, 972
WOOSTER, W. A. and ELLIS, C. D., Note on the heating effect of the γ-rays from Ra B and Ra C, 595
—— —— The atomic number of a radioactive element at the moment of emission of the γ-rays, 844
—— —— The β-ray type of disintegration, 849
WOOSTER, W. A. and RUTHERFORD, E., The natural X-ray spectrum of radium B, 834

X-ray, natural, spectrum of radium B (RUTHERFORD and WOOSTER), 834
X-rays, oscillographic study of the production of, by a Coolidge tube (CROWTHER), 70

Young's modulus for rods, optical interference method of measuring (SEARLE), 475

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CONTENTS.

PAGE

Notes on optical constants. By Mr C. G. **Darwin**, F.R.S., Tait Professor of Natural Philosophy in the University of Edinburgh. I. *The optical behaviour of certain atomic models* 817

Notes on optical constants. By Mr C. G. **Darwin**, F.R.S., Tait Professor of Natural Philosophy in the University of Edinburgh. II. *The lateral scattering from a gas* 824

The β-ray spectrum of the natural L-radiation from radium B. By D. H. **Black**, M.Sc., Emmanuel College. (Communicated by Prof. Sir E. **Rutherford**) 832

The natural X-ray spectrum of radium B. By Professor Sir **Ernest Rutherford**, F.R.S., and W. A. **Wooster**, B.A., Peterhouse. (With One Text-figure) 834

The analysis of the β-ray spectrum due to the natural L radiation of radium B. By D. H. **Black**, M.Sc., Emmanuel College. (Communicated by Prof. Sir E. **Rutherford**) 838

The atomic number of a radioactive element at the moment of emission of the γ-rays. By Dr C. D. **Ellis**, Trinity College, and W. A. **Wooster**, B.A., Peterhouse. (With One Text-figure) 844

The β-ray type of disintegration. By Dr C. D. **Ellis**, Trinity College, and W. A. **Wooster**, B.A., Peterhouse 849

Assemblies of imperfect gases by the method of partition functions. By Mr R. H. **Fowler**, Trinity College 861

An extended form of Kronecker's theorem with an application which shows that Burgers' theorem on adiabatic invariants is statistically true for an assembly. By L. H. **Thomas**, B.A., Trinity College. (Communicated by Mr R. H. **Fowler**) 886

Doublet and triplet separations in optical spectra as evidence whether orbits penetrate into the core. By Mr D. R. **Hartree**, St John's College. (With Three Text-figures) 904

On the theorem of Pappus. By Mr M. H. A. **Newman**, St John's College 919

On differentiation with respect to a function. By Mr E. C. **Francis**, Peterhouse 924

The Lebesgue-Stieltjes integral. By Mr E. C. **Francis**, Peterhouse . 935

The enumeration of the partitions of multipartite numbers. By Major P. A. **MacMahon**, Sc.D., St John's College 951

On the substitution of Wallis's postulate of similarity for Euclid's postulate of parallels. By Professor M. J. M. **Hill**, Peterhouse. (With Eight Text-figures) 964

Proceedings at the Meetings held during the Session 1924—1925 . . 970

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